Approximation by PDEs

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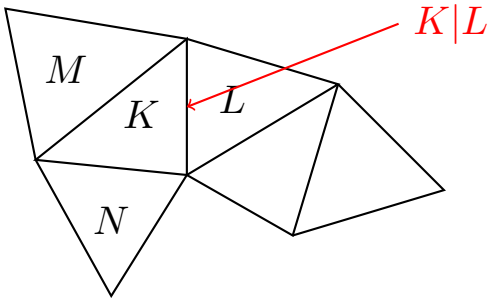


Figure 1: A mesh in $\mathbb{R}^2$

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Abstract

In this course, we will see how to understand and describe the large scale limit of various discrete evolution systems (random and deterministic) with the help of partial differential equations. This will be the occasion to use, and discover, some standard tools from the theory of PDEs, of numerical analysis, and of statistical physics.

1 Introduction

1.1 Discrete conservation laws

Suppose that we are given a family $\mathcal{T}$ of open polyhedral sets forming a partition of the space $\mathbb{R}^d$: for all distinct $K, L \in \mathcal{T}$, we assume that $K \cap L = \emptyset$ and that $\overline{K} \cap \overline{L}$ is contained in an hyperplane of $\mathbb{R}^d$. The partition is understood up to a negligible set: the Lebesgue measure of $\mathbb{R}^d \setminus \bigcup_{K\in\mathcal{T}} K$ is zero. The picture 1 below gives the example of a triangulation of the plane.
We consider the following evolution of an extensive quantity u: let $0 = t_0 < t_1 < \cdots < t_n < \cdots$ be some discrete times, let U_K^n denote the amount of the quantity u in the cell K at time t_n. We assume that U_K^{n+1} is given by the formula

$$U_K^{n+1} = U_K^n + \Delta t_n \sum_{L\in\mathcal{N}(K)} |K|L|Q_{L\to K}^n. \tag{1.1}$$

The notations used in (1.1) are the following ones: Δt_n is the length $t_{n+1} - t_n$ of the time interval, $\mathcal{N}(K)$ is the set of neighbors of K: $L \in \mathcal{T}$ is a neighbor of K if $K|L := \overline{K} \cap \overline{L}$ is non-empty

and of finite $(N-1)$-dimensional Hausdorff measure $|K|L|$ (in particular, K is not a neighbor of K). The quantity $\Delta t_n |K|L| Q^n_{L\to K}$ represents a certain flux of the quantity u that has passed through the interface $K|L$ from the cell L to the cell K between the times t_n and t_{n+1}. We have put in factor the term $\Delta t_n |K|L|$ because we prefer to work with densities, rather than with scale-dependent quantities (the typical scales here depend on the size of the cells and of Δt_n and will tend to zero at some point later on). For the same reason, it is more appropriate to introduce $|K|$, the Lebesgue measure of the cell K, and to work with the scaled quantity $u^n_K = U^n_K/|K|$, which satisfies the equation

$$u^{n+1}_K = u^n_K + \frac{\Delta t_n}{|K|} \sum_{L\in\mathcal{N}(K)} |K|L|Q^n_{L\to K}. \tag{1.2}$$

Assume that the densities of flux $Q^n_{L\to K}$ satisfy the following condition:

$$Q^n_{L\to K} = -Q^n_{K\to L}, \tag{1.3}$$

for all $n \in \mathbb{N}$, for all $K, L \in \mathcal{T}$ being neighbors. The condition (1.3) ensures that the (algebraic) quantity of u that was given by the cell K to the cell L is the quantity of u received by the cell L from the cell K. Under (1.3), the evolution given by (1.2) is conservative: we will show in particular that, when it makes sense, the quantity

$$\sum_{K\in\mathcal{T}} |K| u^n_K$$

is constant with respect to n. Our objective will be to explain what is the limit of (u^n_K) when Δt_n and $|K|$ tends to 0. We need to be more specific on our framework to achieve this goal. Let us simply say for the moment that what we will obtain in the end are some conservation laws

$$\partial_t u + \mathrm{div}_x(Q) = 0, \tag{1.4}$$

where $Q(x)$ is a function of x, $u(x)$ and $\nabla u(x)$. The derivation of (1.4) is related to the analysis of the Finite Volume method, which is used to compute the solution of conservation laws such as (1.4) with the help of the discrete formulation (1.2).

1.2 The symmetric simple exclusion process

Let $0 < N < L$ be some integers. Consider N particles located at one of the site $1, \dots, L-1$ that evolve according to the following process: there is always one particle at site 0 and, for each site $\mathtt{x} \in \{1, \dots, L-1\}$, we draw a random time $T_{\mathtt{x}}$ that follows an exponential law of parameter $\lambda > 0$, so that the family $\{T_{\mathtt{y}}\}$ is independent. Consider the point $\mathtt{x}_*$ at which $\mathtt{x} \mapsto T_{\mathtt{x}}$ is minimal and let the particle at $\mathtt{x}_*$ jump from its original site $\mathtt{x}$ to a new site $\mathtt{y}$ with probability $p(\mathtt{x}_*, \mathtt{y})$, the jump occurring under the restriction that the arrival site $\mathtt{y}$ is vacant. Then start over. This process is called an exclusion process for the reason that jumps to occupied sites are excluded. It is termed *simple* to make the distinction with some more complicated situations, where the probability of a jump from $\mathtt{x}$ to $\mathtt{y}$ may depend not only on $\mathtt{x}$ and $\mathtt{y}$, but on the whole interval $[\mathtt{x}, \mathtt{y}]$ and on the disposition of particles in this interval. We also call the process *symmetric* when $p(\mathtt{x}, \mathtt{x}+l) = p(\mathtt{x}, \mathtt{x}-l)$, whenever the quantities are well defined. Here we will consider the case $p(\mathtt{x}, \mathtt{y}) = 0$ if $|\mathtt{x} - \mathtt{y}| \neq 1$, so that only jumps to left or right immediate neighboring site are possible, and equi-probable. At the boundary, we assume $p(0,1) = 0$, $p(L-1, L-2) = 1$. We can put in correspondence this evolution of particles with the evolution of a random interface

described as follows: we set $H(0) = 0$ and, for $\mathtt{x} \in \{1, \dots, L\}$, define H as the discrete primitive function

$$H(\mathtt{x}) = \sum_{\mathtt{y}=0}^{\mathtt{x}-1} (2\eta(\mathtt{y}) - 1), \tag{1.5}$$

where $\eta(\mathtt{y}) \in \{0, 1\}$ is the number of particle at $\mathtt{y}$. Then we interpolate linearly between those points. Conversely, we deduce $\eta(\mathtt{x})$ from H by the "differentiation" formula $\eta(\mathtt{x}) = [1 + H(\mathtt{x} + 1) - H(\mathtt{x})]/2$. In the situation where the site $\mathtt{x}$ is occupied and the site $\mathtt{x}+1$ is vacant, the shape (above $\{\mathtt{x}, \mathtt{x}+1, \mathtt{x}+2\}$) of the function H is $\wedge$. If the particle at $\mathtt{x}$ jumps at $\mathtt{x}+1$, it becomes $\vee$, – and conversely. We consider then the following problem: assume that L and N are very large. For definiteness, we will take $L = 2N$, which ensures that $H(L) = 0$. Consider the change of scale

$$\mathrm{h}_t^L(x) = L^{-1} H_t(Lx), \quad x \in (0, 1). \tag{1.6}$$

What can we say about the evolution of the profile $t \mapsto \mathrm{h}_t^L$, for, possibly, t very large? We will see that, under adequate conditions on the initial data, and after the following parabolic change of time scale:

$$h_t^L(x) = \mathrm{h}_{L^2 t}^L = L^{-1} H_{L^2 t}(Lx), \quad x \in (0, 1), t > 0, \tag{1.7}$$

we have a kind of law of large numbers: for all final time $T > 0$, h^L is converging in probability in $L^\infty(0, T; L^2(0, 1))$ to a deterministic profile h which is completely determined as a solution of the heat equation with homogeneous Dirichlet boundary conditions.

1.3 Interacting particle systems

We will now consider a problem similar to the previous one, with the difference that it is multi-dimensional and that jumps to occupied sites are not excluded. Let Λ_N be a finite subset of $\mathbb{Z}^d$. We consider a system of particles scattered on Λ_N, which interact as follows: let $\mathtt{x}$ denote a typical site of Λ_N and let $\eta_t(\mathtt{x})$ denote the number of particles located at site $\mathtt{x}$ at time t. We will be interested in the evolution in time of the functions $\mathtt{x} \mapsto \eta_t(\mathtt{x})$. The state space is therefore $\mathcal{E}_N := \mathbb{N}^{\Lambda_N}$, the set of functions $\Lambda_N \to \mathbb{N}$. The evolution is described by the following algorithm: each site $\mathtt{x}$ has its own clock that is independent from the clocks at other sites, and that rings after a time $T_{\mathtt{x}}$ which is a random variable of exponential law of parameter $\lambda(\eta(\mathtt{x}))$. Assume that it is at the site $\mathtt{x}_*$ that a clock is ringing first. If $\eta(\mathtt{x}_*) > 0$, then one particle of the site $\mathtt{x}_*$ jumps to an other site $\mathtt{y}$ chosen at random in Λ_N, according to a transition probability $p(\mathtt{x}_*, \mathtt{y})$ (possibly, at that stage, some exclusion rules may be added, see Section **??**). Then we start over. Let us consider the case where Λ_N is the discrete torus $\mathbb{T}_N^d = \mathbb{Z}^d / N\mathbb{Z}^d$ and p is compatible and translation invariant: for all $l \in N\mathbb{Z}^d$, $m \in \mathbb{Z}^d$,

$$p(\mathtt{x} + l, \mathtt{y}) = p(\mathtt{x}, \mathtt{y}), \quad p(\mathtt{x} + m, \mathtt{y} + m) = p(\mathtt{x}, \mathtt{y}). \tag{1.8}$$

Let us zoom out (*cf.* (1.6)) by considering the function

$$[0, 1)^d \ni x \mapsto N^{-1} \eta_t([Nx]) \tag{1.9}$$

extended by periodicity. In (1.9), $[Nx]$ is the element $\mathtt{x}$ of $\mathbb{T}_N^d \simeq \{0, \cdots, N-1\}^d$ such that $\mathtt{x}_i \le Nx_i < \mathtt{x}_i + 1$ for all $i = 1, \dots, d$. May it be the case that, possibly after a change of time scale (*cf.* (1.7)), some averaging phenomena would lead to a given deterministic behaviour? We will see that the question has to be refined, before being answered positively (at least in certain cases).

2 Martingales in continuous time

2.1 Conditional expectation

Proposition 2.1 (Conditional expectancy)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Let X be real-valued random variable which is integrable: $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists a unique $\mathcal{G}$-measurable and integrable random variable Z such that*

$$\mathbb{E}(\mathbf{1}_A X) = \mathbb{E}(\mathbf{1}_A Z), \quad \forall A \in \mathcal{G}. \tag{2.1}$$

We call Z the conditional expectancy of X knowing $\mathcal{G}$, denoted $\mathbb{E}(X|\mathcal{G})$.

Roughly speaking, $\mathbb{E}(X|\mathcal{G})$ is the average of X with respect to all the events not relative to $\mathcal{G}$. The following facts or examples illustrate this fact.

Fact 1. If $\mathcal{G} = \mathcal{F}$, then $\mathbb{E}(X|\mathcal{F}) = X$ a.s. If $\mathcal{G}$ is the trivial σ-algebra $\{\emptyset, \Omega\}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$.

Example 1. When $\mathcal{G}$ is the σ-algebra generated by an event $A \in \mathcal{F}$, $\mathcal{G} = \{\emptyset, A, A^c, \Omega\}$, then

$$\mathbb{E}(X|\mathcal{G}) = \frac{\mathbb{E}(\mathbf{1}_A X)}{\mathbb{P}(A)}\mathbf{1}_A + \frac{\mathbb{E}(\mathbf{1}_{A^c} X)}{\mathbb{P}(A^c)}\mathbf{1}_{A^c}.$$

If $X = \mathbf{1}_B$ where $B \in \mathcal{F}$, this gives $\mathbb{E}(\mathbf{1}_B|\mathcal{G}) = \mathbb{P}(B|A)\mathbf{1}_A + \mathbb{P}(B|A^c)\mathbf{1}_{A^c}$.

Fact 2. One has the following tower property: if $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$, then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H}) \text{ a.s.} \tag{2.2}$$

As a particular case, when $\mathcal{H} = \{\emptyset, \Omega\}$, we obtain $\mathbb{E}[\mathbb{E}(X|\mathcal{G})] = \mathbb{E}[X]$.

Example 2. Let X, Y be two independent random variable and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel function. Then $Z = \mathbb{E}(f(X,Y)|\sigma(Y))$ is $\sigma(Y)$-measurable, and it is known that such a function can be written $h(Y)$, where h is Borel. In general, when saying that a $\sigma(Y)$-measurable function has the form $h(Y)$, we have no particular information on h. Here, however, we know very well what is h: it is the function obtained by averaging with respect to "all that is not Y", *i.e.*

$$\mathbb{E}(f(X,Y)|\sigma(Y)) = h(Y), \quad h(y) := \mathbb{E}(f(X,y)). \tag{2.3}$$

Example 3. Let D denote the set of dyadic cubes in $[0,1)^d$, and for $n \in \mathbb{N}$, let D_n denote the subset of dyadic cubes of length 2^{-n}: all cubes in D_n are translation by an element of $2^{-n}\mathbb{Z}^d$ of the basic cube $[0, 2^{-n})^d$. Let $f\colon [0,1)^d \to \mathbb{R}$ be integrable. The piecewise-constant function f_n equal to the averaged value of f over each cube Q in D_n can be seen as the conditional expectancy $\mathbb{E}(f|\mathcal{F}_n)$ by taking $\Omega = [0,1)^2$, $\mathbb{P}$ being the Lebesgue measure, $\mathcal{F}$ the Borel σ-algebra, and $\mathcal{F}_n$ being the σ-algebra generated by all the cubes in D_n (verification left as en exercise). There is a consistency property in this approximation process, which is the following one: for all $m < n$, averaging the finer approximation f_n over the coarser grid corresponding to D_m gives f_m:

$$\mathbb{E}(f_n|\mathcal{F}_m) = f_m \text{ a.s.} \tag{2.4}$$

The property (2.4) follows from the tower property (2.2) for example. It is an instance of a martingale property.

2.2 Martingales

Definition 2.1 (Filtration)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$ is said to be a *filtration* if the family is increasing with respect to t: $\mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \leq s \leq t$. The space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is called a filtered space.

Definition 2.2 (Adapted process)**.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered space. A real-valued process $(X_t)_{t\geq 0}$ is said to be *adapted* if, for all $t \geq 0$, X_t is $\mathcal{F}_t$-measurable.

Definition 2.3 (Martingale)**.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered space. Let $(X_t)_{t\geq 0}$ be an adapted real-valued process such that, for all $t \geq 0$, $X_t \in L^1(\Omega)$. The process $(X_t)_{t\geq 0}$ is said to be a *martingale* if, for all $0 \leq s \leq t$, $X_s = \mathbb{E}(X_t|\mathcal{F}_s)$ a.s.

Remark 2.1. A martingale with continuous (*resp.*, càdlàg[1]) trajectories is said to be a continuous (*resp.*, càdlàg) martingale.

Remark 2.2. With respect to a fixed time $t > 0$, conditioning on $\mathcal{F}_s$ with $s \leq t$ is a way to average X_t over all events which occurred between times s and t. For a martingale X, this will let the position X_s unchanged. We expect a martingale not to wander too much therefore. We will see and use several instance of this general principle. See Section 2.3 for a first example. Let us also state the following result.

Theorem 2.2 (Doob's martingale inequality)**.** *Let $p > 1$. Let $(M_t)_{t\in[0,T]}$ be a càdlàg, real-valued martingale, such that $\mathbb{E}|M_T|^p < +\infty$. Then the inequality*

$$\mathbb{E}\left[(M_T^*)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_T|^p, \quad M_T^* = \sup_{t\in[0,T]} |M_t|, \tag{2.5}$$

is satisfied.

2.3 A digression on the Calderón-Zygmund decomposition

2.3.1 The Calderón-Zygmund decomposition

Let $f\colon [0,1)^d \to \mathbb{R}$ be a non-negative, integrable function. Let $\lambda > 0$ be a fixed threshold such that the integral of f over $[0,1)^d$ is smaller than $\lambda/2$. In terms of Example 3. in Section 2.1, this means $\mathbb{E}[f] \leq \lambda/2$. Consider (see Remark 2.2) that being below λ is "not wandering too much", while being above λ is "wandering too much". What is the behavior of the martingale (f_n) defined in the Example 3. in Section 2.1? Let T be the stopping time $T = \inf\{n \geq 0; f_n > \lambda\}$. We know that $T > 0$ almost surely. If $T = +\infty$, then $f_n \leq \lambda$ for all n, and thus $f = \lim f_n \leq \lambda$. Here we use the intuitive fact that $f = \lim f_n$. We have to specify the mode of convergence however and to justify the convergence. The convergence is almost sure. One can use the martingale convergence theorem for example (probabilistic approach) or the dyadic version of the Lebesgue differentiation theorem (analyst's approach). In any case, we obtain: $f \leq \lambda$ a.s. on $\{T = +\infty\}$. The set $\{T < +\infty\}$ can be written as an at most countable collection $(Q_i)_{i\in I}$ of dyadic cubes. Indeed, it is the union over $n \geq 1$ of the sets $\{T = n\}$, and $\{T = n\}$ is a union of dyadic cubes in D_n (because f_n is constant on each $Q \in D_n$). If Q is one of the cubes that enter in the decomposition of $\{T = n\}$, and if $Q' \in D_{n-1}$ is the twice bigger cube containing Q, then the averaged value of f on Q' is smaller than λ (otherwise $T < n$). It follows that

$$\lambda \leq \frac{1}{|Q|}\int_Q f(x)dx \leq \frac{1}{|Q|}\int_{Q'} f(x)dx = \frac{2^d}{|Q'|}\int_{Q'} f(x)dx \leq 2^d\lambda. \tag{2.6}$$

[1] càdlàg meaning "continue à droite avec limites à gauche", i.e. "continuous from the right with left limits" at each points

From these considerations on martingales, we can deduce the following statement.

Lemma 2.3 (Calderón-Zygmund)**.** *Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a non-negative, integrable function. Let $\lambda > 0$. There exists an at most countable family $(Q_i)_{i\in I}$ of dyadic cubes such that*

$$\forall i \in I, \quad \lambda \leq \frac{1}{|Q_i|}\int_{Q_i} f(x)dx \leq 2^d\lambda, \tag{2.7}$$

and $f \leq \lambda$ a.e. on the complementary set $\mathbb{R}^d \setminus \cup_{i\in I} Q_i$.

Proof of Lemma 2.3. fix N large enough such that $2^{-Nd}\|f\|_{L^1(\mathbb{R}^d)} \leq \lambda/2$. Consider the countable decomposition of $\mathbb{R}^d$ by all the dyadic cubes of size 2^N. On each such cube R, we apply the analysis performed before the statement of the lemma. This analysis was done with the starting cube $R = [0,1)^d$, but can be readily adapted to the general case. The final family of cube $(Q_i)_{i\in I}$ is then the union of the families obtained on each such cube R. □

The Calderón-Zygmund lemma is applied to obtain a decomposition of $f = g + b$, where

$$g = \sum_{i\in} \left[\frac{1}{|Q_i|}\int_{Q_i} f(x)dx\right]\mathbf{1}_{Q_i} + f\mathbf{1}_{\mathbb{R}^d\setminus\cup_{i\in I}Q_i} \tag{2.8}$$

and

$$b = f - g = \sum_{i\in I} b_i, \quad b_i = \left[f - \frac{1}{|Q_i|}\int_{Q_i} f(x)dx\right]\mathbf{1}_{Q_i}. \tag{2.9}$$

The function g is considered as the good part, since it is controlled in size by λ; more precisely, $|g(x)| \leq (2^d+1)\lambda$. The function b is considered as the "bad" part. It is not controlled in size but has the properties that b_i is supported in Q_i and has zero integral. The Calderón-Zygmund is fundamental in harmonic analysis. Note that, if we come back again to the probabilistic approach (and restrict things to $[0,1)^d$), then g is simply f_T, while $b = f - f_T$.

2.3.2 Application to elliptic estimates

Let U be an open subset of $\mathbb{R}^d$, $d \geq 2$. Let $f\colon U \to \mathbb{R}$ be measurable. The *Newtonian potential* of f in U is the function u defined by the convolution product

$$u(x) = \int_U G(x-y)f(y)dy, \quad x \in \mathbb{R}^d, \tag{2.10}$$

where the function G is defined by

$$G(x) = \begin{cases} -\frac{1}{2\pi}\ln|x| & \text{if } d = 2, \\ \dfrac{1}{d(d-2)\omega_d}\dfrac{1}{|x|^{d-2}} & \text{if } d \geq 3, \end{cases} \tag{2.11}$$

where ω_d is the d-dimensional Lebesgue measure of the unit ball in dimension d. Although G is singular at the origin, the function u is well defined and has some given regularity/integrability properties, depending on the regularity/integrability properties of f. See [13, Chapter 4.]. Since G is the fundamental solution of the Laplace equation in $\mathbb{R}^d$, the function u satisfies $-\Delta u = f$ in U, again under adequate regularity/integrability properties of u and f. The Newtonian potential is also used to express a solution of the Poisson equation $-\Delta v = f$ as the sum $u + w$, where w is harmonic in U (no considerations on boundary conditions here). We will use the Calderón-Zygmund decomposition to prove the gain of regularity of two derivatives in the space L^p.

Theorem 2.4. *Let $1 < p < +\infty$. Let f be bounded and locally Hölder continuous and let u be given by (2.10). Then u is of class C^2 in U, $-\Delta u = f$ in U, and*

$$\|\partial^2_{ij} u\|_{L^p(U)} \leq C\|f\|_{L^p(U)}, \tag{2.12}$$

for all $i, j \in \{1, \dots, d\}$, where the constant C depends on d and p only.

We will focus on (2.12). See [13, p.55] for the proof that u is of class C^2 in U and satisfies the Poisson equation $-\Delta u = f$ in U. Informally, we have

$$\partial^2_{ij} u(x) = \int_{\mathbb{R}^d} K_{ij}(x-y)\tilde{f}(y)dy,$$

where $\tilde{f}$ is the extension of f by 0 outside U, and where K_{ij} has a non-integrable singularity of type $|x|^{-d}$ at the origin. We can also write (again informally), $\partial^2_{ij} u = R_i R_j \tilde{f}$, where R_j is the *Riesz transform*. It is defined a priori as the application $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ given, after conjugation with the Fourier Transform, by

$$\mathcal{F}(R_j f)(\xi) = i\frac{\xi_j}{|\xi|}\mathcal{F}(f)(\xi), \quad \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx. \tag{2.13}$$

We recognize the expression of the operator $f \mapsto \partial_{x_j}(-\Delta)^{1/2} f$. Using the expression of Fourier transform of the homogeneous function $\xi \mapsto \frac{\xi_j}{|\xi|}$, [12], we also obtain (still informally at that stage) the expression

$$R_j f(x) = \int_{\mathbb{R}^d} K_j(x-y)f(y)dy,$$

where $K_j(x) = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \frac{x_j}{|x|^{d+1}}$ has a non-integrable singularity at the origin. We will use the Calderón-Zygmund decomposition to establish the following result.

Theorem 2.5 (Singular Integral)**.** *Let $K \in C(\mathbb{R}^d \setminus \{0\})$ be a given kernel. Assume that, for all $f \in C^\infty_c(\mathbb{R}^d)$, for all $x \in \mathbb{R}^d$, the limit*

$$Tf(x) = \lim_{\varepsilon \to 0} T_\varepsilon f(x), \quad T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x-y)f(y)dy, \tag{2.14}$$

exists, that there is a constant $A \geq 0$ such that

$$\|Tf\|_{L^2(\mathbb{R}^d)} \leq A\|f\|_{L^2(\mathbb{R}^d)}, \tag{2.15}$$

for all $f \in C^\infty_c(\mathbb{R}^d)$, and

$$\sup_{y \in B} \int_{(2B)^c} |K(z) - K(z-y)|dz \leq A, \tag{2.16}$$

for all ball $B = B(0,r)$, $r > 0$, centred at the origin (with $2B = B(0,2r)$). Then, for all $1 < p < +\infty$, there exists a constant $A_p \geq 0$ such that

$$\|Tf\|_{L^p(\mathbb{R}^d)} \leq A_p\|f\|_{L^p(\mathbb{R}^d)}, \tag{2.17}$$

for all $f \in C^\infty_c(\mathbb{R}^d)$.

Let us do few comments on the result. First, see [21, p.19] for a more general statement. Second, note that the constant A does not depend on the radius r in the regularity condition (2.16). Let us consider the case of the Riesz transform. Using the definition (2.13) and the expression $\mathcal{F}(|x|^{-(d-1)}) = \alpha_d |\xi|^{-1}$ for the Fourier Transform of the tempered distribution $x \mapsto |x|^{-(d-1)}$ (where α_d is a constant depending on d only), we see that

$$R_j f(x) = \beta_d \int_{\mathbb{R}^d} |x-y|^{-(d-1)} \partial_j f(y) dy, \tag{2.18}$$

(where β_d also denotes a constant depending on d only). Set $K(z) = \beta_d \frac{z_j}{|z|^{d+1}}$. Let $\nu_\varepsilon(x,y)$ denote the outward unit normal to the ball $B(x,\varepsilon)$ at a point y of the boundary and let $(e_i)_{1,d}$ denote the canonical basis of $\mathbb{R}^d$. We use the identity $\partial_j \varphi = \mathrm{div}(\varphi e_j)$ and the Green formula to obtain

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x-y) f(y) dy = \beta_d \int_{|x-y|>\varepsilon} |x-y|^{-(d-1)} \partial_j f(y) dy + r_\varepsilon,$$

where

$$r_\varepsilon = \beta_d \int_{|x-y|=\varepsilon} |x-y|^{-(d-1)} \nu_\varepsilon(x,y) \cdot e_j \partial_j f(y) d\sigma(y).$$

We have $\nu_\varepsilon(x,y) = \frac{y-x}{|y-x|}$, hence

$$r_\varepsilon = \beta_d \varepsilon^{-(d-1)} \int_{|x-y|=\varepsilon} \frac{y-x}{|y-x|} \cdot e_j \partial_j f(y) d\sigma(y) = \beta_d \int_{|z|=1} \frac{z}{|z|} \cdot e_j \partial_j f(x+\varepsilon z) d\sigma(z).$$

Since

$$\int_{|z|=1} \frac{z}{|z|} \cdot e_j d\sigma(z) = 0$$

by symmetry, we obtain, for f smooth enough,

$$r_\varepsilon = \beta_d \int_{|z|=1} \frac{z}{|z|} \cdot e_j [\partial_j f(x+\varepsilon z) - \partial_j f(x)] d\sigma(z) = \mathcal{O}(\varepsilon).$$

This shows that the limit in (2.14) exists indeed. The property (2.15) is a direct consequence of the definition (2.13). We use the homogeneity properties of $K(z) = \beta_d \frac{z_j}{|z|^{d+1}}$ and the change of variable $z' = rz$, $y' = ry$ to see that (2.16) is equivalent to

$$\sup_{y \in B_1} \int_{(2B_1)^c} |K(z) - K(z-y)| dz < +\infty, \tag{2.19}$$

where $B_1 = B(0,1)$ is the unit ball. Since $|z| > 2$, $|z-y| > 1$ in (2.19), and $|K(x)| \leq \beta_d |x|^{-d}$, we must check the integrability at infinity in (2.19). For $y \in B$ and $z \in (2B_1)^c$, we have

$$\begin{aligned} |K(z) - K(z-y)| &= \left| \int_0^1 (\nabla K)(z-ty) \cdot y dt \right| \\ &\leq \int_0^1 |\nabla K(z-ty)| dt \leq C(d) \int_0^1 \frac{1}{|z-ty|^{d+1}} dt \leq \frac{C'(d)}{|z|^{d+1}}. \end{aligned} \tag{2.20}$$

Indeed, $|z| \leq |z-ty| + 1 \leq |z-ty| + |z|/2$, which gives $|z| \leq 2|z-ty|$ for $t \in [0,1]$. The bound in (2.20) gives the desired result.

Proof of Theorem 2.5. We use the Marcinkiewicz' interpolation Theorem (see the footnote in [21, page 12]) and a duality argument to reduce the proof to the weak-$(1,1)$ estimate

$$|\{|Tf| > \alpha\}| \leq \frac{A_1}{\alpha}\|f\|_{L^1(\mathbb{R}^d)}, \tag{2.21}$$

for $\alpha > 0$, $f \in C_c^\infty(\mathbb{R}^d)$, where $|E|$ denotes the d-dimensional measure of a Borel set E. Let us apply the Calderón-Zygmund decomposition:

$$|\{|Tf| > \alpha\}| \leq |\{|Tg| > \alpha/2\}| + |\{|Tb| > \alpha/2\}|. \tag{2.22}$$

Although f is smooth, g and b may be not smooth. All the computations below can be justified by working first we T_ε, defined in (2.14), and then letting $\varepsilon \to 0$. The first piece in (2.22) is easy to estimate: we use the Chebychev inequality and the L^2-estimate (2.15) to obtain

$$|\{|Tg| > \alpha/2\}| \leq \frac{4}{\alpha^2}\|Tg\|_{L^2}^2 \leq \frac{4A^2}{\alpha^2}\|g\|_{L^2}^2.$$

We have seen that the pointwise bound $|g(x)| \leq (1+2^d)\lambda$ is satisfied. We have also $\|g\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}$ as a direct application of (2.8). Therefore $\|g\|_{L^2}^2$ is bounded by $(1+2^d)\lambda\|f\|_{L^1(\mathbb{R}^d)}$ and we conclude that

$$|\{|Tg| > \alpha/2\}| \leq (2^d+1)\frac{4A^2}{\alpha^2}\lambda\|f\|_{L^1(\mathbb{R}^d)}. \tag{2.23}$$

To estimate the second term in (2.22), we first note that, given a Borel set E, we have, using the Markov inequality,

$$|\{|Tb| > \alpha/2\}| \leq |\{|Tb| > \alpha/2\} \cap E| + |E^c| \leq \frac{2}{\alpha}\int_E |Tb(x)|dx + |E^c|.$$

By the decomposition (2.9), we deduce that

$$|\{|Tb| > \alpha/2\}| \leq \frac{2}{\alpha}\sum_{i\in I}\int_E |Tb_i(x)|dx + |E^c|. \tag{2.24}$$

Let x_i denote the center of Q_i. Let B_i denote the ball with same center as Q_i and diameter $\mathrm{diam}(Q_i)$. We have $|B_i| = c_d|Q_i|$ for a given constant c_d. We use the cancellation property satisfied by b_i to write

$$Tb_i(x) = \int_{B_i}(K(x-y) - K(x-x_i))b_i(y)dy.$$

With Fubini's theorem, we deduce that

$$\int_E |Tb_i(x)|dx \leq \int_{B_i}\int_E |K(x-y) - K(x-x_i)|dx|b_i(y)|dy.$$

Take $E = \cap_{j\in I}(2B_j)^c$. Then

$$\int_E |K(x-y)-K(x-x_i)|dx \leq \int_{(2B_i)^c}|K(x-y)-K(x-x_i)|dx = \int_{(2B_i')^c}|K(z+x_i-y)-K(z)|dz,$$

where $B_i = x_i + B_i'$. Using (2.16) and (2.9) gives us

$$|\{|Tb| > \alpha/2\}| \leq \frac{2A}{\alpha}\sum_{i\in I}\int_{B_i}|b_i(y)|dy + |E^c| \leq \frac{2A}{\alpha}\|f\|_{L^1(\mathbb{R}^d)} + |E^c|. \tag{2.25}$$

By (2.6), we also have

$$|E^c| \leq \sum_{i\in I} |2B_i| = 2^d \sum_{i\in I} |B_i| \leq \frac{2^d c_d^{-1}}{\lambda} \|f\|_{L^1(\mathbb{R}^d)}.$$

The final estimate

$$|\{|Tf| > \alpha\}| \leq C(A,d)(\alpha^{-2}\lambda + \alpha^{-1} + \lambda^{-1})\|f\|_{L^1(\mathbb{R}^d)}$$

follows from (2.23) and (2.25). Taking $\lambda = \alpha$, we conclude to (2.21). □

3 Markov processes

We consider Markov processes taking values in a Polish space E. Recall that a process $X = (X_t)_{t\geq 0}$ is a collection of random variables: for each $t \geq 0$, $X_t \colon (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable (on E we always consider the Borel σ-algebra, denoted $\mathcal{B}(E)$. The law of the process is obtained by considering the random variable $X \colon \Omega \to E^{\mathbb{R}_+}$, where $E^{\mathbb{R}_+}$ is the set of functions from $\mathbb{R}_+$ to E. On $E^{\mathbb{R}_+}$, we consider the cylindrical σ-algebra, which is the σ-algebra generated by the evaluations maps $\pi_t \colon E^{\mathbb{R}_+} \to E$, $\pi_t(f) := f(t)$, [22, Chapter 2.4]. Then, the law of X is described by the set of finite-dimensional distributions $\mathbb{P}(A)$, where A is a cylindrical set of the form

$$A = \{X_{t_1} \in B_1, \dots, X_{t_n} \in B_n\}, \tag{3.1}$$

for some $n \in \mathbb{N}$, $B_1, \dots, B_n$ some Borel subsets of E and $t_1, \dots, t_n \geq 0$. The filtration $(\mathcal{F}_t^X)$ denotes the filtration generated by X: for a given $t \geq 0$, $\mathcal{F}_t^X$ is the σ-algebra generated by the sets A of the form (3.1), with all times $t_i \leq t$.
We will also denote by $\mathrm{BM}(E)$ the Banach space of bounded Borel-measurable functions on E with the sup-norm

$$\|\varphi\|_{\mathrm{BM}(E)} = \sup_{x\in E} |\varphi(x)|. \tag{3.2}$$

The set $\mathrm{BC}(E)$ is the subspace of continuous bounded functions.

3.1 Definition

By Markov process, we mean the triplet constituted of a Markov semi-group, some probability kernels, and the associated Markov processes. More precisely, we suppose first that we are given:

1. a Markov semi-group $\mathbf{P} = (P_t)_{t\geq 0}$, which is defined a priori as a family of endomorphisms of the space $\mathrm{BM}(E)$ that satisfy the initial condition $P_0 = \mathrm{Id}$, the semi-group property $P_t \circ P_s = P_{t+s}$ for $t, s \geq 0$, the preservation of positivity $P_t\varphi \geq 0$ when $\varphi \geq 0$, while fixing the constant function $\mathbf{1}$ equal to 1 everywhere: $P_t\mathbf{1} = \mathbf{1}$ for all $t \geq 0$,

2. a probability kernel $Q(t, x, B)$: for all $\varphi \in \mathrm{BM}(E)$, for all $x \in E$,

$$P_t\varphi(x) = \int_E \varphi(y) Q(t, x, dy), \tag{3.3}$$

where, for every $t \geq 0$, for every $x \in E$, $Q(t, x, \cdot)$ is a probability measure and the dependence in x is measurable, in the sense that the right-hand side of (3.3) is a measurable function of x,

3. a set $\mathbf{X} = \{(X_t^x)_{t\geq 0}; x \in E\}$ of Markov processes indexed by their starting points x: $X_0^x = x$ almost surely, such that:

   - the finite-dimensional distributions of $(X_t^x)_{t\geq 0}$ are given by

$$\mathbb{P}(X_0^x \in B_0, X_{t_1}^x \in B_1, \dots, X_{t_k}^x \in B_k) = \int_{B_0} \cdots \int_{B_{k-1}} Q(t_k - t_{k-1}, y_{k-1}, B_k) \times Q(t_{k-1} - t_{k-2}, y_{k-2}, dy_{k-1}) \cdots Q(t_1, y_0, dy_1)\mu(dy_0), \quad (3.4)$$

     where $0 \leq t_1 \leq \cdots \leq t_k$, $B_0, \dots, B_k \in \mathcal{B}(E)$ and $\mu = \delta_x$ (Dirac mass),

   - the Markov property

$$\mathbb{E}\left[\varphi(X_{t+s}^x)|\mathcal{F}_t^X\right] = P_s\varphi(X_t^x) \quad (3.5)$$

     is satisfied for all $s, t \geq 0$, $\varphi \in \mathrm{BM}(E)$.

There are a lot of redundancies in the definition above, that we will now analyse. It is not limiting, however, to assume that all these elements are given altogether, all the more since the processes $(X_t^x)_{t\geq 0}$ will generally have additional pathwise properties, being typically continuous or càdlàg. They may also satisfy the Markov property (3.5) with respect to a given filtration $(\mathcal{F}_t)$ larger than $(\mathcal{F}_t^X)$. First, we need to define an appropriate mode on convergence of functions in $\mathrm{BM}(E)$.

Definition 3.1. We say that there is bounded pointwise convergence of a sequence (φ_n) in $\mathrm{BM}(E)$ to $\varphi \in \mathrm{BM}(E)$ if $\sup_n \|\varphi_n\|_{\mathrm{BM}(E)} < +\infty$ and $\varphi_n(x) \to \varphi(x)$ for all $x \in E$. This mode of convergence is denoted $\varphi_n \overset{\text{b.p.c.}}{\longrightarrow} \varphi$.

Proposition 3.1. *Let $(P_t)_{t\geq 0}$ be a semi-group as in 1. Assume that, for each $t \geq 0$,*

$$[\varphi_n \overset{\text{b.p.c.}}{\longrightarrow} \varphi] \quad \Rightarrow \quad [P_t\varphi_n \overset{\text{b.p.c.}}{\longrightarrow} P_t\varphi]. \quad (3.6)$$

Then there exists a probability kernel as in 2. such that (3.3) is satisfied.

Proof of Proposition 3.1. We give the main ideas of the proof. First observe that (3.3) implies the continuity property (3.6): it is natural to assume (3.6) therefore. Set

$$Q(t, x, A) = (P_t\mathbf{1}_A)(x) \quad (3.7)$$

For each fixed t, x, this defines a non-negative set function $Q(t, x, \cdot)$ such that $Q(t, x, E) = 1$. The only delicate point to show that $Q(t, x, \cdot)$ is a probability measure is the countable additivity. It follows from (3.6) and the convergence $\varphi_n \overset{\text{b.p.c.}}{\longrightarrow} \varphi$, when $A_1, \dots, A_n \dots$ are disjoint Borel sets in E, $\varphi_n = \mathbf{1}_{\cup_{1\leq k\leq n} A_k}$, $\varphi = \mathbf{1}_{\cup_{1\leq k} A_k}$. By (3.7), (3.3) is satisfied when φ is a characteristic function. By linearity, this remains true for simple functions. Any $\varphi \in \mathrm{BM}(E)$ can be approached for bounded pointwise convergence by a sequence of simple functions, this gives the relation (3.3) in all its generality. □

Proposition 3.2. *Let Q be a probability kernel as in 2. Assume that*

$$Q(t + s, x, A) = \int_E Q(s, y, A)Q(t, x, dy), \quad (3.8)$$

is satisfied for every $t, s \geq 0$, $x \in E$, $A \in \mathcal{B}(E)$. Then there exists a Markov semi-group as in 1. such that (3.3) is satisfied.

The relation (3.8) is called the Chapman Kolmogorov relation.

Proof of Proposition 3.2. We define P_t by the relation (3.3). Then $(P_t)_{t\geq 0}$ has all the desired properties listed in 1. The only point that must be studied carefully is the semi-group property. It is sufficient to establish that $P_{t+s}\varphi = P_t(P_s\varphi)$ is satisfied for characteristic functions, but then, this is equivalent to (3.8). □

Let us now study the relation between the process given in 3. and the probability kernel Q. First, we state without proof the following result. See, *e.g.* [7, Theorem 1.1 p.157], for the proof.

Proposition 3.3. *Let Q be a probability kernel as in 2. Then there exists a measurable space $(\Omega, \mathcal{F})$, a process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F})$ such that: for all probability measure μ on E, there exists a probability measure $\mathbb{P}_\mu$ on $(\Omega, \mathcal{F})$, such that, under $\mathbb{P}_\mu$, $(X_t)_{t\geq 0}$ has the finite-dimensional distributions given by (3.4): for all $0 \leq t_1 \leq \cdots \leq t_k$, $B_0, \ldots, B_k \in \mathcal{B}(E)$, the probability*

$$\mathbb{P}_\mu(X_0 \in B_0, X_{t_1} \in B_1, \ldots, X_{t_k} \in B_k) \tag{3.9}$$

is given by the right-hand side of (3.4).

We denote by $\mathbb{E}_\mu$ the expectancy operator associated to $\mathbb{P}_\mu$. When $\mu = \delta_x$, we use the notations $\mathbb{P}_x$ and $\mathbb{E}_x$. The Kolmogorov extension theorem, [22], can be used to construct the measure $\mathbb{P}_\mu$. The probability space is the *path space*: $\Omega = E^{\mathbb{R}_+}$. The σ-algebra $\mathcal{F}$ is the cylindrical σ-algebra (called product σ-algebra in [22, Chapter 2.4]). The process X is then the *canonical process* $X_t(\omega) = \omega(t)$.
In the following two results, we establish the link between the Markov property and the Chapman-Kolmogorov (or semi-group) property.

Proposition 3.4. *Under the hypotheses of Proposition 3.3, assume that the Chapman-Kolmogorov property (3.8) is satisfied. Then the process $(X_t, \mathbb{P}_\mu)$ constructed in Proposition 3.3 is Markov.*

Proof of Proposition 3.4. Our aim is to show the identity

$$\mathbb{E}_\mu\left[\varphi(X_{t+s})|\mathcal{F}_t^X\right] = P_s\varphi(X_t) \tag{3.10}$$

for all $s, t \geq 0$, $\varphi \in \mathrm{BM}(E)$, where P_t is defined by (3.3) (in particular, P_t satisfies (3.6)). The meaning of (3.10) is

$$\mathbb{E}_\mu\left[\varphi(X_{t+s})\mathbf{1}_A\right] = \mathbb{E}_\mu\left[P_s\varphi(X_t)\mathbf{1}_A\right], \tag{3.11}$$

for all $A \in \mathcal{F}_t^X$. We can see both members of (3.11) as measures in A. Since $\mathcal{F}_t^X$ is generated by cylindrical sets that form a π-system, it is sufficient, [3, Theorem 3.3], to establish (3.11) for A of the form

$$A = \{X_0 \in B_0, X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\}, 0 < t_1 < \cdots < t_n \leq t. \tag{3.12}$$

Since (3.11) is linear in φ and the continuity property (3.6) is satisfied we can also reduce the proof of (3.11) to the case where φ is a characteristic function $\mathbf{1}_B$, with $B \in \mathcal{B}(E)$. Alternatively, the same kind of argument shows that, for all $\psi \in \mathrm{BM}(E)$ and A of the form (3.12), we have

$$\mathbb{E}_\mu\left[\psi(X_t)\mathbf{1}_A\right] = \int_{B_0} \cdots \int_{B_n} P_{t-t_n}\psi(y_n)Q(t_n - t_{n-1}, y_{n-1}, dy_n)\cdots Q(t_1, y_0, dy_1)\mu(dy_0), \tag{3.13}$$

since (3.4) and (3.3) show that (3.13) is true when $\psi = \mathbf{1}_{B_{n+1}}$. We will use (3.13) and the Chapman-Kolmogorov (or, more precisely, semi-group) property to conclude. Taking $\psi = P_s\varphi$ in (3.13) and using the semi-group property, we see that the right-hand side of (3.11) is

$$\int_{B_0} \cdots \int_{B_n} P_{t+s-t_n}\varphi(y_n) Q(t_n - t_{n-1}, y_{n-1}, dy_n) \cdots Q(t_1, y_0, dy_1)\mu(dy_0). \tag{3.14}$$

Then, using (3.13) again with $t+s$ instead of t shows that (3.14) coincides with the left-hand side of (3.11). □

Proposition 3.5. *Assume that the process $(X_t, \mathbb{P}_\mu)$ constructed in Proposition 3.3 is Markov. Then Q satisfies the Chapman-Kolmogorv property (3.8).*

Proof of Proposition 3.5. We will establish the equivalent semi-group property for (P_t). Let $\varphi \in \mathrm{BM}(E)$. By the tower property (2.2), we have

$$P_{t+s}\varphi(x) = \mathbb{E}_x\left[\varphi(X_{t+s})\right] = \mathbb{E}_x\left[\mathbb{E}_x\left[\varphi(X_{t+s})|\mathcal{F}_t^X\right]\right].$$

The Markov property then gives

$$P_{t+s}\varphi(x) = \mathbb{E}_x\left[P_s\varphi(X_t)\right] = (P_t \circ P_s)\varphi(x),$$

which is the desired identity. □

3.2 Invariant measures and weak convergence of probability measures

Let $(\mathbf{P}, Q, \mathbf{X})$ be a Markov process as in Section 3.1. If μ is a probability measure on E, we denote by $P_t^*\mu$ the law at time t of X_t, when $X_0 \sim \mu$:

$$\langle P_t^*\mu, \varphi\rangle := \mathbb{E}_\mu\left[\varphi(X_t)\right].$$

The notation can be justified as follows: using (3.13) with $A = \Omega$, *i.e.* $B_0 = \cdots = B_n = E$, we see that

$$\mathbb{E}_\mu\left[\varphi(X_t)\right] = \int_E P_t\psi(y)\mu(dy) = \langle \mu, P_t\varphi\rangle.$$

This establishes the expected formula

$$\langle P_t^*\mu, \varphi\rangle = \langle \mu, P_t\varphi\rangle. \tag{3.15}$$

Definition 3.2 (Invariant measure)**.** A probability measure μ on E is said to be an invariant measure if $P_t^*\mu = \mu$ for all $t \geq 0$.

To find an invariant measure μ, one must choose X_0 conveniently, to ensure that X_t follows the same law μ for all $t \geq 0$: an invariant measure is a fixed-point for the evolution in distribution of the Markov process.
As far as the evolution of the distribution $P_t^*\mu$ of the Markov process is concerned, we can wonder what are the continuity property of $t \mapsto P_t^*\mu$. The space $\mathcal{P}_1(E)$ of Borel probability measures on E is a subset of the dual space to the Banach space $\mathrm{BC}(E)$ (the norm being the sup norm (3.2)). We consider the weak-$*$ topology on $\mathcal{P}_1(E)$[2]. A sequence (μ_n) converges to μ in $\mathcal{P}_1(E)$ if

$$\langle \mu_n, \varphi\rangle \to \langle \mu, \varphi\rangle, \tag{3.16}$$

[2]This topology on $\mathcal{P}_1(E)$ is metrizable: this is a non-obvious fact, [4, Theorem 6.8], in particular we cannot simply build a metric based on a dense countable subset of $\mathrm{BC}(E)$ since $\mathrm{BC}(E)$ may be not separable (see the argument in [14, p.6] however)

for all $\varphi \in \mathrm{BC}(E)$. When (3.17) is realized, it is customary to say that "(μ_n) converges weakly to μ".
Let us state without proof the following version of the Portmanteau Theorem (see [14, p.4,5] for the proof).

Theorem 3.6. *The following five statements are equivalent.*

(i) (μ_n) *converges weakly to* μ,

(ii) (3.16) *is satisfied for all* uniformly continuous *and bounded functions* φ *on* E,

(iii) $\limsup \mu_n(F) \leq \mu(F)$ *for all closed set* F,

(iv) $\liminf \mu_n(G) \geq \mu(G)$ *for all open set* G,

(v) $\lim \mu_n(A) = \mu(A)$ *for all Borel set* A *such that* $\mu(\partial A) = 0$.

Coming back to $t \mapsto P_t^* \mu$, we see that $P_t^* \mu \to P_s^* \mu$ if

$$\lim_{t \to s} \langle P_t^* \mu, \varphi \rangle = \langle P_s^* \mu, \varphi \rangle, \tag{3.17}$$

for all $\varphi \in \mathrm{BC}(E)$.

Definition 3.3 (Stochastic continuity of the semi-group)**.** Let $s \geq 0$. The semi-group $(P_t)_{t \geq 0}$ is said to be *stochastically continuous* at s if $P_t \varphi \xrightarrow{\text{b.p.c.}} P_s \varphi$ when $t \to s$ for every $\varphi \in \mathrm{BC}(E)$. We say that $(P_t)_{t \geq 0}$ is stochastically continuous if it is stochastically continuous at very point.

The stochastic continuity of $(P_t)_{t \geq 0}$ at s is *equivalent* to the weak convergence (3.17). Indeed, if (3.17) is satisfied, then $\langle \mu, P_t \varphi \rangle \to \langle \mu, P_s \varphi \rangle$ by the duality formula (3.15). Taking $\mu = \delta_x$ especially, we obtain $P_t \varphi(x) \to P_s \varphi(x)$. Since $|P_t \varphi(x)| \leq \|\varphi\|_{\mathrm{BC}(E)}$, we obtain the b.p. convergence. Reciprocally, $\langle \mu, P_t \varphi \rangle \to \langle \mu, P_s \varphi \rangle$ follows from the stochastic continuity by dominated convergence.
There is also a notion of stochastic continuity for processes: a stochastic process (X_t) is stochastically continuous at s if $X_t \to X_s$ in probability for the topology of E: for all $\delta > 0$,

$$\lim_{t \to s} \mathbb{P}(d(X_t, X_s) > \delta) = 0, \tag{3.18}$$

where d is the distance on E.

Exercise 3.4 (Stochastic continuity)**.** Let (X_t) be a Markov process, and let P_t be defined by $P_t \varphi(x) = \mathbb{E}_x[\varphi(X_t)]$. Show that stochastic continuity of (X_t) at s implies stochastic continuity of (P_t) at s (*Hint:* use *(ii)* in Theorem 3.6). *The solution to Exercise 3.4 is here.*

Definition 3.5 (Feller semi-group)**.** A semi-group $(P_t)_{t \geq 0}$ is said to be Feller if

$$P_t \colon \mathrm{BC}(E) \to \mathrm{BC}(E),$$

for all $t \geq 0$: $P_t \varphi \in \mathrm{BC}(E)$ when $\varphi \in \mathrm{BC}(E)$.

If $(P_t)_{t \geq 0}$ is Feller and (μ_n) converges weakly to μ, then we can test (3.16) against $P_t \varphi$. Using the duality relation (3.15), we conclude that $(P_t^* \mu_n)$ converges weakly to $P_t^* \mu$.

The following exercises are all about invariant measures.

Exercise 3.6 (Invariant measure for a discrete Ornstein-Uhlenbeck process)**.** Let $X_0, X_1, \dots$ be the sequence of random variables on $\mathbb{R}$ defined as follows: X_0 is chosen at random, according to a law μ_0, then, X_N being known, a random variable Z_{N+1} taking the values $+1$ or -1 with equi-probability is drawn independently on $X_0, \dots, X_N$ and X_{N+1} given by

$$X_{N+1} = \frac{1}{2} X_N + Z_{N+1}.$$

1. What means $\mu_0 = \delta_0$? What are then the law μ_1, μ_2 of X_1 and X_2 respectively?
2. Consider the case $\mu_0 = \frac{1}{2}\delta_{-2} + \frac{1}{2}\delta_{+2}$. Compute μ_1, μ_2, μ_3. Can you guess a general formula for μ_N?
3. Find an invariant measure.

The solution to Exercise 3.6 is here.

Exercise 3.7 (Invariant measure by Cesàro convergence)**.** Suppose that $(P_t)_{t\geq 0}$ is stochastically continuous and Feller. For $T > 0$, and $\mu \in \mathcal{P}_1(E)$, let $\bar{\mu}_T$ be the probability measure defined by

$$\langle \bar{\mu}_T, \varphi \rangle = \frac{1}{T} \int_0^T \langle P_t^* \mu, \varphi \rangle dt.$$

Suppose that there exists a probability measure ν on E such that, for at least on $\mu \in \mathcal{P}_1(E)$, $\bar{\mu}_T$ converge weakly to ν when $T \to +\infty$. Show that ν is an invariant measure.
The solution to Exercise 3.7 is here.

Exercise 3.8 (Invariant measures for deterministic systems)**.** Let $(\Phi_t)_{t\geq 0}$ denote the flow associated to the ordinary differential equation $\dot{x} = F(x)$. Here $F \colon \mathbb{R}^d \to \mathbb{R}^d$ is a (globally) Lipschitz continuous function.

1. Show that $P_t\varphi := \varphi \circ \Phi_t$ defines a Markov semi-group on $\mathrm{BM}(\mathbb{R}^d)$.
2. *Punctual equilibria.* Let $x_1, \dots, x_n$ be some zeros of F. Show that any convex combination of the Dirac masses $\delta_{x_1}, \dots, \delta_{x_n}$ is an invariant measure.
3. *Hamiltonian system.* Suppose that $d = n + n$, $x = (p, q)$ and

$$F\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} D_q H(p,q) \\ -D_p H(p,q) \end{pmatrix},$$

where $H \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is of class C^2.

   (a) Show that $t \mapsto H \circ \Phi_t(x)$ is constant for all x.
   (b) Assume that $e^{-\beta H} \in L^1(\mathbb{R}^d)$ for all $\beta > 0$. We introduce the Gibbs measure μ_β, which is the measure of density $Z(\beta)^{-1} e^{-\beta H}$ with respect to the Lebesgue measure ($Z(\beta) = \int e^{-\beta H(x)} dx$ is a normalizing factor). Show that μ_β is an invariant measure.

The solution to Exercise 3.8 is here.

3.3 Infinitesimal generator

Given a Markov process as in Section 3.1, we would like to define the associated infinitesimal generator. There are various possible approaches. In [1] for example, it is assumed that the process admits an invariant measure μ. The semi-group can then be extended as a contraction semi-group on $L^2(\mu)$. By assuming additionally that this extension gives rise to a strongly continuous semi-group, [1, Property (vi), p.11], one can use the standard theory of strongly continuous semi-group, [19], to define the infinitesimal generator. One may wonder why not simply working in $\mathrm{BM}(E)$, or $\mathrm{BC}(E)$, which are Banach spaces, to apply the standard theory of strongly continuous semi-group. The difficulty is that the continuity property $P_t\varphi \to \varphi$ when $t \to 0$ is too stringent in that context, at least when E is infinite-dimensional. Consider for example the simple deterministic case where $P_t\varphi$ is given as the composition $\varphi \circ \Phi_t$ with a flow (Φ_t). Let E be the Hilbert space $E = \ell^2(\mathbb{N})$, with orthonormal basis $(e_n)_{n\in\mathbb{N}}$, and let Φ_t be given by

$$\Phi_t(x) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\langle x, e_n\rangle e_n,$$

where (λ_n) is an increasing sequence converging to $+\infty$. In general, one cannot control the distance $\|\Phi_t(x) - x\|_{\ell^2(\mathbb{N})}$ uniformly in x (this is possible when x is restricted to a compact set), so even if φ is uniformly continuous, one does not expect the convergence

$$\lim_{t\to 0} \sup_{x\in E} |P_t\varphi(x) - \varphi(x)| = 0.$$

We consider a different mode of convergence therefore, the bounded pointwise convergence (Definition 3.1). A function $\varphi \in \mathrm{BM}(E)$ is in the domain $D(\mathscr{L})$ of the infinitesimal generator $\mathscr{L}$ of (P_t) if there exists $\psi \in \mathrm{BM}(E)$ such that

$$\frac{P_t\varphi - \varphi}{t} \overset{\text{b.p.c.}}{\longrightarrow} \psi, \tag{3.19}$$

when $t \to 0$. We then set $\mathscr{L}\varphi = \psi$. Note that, on the elements $\varphi \in D(\mathscr{L})$, the property of continuity

$$P_t\varphi \overset{\text{b.p.c.}}{\longrightarrow} \varphi \tag{3.20}$$

when $t \to 0$, is satisfied. By the semi-group property, (3.20) implies more generally the property of continuity from the right $P_t\varphi \overset{\text{b.p.c.}}{\longrightarrow} P_{t_*}\varphi$ when $t \downarrow t_*$, for every $t_* \geq 0$. The semi-group property, and the continuity of P_t with respect to b.p. convergence, that can be deduced from (3.3) (see (3.6)), have also the following consequence: if $\varphi \in D(\mathscr{L})$, then $P_t\varphi \in D(\mathscr{L})$ for all $t \geq 0$ and

$$\frac{P_{t+h}\varphi - P_t\varphi}{h} \overset{\text{b.p.c.}}{\longrightarrow} P_t\mathscr{L}\varphi = \mathscr{L}P_t\varphi, \tag{3.21}$$

when $h \to 0^+$.

3.4 Martingale property of Markov processes

Consider a Markov process as in Section 3.1, which is Markov with respect to a filtration $(\mathcal{F}_t)$, and has a generator $\mathscr{L}$, as defined as in Section 3.3. We make the following hypotheses:

1. *stochastic continuity:* we have $P_t\varphi \overset{\text{b.p.c.}}{\longrightarrow} P_{t_*}\varphi$ when $t \to t_*$ for every $\varphi \in \mathrm{BC}(E)$ and every $t_* \geq 0$,

2. *measurability:* the application $(\omega, t) \mapsto X_t(\omega)$ is measurable $\Omega \times \mathbb{R}_+ \to E$.

We have then the following result.

Theorem 3.7. *Let $\varphi \in D(\mathscr{L}) \cap \mathrm{BC}(E)$. Then*

$$M_t := \varphi(X_t) - \varphi(X_0) - \int_0^t \mathscr{L}\varphi(X_s)ds \tag{3.22}$$

is a $(\mathcal{F}_t)$-martingale. If furthermore $|\varphi|^2$ is in the domain of $\mathscr{L}$, then the process (Z_t) defined by

$$Z_t := |M_t|^2 - \int_0^t (\mathscr{L}|\varphi|^2 - 2\varphi\mathscr{L}\varphi)(X_s)ds, \tag{3.23}$$

is a $(\mathcal{F}_t)$-martingale.

Remark 3.1 (Quadratic variation). If (X_t) has continuous trajectories, then

$$A_t := \int_0^t (\mathscr{L}|\varphi|^2 - 2\varphi\mathscr{L}\varphi)(X_s)ds \tag{3.24}$$

is the quadratic variation $\langle M, M\rangle_t$, [15, p.38], of (M_t). In the general case where (X_t) is càdlàg, (A_t) is the compensator, [15, p.32], of the quadratic variation $[M, M]_t$, [15, p.51], of (M_t). For instance, if $(X_t = N_t)$ is a Poisson Process of rate λ, then $\mathscr{L}\varphi(n) = \lambda(\varphi(n+1) - \varphi(n))$ and

$$A_t := \lambda \int_0^t (\varphi(N_s + 1) - \varphi(N_s))^2 ds.$$

Taking $\varphi = \mathrm{Id}$, gives the standard fact that $(N_t - \lambda t)$ is a martingale.

Proof of Theorem 3.7. Let $0 \le s \le t$. By the Markov property, we have

$$\begin{aligned}\mathbb{E}[M_t|\mathcal{F}_s] - M_s &= \mathbb{E}[M_t - M_s|\mathcal{F}_s] \\ &= P_{t-s}\varphi(X_s) - \varphi(X_s) - \int_s^t [P_{\sigma-s}\mathscr{L}\varphi](X_s)d\sigma. \end{aligned} \tag{3.25}$$

To establish (3.25) we have used the fact that

$$\mathbb{E}\left[\int_s^t \psi(\sigma)d\sigma\,|\mathcal{F}_s\right] = \int_s^t \mathbb{E}\left[\psi(\sigma)\,|\mathcal{F}_s\right]d\sigma, \tag{3.26}$$

with $\psi(\omega, \sigma) := \mathscr{L}\varphi(X_\sigma(\omega))$, which is a measurable function. The identity (3.26) follows from the linearity of the conditional expectation when ψ is a simple function, and standard arguments give the general case. From (3.25) and the identity

$$P_t\varphi(x) - \varphi(x) = \int_0^t P_s\mathscr{L}\varphi(x)ds, \tag{3.27}$$

for all $\varphi \in D(\mathscr{L})$, $x \in E$, $t \ge 0$, we conclude that $\mathbb{E}[M_t|\mathcal{F}_s] - M_s = 0$. To establish (3.27), we notice that

$$\beta(t) := P_t\varphi(x) - \varphi(x)$$

is a continuous function (here we use the stochastic continuity of (P_t)), which is right-differentiable at every point, with right-differential $\beta'(t+) := P_t\mathscr{L}\varphi(x)$ which is a bounded (by stochastic

continuity of (P_t), it is even continuous if $\mathscr{L}\varphi \in \mathrm{BC}(E)$ – but this is not assumed a priori). Lemma 3.8 below then gives the result.
The proof of the martingale property for (3.23) is divided in several steps. First, we fix two times $0 \leq \tau < \tau' \leq T$. We fix a subdivision $\sigma = (t_i)_{0,n}$ of $[0, \tau']$, chosen in such a way that τ is always one of the t_i, say $\tau = t_l$ (the index l may hence vary with σ). By $C(\varphi)$, we will denote any constant that depend on φ and is independent on σ and may vary from lines to lines. We also denote by $A = \mathcal{O}(B)$ any estimate of the form $|A| \leq C(\varphi)|B|$. At last, we introduce the following notations: we denote by $\delta_{t_i} K$ the increment $K_{t_{i+1}} - K_{t_i}$ of a function $t \mapsto K_t$. We also denote by $\mathbb{E}_{t_i}$ the conditional expectation with respect to $\mathcal{F}_{t_i}$. Our aim is to show that

$$A_{\tau'} = \lim_{|\sigma| \to 0} \sum_{i=0}^{n-1} \mathbb{E}_{t_i} \left[|\delta_{t_i} M|^2 \right], \tag{3.28}$$

where the limit is taken in $L^2(\Omega)$. Indeed, taking (3.28) for granted, $\mathbb{E}[Z_{\tau'} - Z_\tau | \mathcal{F}_\tau]$ is the limit when $|\sigma| \to 0$ of the quantity

$$\mathbb{E}\Big[|M_{t_n}|^2 - |M_{t_l}|^2 - \sum_{i=l}^{n-1} \mathbb{E}_{t_i} \left[|\delta_{t_i} M|^2 \right] \Big| \mathcal{F}_\tau \Big]. \tag{3.29}$$

Let us show that (3.29) $= 0$. To simplify the presentation[3], we will treat the case $t_l = \tau = 0$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ (it makes sense to consider that $\mathcal{F}_0$ is the trivial sigma algebra since $M_0 = 0$). We have

$$\mathbb{E}\left[|M_{t_n}|^2 \right] = \mathbb{E}\left[\left| \sum_{i=0}^{n-1} \delta_{t_i} M \right|^2 \right]. \tag{3.30}$$

In (3.30), we can expand the square. The contribution of the double products is zero since, if $j > i$, then, using the fact that $\delta_{t_i} M$ is $\mathcal{F}_{t_j}$-measurable, we have

$$\mathbb{E}\left[\delta_{t_i} M \delta_{t_j} M \right] = \mathbb{E}\left[\mathbb{E}_{t_j} \left[\delta_{t_j} M \right] \delta_{t_i} M \right] = 0.$$

The last identity follows from the martingale property $\mathbb{E}_{t_j}\left[\delta_{t_j} M \right] = 0$. This implies (3.29) $= 0$, and thus $\mathbb{E}[Z_{\tau'} - Z_\tau | \mathcal{F}_\tau] = 0$. The proof of (3.28) is divided into three steps.
Step 1. We show that $A_{\tau'} = \lim_{|\sigma| \to 0} \sum_{i=0}^{n-1} \mathbb{E}_{t_i} \left[\delta_{t_i} A \right]$, with a convergence in $L^2(\Omega)$. Since $A_{\tau'} = \sum_{i=0}^{n-1} \delta_{t_i} A$, we have to show that $\lim_{|\sigma| \to 0} \sum_{i=0}^{n-1} \zeta_i = 0$ in $L^2(\Omega)$, where $\zeta_i := \delta_{t_i} A - \mathbb{E}_{t_i} \left[\delta_{t_i} A \right]$. The method is similar to the analysis of (3.29) above: we decompose

$$\mathbb{E}\left| \sum_{i=0}^{n-1} \zeta_i \right|^2 = \sum_{i=0}^{n-1} \mathbb{E}\left[|\zeta_i|^2 \right] + 2 \sum_{0 \leq i < j < n} \mathbb{E}\left[\zeta_i \zeta_j \right]. \tag{3.31}$$

By conditioning with respect to $\mathcal{F}_{t_j}$, we get that each term in the last sum in (3.31) is trivial. Since $\zeta_i = \mathcal{O}(\delta t_i)$, the first sum in the right-hand side of (3.31) is $\mathcal{O}(|\sigma|)$. This gives the result.
Step 2. We show that $\mathbb{E}_{t_i}\left[|\delta_{t_i} \varphi(X)|^2 \right] = \mathcal{O}(\delta t_i)$. First, by the Markov Property, we have $\mathbb{E}_{t_i}\left[\delta_{t_i} M \right] = 0$. Using (3.22), which implies

$$\delta_{t_i} M = \delta_{t_i} \varphi(X) - \int_{t_i}^{t_{i+1}} \mathscr{L}\varphi(X_s) ds, \tag{3.32}$$

[3] the general case is left as an exercise, use the tower property (2.2)

we deduce $\mathbb{E}_{t_i}\left[\delta_{t_i}\varphi(X)\right] = \mathcal{O}(\delta t_i)$. We can apply the previous estimate to $|\varphi|^2$, since $|\varphi|^2$ is in the domain of $\mathscr{L}$ by hypothesis (*cf.* (3.34) below), to get $\mathbb{E}_{t_i}\left[\delta_{t_i}|\varphi|^2(X)\right] = \mathcal{O}(\delta t_i)$. On the other hand, we have also the identity

$$|\delta_{t_i}\varphi(X)|^2 = \delta_{t_i}|\varphi|^2(X) - 2\varphi(X_{t_i})\delta_{t_i}\varphi(X). \tag{3.33}$$

Taking expectation with respect to $\mathcal{F}_{t_i}$ in (3.33) and using the fact that

$$\mathbb{E}_{t_i}\left[\varphi(X_{t_i})\delta_{t_i}\varphi(X)\right] = \varphi(X_{t_i})\mathbb{E}_{t_i}\left[\delta_{t_i}\varphi(X)\right]$$

gives the desired estimate $\mathbb{E}_{t_i}\left[|\delta_{t_i}\varphi(X)|^2\right] = \mathcal{O}(\delta t_i)$. We can insert this result in (3.32) to obtain also $\mathbb{E}_{t_i}\left[|\delta_{t_i}M|^2\right] = \mathcal{O}(\delta t_i)$.

Step 3. We conclude the proof. First, we note that (3.32) applied to φ^2 gives

$$\delta_{t_i}M^{(2)} = \delta_{t_i}|\varphi|^2(X) - \int_{t_i}^{t_{i+1}} \mathscr{L}|\varphi|^2(X_s)ds, \tag{3.34}$$

where $M_t^{(2)} := |\varphi|^2(X_t) - |\varphi|^2(x) - \int_0^t \mathscr{L}|\varphi|^2(X_s)ds$. We combine (3.32), (3.33) and (3.34) to obtain the identity

$$\begin{aligned}|\delta_{t_i}M|^2 + 2\delta_{t_i}M\int_{t_i}^{t_{i+1}}\mathscr{L}\varphi(X_s)ds + \left|\int_{t_i}^{t_{i+1}}\mathscr{L}\varphi(X_s)ds\right|^2 \\ = \delta_{t_i}M^{(2)} + \int_{t_i}^{t_{i+1}}\mathscr{L}|\varphi|^2(X_s)ds - 2\varphi(X_{t_i})\left(\delta_{t_i}M + \int_{t_i}^{t_{i+1}}\mathscr{L}\varphi(X_s)ds\right).\end{aligned} \tag{3.35}$$

Taking the conditional expectation $\mathbb{E}_{t_i}$ in (3.35) and using the Markov property gives us

$$\mathbb{E}_{t_i}\left[|\delta_{t_i}M|^2\right] = \mathbb{E}_{t_i}\left[\delta_{t_i}A\right] - 2\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}}(\varphi(X_{t_i}) - \varphi(X_s))\mathscr{L}\varphi(X_s)ds\right] + \mathcal{O}(|\delta t_i|^{3/2}). \tag{3.36}$$

Indeed, we have discarded the terms

$$\mathbb{E}_{t_i}\left[2\delta_{t_i}M\int_{t_i}^{t_{i+1}}\mathscr{L}\varphi(X_s)ds\right] \text{ and } \mathbb{E}_{t_i}\left[\left|\int_{t_i}^{t_{i+1}}\mathscr{L}\varphi(X_s)ds\right|^2\right],$$

which are respectively $\mathcal{O}(|\delta t_i|^{3/2})$ and $\mathcal{O}(|\delta t_i|^2)$. To obtain the $\mathcal{O}(|\delta t_i|^{3/2})$-estimate, we use the bound

$$\left|2\delta_{t_i}M\int_{t_i}^{t_{i+1}}\mathscr{L}\varphi(X_s)ds\right| \le \eta|\delta_{t_i}M| + \eta^{-1}\left|\int_{t_i}^{t_{i+1}}\mathscr{L}\varphi(X_s)ds\right|^2,$$

then Step 2, and then we choose $\eta = (\delta t_i)^{1/2}$. We can repeat Step 2, where we consider the time interval $[t_i, s]$ instead of $[t_i, t_{i+1}]$, to obtain the estimate $\mathbb{E}_{t_i}\left[|\varphi(X_{t_i}) - \varphi(X_s)|^2\right] = \mathcal{O}(\delta t_i)$, when $t_i \le s \le t_{i+1}$. Consequently, the last term in (3.36) is also $\mathcal{O}(|\delta t_i|^{3/2})$. By summing with respect to i in (3.36), we deduce finally that

$$\sum_{i=0}^{n-1}\mathbb{E}_{t_i}\left[\delta_{t_i}A\right] = \sum_{i=0}^{n-1}\mathbb{E}_{t_i}\left[|\delta_{t_i}M|^2\right] + \mathcal{O}(|\sigma|^{1/2}).$$

This equality, combined with Step 1, yields (3.28). This achieves the proof. □

Lemma 3.8. *Let $\beta\colon \mathbb{R}_+ \to \mathbb{R}$ be a continuous function, right-differentiable at every point, such that $t \mapsto \beta'(t+)$ is bounded. Then*

$$\beta(t) - \beta(0) = \int_0^t \beta'(s+)ds, \tag{3.37}$$

for all $t \geq 0$.

Proof of Lemma 3.8. Note first that

$$\beta'(t+) = \lim_{n\to+\infty} n(\beta(t+n^{-1}) - \beta(t))$$

defines a measurable function of t as limit of measurable functions. Since it is bounded by hypothesis, it is integrable. Let $T > 0$. Let $m, M \in \mathbb{R}$ be such that $m \leq \beta'(t_+) \leq M$ for every $t \in [0,T]$. We will show that

$$m \leq \frac{\beta(t) - \beta(0)}{t} \leq M, \tag{3.38}$$

for all $t \in [0,T]$. This gives the conclusion by considering

$$\tilde{\beta}(t) = \beta(t) - \beta(0) - \int_0^t \beta'(s+)ds,$$

and applying (3.38) with $m = M = 0$. For $t \in (0,T]$, let us denote by $\Gamma(t)$ the quotient in (3.38). We set $\Gamma(0) = \beta'(0+)$ to extend Γ by continuity at 0. Let $\delta > 0$. Assume that there exists $t_1 \in [0,T]$ such that $\Gamma(t_1) > M + \delta$. By continuity of Γ, we have $t_1 > 0$. By restricting things to $[0,t_1]$ if necessary, we can assume $t_1 = T$. Let now $D(t) = \beta(0) + (M + \delta/2)t$ be a parametrization of the straight-line with slope $M + \delta/2$ having the same origin as the graph of β. We have $\beta(T) > D(T)$. Let τ denote the infimum of the points $t \in [0,T]$ such that $\beta > D$ on $[t,T]$. By continuity, τ is well defined, $\tau \in [0,T)$ and $\beta(\tau) = D(\tau)$. At this stage, a picture is useful: at the point τ, the graph of β crosses the straight line $t \mapsto \beta(0) + (M + \delta/2)t$ and is above this straight line on $[\tau,T]$. It is clear that this contradicts the fact that $\beta'(\tau+) \leq M$ and, indeed, the inequality

$$\frac{\beta(\tau+h) - \beta(\tau)}{h} = \frac{\beta(\tau+h) - D(\tau)}{h} \geq \frac{D(\tau+h) - D(\tau)}{h},$$

for $h > 0$ small enough gives, at the limit $h \to 0+$, the contradiction $\beta'(\tau+) \geq M + \delta/2$. □

Exercise 3.9 (Carré du champ)**.** The operator

$$\Gamma\colon \varphi \mapsto \frac{1}{2}\mathscr{L}|\varphi|^2 - \varphi\mathscr{L}\varphi$$

is called the "carré du champ", [1, p.viii].

1. Let $F\colon \mathbb{R}^d \to \mathbb{R}^d$ be Lipschitz continuous. Let (X_t^x) be given as the solution to the Cauchy Problem
$$\frac{d}{dt}X_t^x = F(X_t^x), \quad X_0^x = x.$$
Compute Γ on $C_b^1(\mathbb{R}^d)$ (bounded C^1 function with bounded first derivatives).

2. (For those who know SDEs). Let F be as above and $\sigma\colon \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ be Lipschitz continuous. Let (X_t^x) be given as the solution to the Cauchy Problem

$$dX_t^x = F(X_t^x)dt + \sigma(X_t^x)dW_t, \quad X_0^x = x, \tag{3.39}$$

where (W_t) is a d-dimensional Wiener process. Compute Γ on $C_b^2(\mathbb{R}^d)$ (bounded C^2 function with bounded first and second derivatives).

3. In the general case, show that $\Gamma(\varphi) \geq 0$ for all φ.

The solution to Exercise 3.9 is here.

The results of Theorem 3.7 can be extended to the case where the test function φ also depends on t. We will need this result only in the simple case where the test function has the form $\theta(t)\varphi(x)$.

Corollary 3.9. *Let $\varphi \in D(\mathscr{L})$ satisfies $|\varphi|^2 \in D(\mathscr{L})$. Let $\theta \in C^1(\mathbb{R}_+)$ and let $\psi(t,x) = \theta(t)\varphi(x)$. Then, the process*

$$M_t := \psi(t, X_t) - \psi(0, X_0) - \int_0^t (\partial_t + \mathscr{L})\psi(s, X_s)ds \tag{3.40}$$

is a $(\mathcal{F}_t)$-martingale and the process (Z_t) defined by

$$Z_t := |M_t|^2 - \int_0^t ((\partial_t + \mathscr{L})|\psi|^2 - 2\psi(\partial_t + \mathscr{L})\psi)(s, X_s)ds, \tag{3.41}$$

is a $(\mathcal{F}_t)$-martingale.

Proof of Corollary 3.9. By the Markov property, we have

$$\begin{aligned}\mathbb{E}\left[M_t - M_s|\mathcal{F}_s\right] \\ = \theta(t)(P_{t-s}\varphi)(X_s) - \theta(s)\varphi(X_s) - \int_s^t (\theta'(\sigma)(P_{\sigma-s}\varphi)(X_s) + \theta(\sigma)\frac{d}{d\sigma}(P_{\sigma-s}\varphi)(X_s))d\sigma.\end{aligned}$$

By explicit integration, we see that (M_t) is a $(\mathcal{F}_t)$-martingale. We compute then

$$(\partial_t + \mathscr{L})|\psi|^2 - 2\psi(\partial_t + \mathscr{L})\psi = \theta^2(\mathscr{L}|\varphi|^2 - 2\varphi\mathscr{L}\varphi).$$

Let us examine the proof of the second part of Theorem 3.7. Since θ is locally Lipschitz continuous, we have $\theta(t) = \theta(t_i) + \mathcal{O}(\delta t_i)$, for $t \in [t_i, t_{i+1}]$. Using this approximation, it is easy to show, by adapting the proof of Theorem 3.7, that, in our context, (Z_t) is a martingale. □

Exercise 3.10 (Markov process with finite state space)**.** Let $(\mathbf{P}, Q, \mathbf{X})$ be a Markov process. Assume that the state space E is finite, $E = \{x_1, \ldots, x_L\}$. We introduce the family of matrices $A(t) = a_{ij}(t)$, with $a_{ij(t)} = Q(t, x_i, \{x_j\})$, *i.e.* $a_{ij}(t) = \mathbb{P}_{x_i}(X(t) = x_j)$.

1. If $\varphi\colon E \to \mathbb{R}$, we still denote by φ the vector $(\varphi(x_i))_{1\leq i\leq L}$. Give the expression of $P_t\varphi$ as a product matrix-vector.

2. If μ is a probability measure on E, we still denote by μ the vector $(\mu(\{x_i\})_{1\leq i\leq L}$. Give the expression of $P_t^*\mu$ as a product matrix-vector.

3. We assume that $t \mapsto A(t)$, from $\mathbb{R}_+$ into $\mathcal{M}_L(\mathbb{R})$ is of class C^1. Show that $A(t) = e^{t\mathscr{L}}$, where $\mathscr{L} = A'(0)$ is the generator.

4. Give the equation satisfied by an invariant measure.

The solution to Exercise 3.10 is here.

Exercise 3.11 (Markov process in discrete time)**.** We consider now a Markov process $(X_n)_{n\geq 0}$ in discrete time.

1. Assume that the state space E is finite. How can you rephrase the questions and answers of the previous exercise 3.10?

2. Give and prove the equivalent statement to Theorem 3.7. More precisely, let $\mathscr{L} = P_1 - \mathrm{Id}$, let $\varphi \in \mathrm{BM}(E)$. Show that

$$M_n = \varphi(X_n) - \varphi(X_0) - \sum_{k=0}^{n-1} \mathscr{L}\varphi(X_k) \tag{3.42}$$

and

$$Z_n := |M_n|^2 - \sum_{k=0}^{n-1} \Gamma[\varphi](X_k) \tag{3.43}$$

are martingales. In (3.43), $\Gamma(\varphi)$ is a certain non-negative expression that you will have to identify.

The solution to Exercise 3.11 is here.

4 Evolution of a random interface

In this part we establish the limit behavior of the symmetric simple exclusion process. More precisely, we show in Theorem 4.1 that, after an adequate change of scales, the random interface associated to the symmetric simple exclusion process converges in probability to the solution of a heat equation.

4.1 Change of scale and limit behavior

Let X_L denote the discrete interval $X_L = \{0, \dots, L\}$. Let E_L be the set of functions $H \colon X_L \to \mathbb{R}$ such that $H(L) = 0$. Let $E_L^{(1)}$ be the convex subset of E_L constituted of the functions H such that $H(0) = 0$ and $|H(\mathtt{x}+1) - H(\mathtt{x})| = 1$ for all $\mathtt{x} \in \{0, \dots, L-1\}$. The space $E_L^{(1)}$ is the state space for the process described in Section 1.2. To $H \in E_L$, we associate a function $\hat{H} \colon [0,1] \to \mathbb{R}$ defined by

$$\hat{H}(x) = L^{-1} H(\lfloor Lx \rfloor), \tag{4.1}$$

where $p = \lfloor y \rfloor$, defined for $y \geq 0$, is the integer such that $p \leq y < p+1$. The map $H \mapsto \hat{H}$ is an isometry $E_L \to L^2(0,1)$ when E_L and $L^2(0,1)$ are endowed with the respective scalar products

$$\langle H, G \rangle_{E_L} = \frac{1}{L^3} \sum_{\mathtt{x} \in X_L} H(\mathtt{x}) G(\mathtt{x}), \quad \langle f, g \rangle_{L^2} = \int_0^1 f(x) g(x) dx. \tag{4.2}$$

Indeed, given $H, G \in E_L$, we compute

$$\langle \hat{H}, \hat{G} \rangle_{L^2(0,1)} = \sum_{\mathtt{x}=0}^{L-1} \int_{\frac{\mathtt{x}}{L}}^{\frac{\mathtt{x}+1}{L}} \hat{H}(y) \hat{G}(y) dy = \sum_{\mathtt{x}=0}^{L-1} \frac{1}{L^3} H(\mathtt{x}) G(\mathtt{x}) = \langle H, G \rangle_{E_L}. \tag{4.3}$$

Since $H \mapsto \hat{H}$ is an isometry, a natural left-inverse is given by calculating the adjoint operator. This is easily done, and we obtain a map $L^2(0,1) \to E_L$, which, to $h \in L^2(0,1)$ associates the function in E_L given by

$$Jh(L) = 0, \quad Jh(\mathtt{x}) = L^2 \int_{\frac{\mathtt{x}}{L}}^{\frac{\mathtt{x}+1}{L}} h(x)dx, \quad \mathtt{x} = 0, \dots, L-1. \tag{4.4}$$

However, we will work preferably with the related application $h \mapsto \check{h}$, defined by $\check{h}(\mathtt{x}) = Lh(L^{-1}\mathtt{x})$. The reason of this modification is apparent in Proposition 4.5. If h is Lipschitz continuous on $[0,1]$ and $h(1) = 0$, then $|Jh(\mathtt{x}) - \check{h}(\mathtt{x})|$ is bounded by $\mathrm{Lip}(h)$. This implies that

$$|\langle \hat{H}, h\rangle_{L^2(0,1)} - \langle H, \check{h}\rangle_{E_L}| \leq \mathrm{Lip}(h) L^{-2} \sup_{\mathtt{x} \in X_L} |H(\mathtt{x})|, \tag{4.5}$$

for all $H \in E_L$. Let h_{in} be a continuous function on $[0,1]$, which is 1-Lipschitz continuous and satisfies the boundary conditions $h_{\mathrm{in}}(0) = h_{\mathrm{in}}(1) = 0$. Given such a function h_{in}, we build an initial datum $H_{\mathrm{in}} \in E_L^{(1)}$ for the evolution of the random interface. We want h_{in} to be close to $\hat{H}_{\mathrm{in}}$ in a certain norm. It is simpler to consider things in the space E_L, in which case we require H_{in} and the function $\check{h}_{\mathrm{in}}$ to be at distance $\mathcal{O}(1)$ for a certain norm. Note that the graph of a profile $H \in E_L^{(1)}$ is a subset of the lattice

$$\mathscr{R} = \{(\mathtt{x}, H); \mathtt{x} \in \{0, \cdots, L\}, H \in \mathbb{Z}_{\mathtt{x}}\},$$

where we have set $\mathbb{Z}_{\mathtt{x}} = 2\mathbb{Z}$ if $\mathtt{x}$ is even, $\mathbb{Z}_{\mathtt{x}} = 2\mathbb{Z}+1$ if $\mathtt{x}$ is odd. To build H_{in}, we draw the graph Gr_L of $\check{h}_{\mathrm{in}}$. Then we choose the closest points of Gr_L in $\mathscr{R}$ to obtain the graph of H_{in}. We check that H_{in} satisfies the constraint $|H_{\mathrm{in}}(\mathtt{x}+1) - H_{\mathrm{in}}(\mathtt{x})| = 1$ (it follows from the fact that h_{in} is 1-Lipschitz continuous). We have then

$$\sup_{\mathtt{x} \in X_L} \left|\check{h}_{\mathrm{in}}(\mathtt{x}) - H_{\mathrm{in}}(\mathtt{x})\right| \leq 1 \Rightarrow \sup_{\mathtt{x} \in X_L} |Jh_{\mathrm{in}}(\mathtt{x}) - H_{\mathrm{in}}(\mathtt{x})| \leq 2, \tag{4.6}$$

hence $\|Jh_{\mathrm{in}} - H_{\mathrm{in}}\|_{E_L} \leq 2L^{-1}$. This gives

$$\|\hat{H}_{\mathrm{in}} - h_{\mathrm{in}}\|_{L^2(0,1)} \leq 2L^{-1}. \tag{4.7}$$

Let (H_t) be the Markov process described in Section 1.2 (we will show below in Section 4.2 that it is a Markov process indeed) that starts from H_{in}. We fix a time $T > 0$ and consider the solution to the heat equation on $[0,1]$ with Dirichlet homogeneous boundary conditions and initial datum h_{in}: this the function $h \in L^2(0,T; H_0^1(0,1))$ such that $\partial_t h \in L^2(0,T; H^{-1}(0,1))$ and

$$\langle \partial_t h(t), g\rangle_{L^2(0,1)} + \langle \partial_x h(t), \partial_x g\rangle_{L^2(0,1)} = 0, \tag{4.8}$$

for all $g \in H_0^1(0,1)$ and a.e. $t \in (0,T)$, and $h(0) = h_{\mathrm{in}}$, see [8, p.374]. We call such a function h a weak solution to the following problem:

$$\partial_t h - \partial_x^2 h = 0 \text{ in } (0,+\infty) \times (0,1), \tag{4.9}$$

$$h(t,x) = 0 \text{ for } (t,x) \in (0,+\infty) \times (\{0\} \cup \{1\}), \tag{4.10}$$

$$h(0,x) = h_{\mathrm{in}}(x) \text{ for } x \in (0,1). \tag{4.11}$$

We will establish the following result.

Theorem 4.1. *Let $h_{\rm in}$ be a 1-Lipschitz continuous function on $[0,1]$ vanishing at 0 and 1 and satisfying $h_{\rm in} \in H^2(0,1)$. Let h be the unique solution to (4.9)-(4.10)-(4.11) (see (4.8)). Let $H_{\rm in} \in E_L^{(1)}$ satisfy (4.7), and let (H_t) be the Markov process described in Section 1.2 that starts from $H_{\rm in}$. Then the rescaled process $(\hat{H}_{L^2 t})$ converges to h in probability when L tends to $+\infty$, in the sense that, for all $T > 0$, for all $\delta > 0$, one has*

$$\lim_{L\to+\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \|\hat{H}_{L^2 t} - h(t,\cdot)\|_{L^2(0,1)} > \delta \right) = 0. \tag{4.12}$$

4.2 Markov property

We will show in this part that the process (H_t) described in Section 1.2 is a Markov process and give the expression of its generator. First, the general procedure using "clocks", that transform discrete-time Markov processes into continuous-time Markov processes is analysed. We begin this step with a section of remainder about Poisson processes. In a second step, we study the discrete-time Markov process that gives rise to (H_t).

4.2.1 Poisson processes

Definition 4.1 (Counting process)**.** A counting process $(X_t)_{t\geq 0}$ is a càdlàg process with values in $\mathbb{N}$ such that

1. $X_0 = 0$ a.s.,
2. every jump of $t \mapsto X_t$ has amplitude $+1$: $X_t - X_{t-} = 1$ if there is a jump at t.

Definition 4.2 (Poisson process)**.** Let $\lambda > 0$. A Poisson process (on $\mathbb{R}_+$) of parameter λ is a counting process $(N(t))_{t\geq 0}$ such that, for all $t, s \geq 0$,

1. $N(t+s) - N(t)$ is independent on $\mathcal{F}_t^N$,
2. $N(t+s) - N(t)$ follows a Poisson's law of parameter λs:

$$\mathbb{P}(N(t+s) - N(t) = k) = e^{-\lambda s} \frac{(\lambda s)^k}{k!}. \tag{4.13}$$

The aim of Exercise 4.3 below is to make the relation between the count of exponential arrival times (these are the clocks that we use in Section 1.2 for instance) and Poisson processes. This relation is only partially established in Exercise 4.3. To complete the analysis, we introduce the notion of Poisson point process in $\mathbb{R}^d$. Exercise 4.5 gives several results and the construction of Poisson point processes. This is used in Exercise 4.6 to complete Exercise 4.3 .

Exercise 4.3 (Poisson process)**.** Let (T_n) be a sequence of i.i.d. random variables with exponential law of parameter $\lambda > 0$: $\mathbb{P}(T_n > t) = e^{-\lambda t}$. We define a sequence of times $S_0, S_1, \ldots$ as follows: $S_0 = 0$ and, for $n \geq 1$, $S_n = T_1 + \ldots + T_n$. Given an interval I of $\mathbb{R}_+$, we denote by $\Gamma(I)$ the number of times S_n that fall in I:

$$\Gamma(I) = \#(\mathcal{S} \cap I), \quad \mathcal{S} = \{S_n; n \in \mathbb{N}\}.$$

1. Compute the (density of the) law of S_n.
2. Let $N(t) = \Gamma([0,t])$. Show that $N(t)$ is a counting process and is Poisson of parameter λt (*hint:* $\mathbb{P}(N(t) = n) = \mathbb{P}(S_n \leq t < S_{n+1})$).

The solution to Exercise 4.3 is here.

Definition 4.4 (Poisson point process)**.** Let μ be a (non-negative) measure on the Borel σ-algebra of $\mathbb{R}^d$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space. A Poisson process Π with intensity μ on $\mathbb{R}^d$ is a map from Ω into the set of countable subsets of $\mathbb{R}^d$ such that

1. for all Borel subset A of $\mathbb{R}^d$,
$$\Gamma(A) := \#\{\Pi \cap A\}$$
is a random variable,
2. for all *disjoint* Borel subsets $A_1, \dots, A_k$ of $\mathbb{R}^d$, the random variables $\Gamma(A_1), \dots, \Gamma(A_k)$ are independent,
3. for all Borel subset A of $\mathbb{R}^d$, $\Gamma(A)$ follows a Poisson distribution of parameter $\mu(A)$:
$$\mathbb{P}(\Gamma(A) = n) = e^{-\mu(A)} \frac{\mu(A)^n}{n!}.$$

In the third item 3, we use the following convention: a random variable X with Poisson's law of parameter 0 is concentrated on 0: $\mathbb{P}(X = 0) = 1$. Similarly, a random variable X with Poisson's law of parameter $+\infty$ is concentrated on $+\infty$: $\mathbb{P}(X = +\infty) = 1$.

Exercise 4.5 (Construction of a Poisson point process)**.**

1. Let Π be a Poisson point process of intensity μ on $\mathbb{R}^d$.
   (a) Show that μ has no atom.
   (b) Suppose that μ is finite: $\mu(\mathbb{R}^d) < +\infty$. Let $A_1, \dots, A_k$ be some disjoints subsets of $\mathbb{R}^d$. Let $n_1, \dots, n_k$ and $n \in \mathbb{N}$ be such that $n_0 := n - \sum_{i=1}^k n_i \geq 0$. Show that
   $$\mathbb{P}(\Gamma(A_1) = n_1, \dots, \Gamma(A_k) = n_k | \Gamma(\mathbb{R}^d) = n) = \frac{n!}{n_0! n_1! \cdots n_k!} [\nu(A_0)]^{n_0} \cdots [\nu(A_k)]^{n_k}, \tag{4.14}$$
   where
   $$A_0 = \mathbb{R}^d \setminus (A_1 \cup \cdots \cup A_k), \tag{4.15}$$
   and ν is the normalized measure defined by
   $$\nu(A) = \frac{\mu(A)}{\mu(\mathbb{R}^d)}. \tag{4.16}$$
   In the right-hand side of (4.14) appears the multinomial distribution with parameters n and
   $$p_0 = \nu(A_0), p_1 = \nu(A_1), \dots, p_k = \nu(A_k).$$
   This link between Poisson point processes and multinomial distribution will be exploited to give a construction of a Poisson point process. It will also be exploited in Exercise 4.6 below to complete Exercise 4.3.
2. Let μ be a finite measure on $\mathbb{R}^d$ with no atoms. Let ν be the probability measure defined by (4.16) and let $X_1, \dots, X_n$ be some iid random variables of law ν.
   (a) Show that, almost-surely, $\Pi_n = \{X_1, \dots, X_n\}$ contains n points.
   (b) Let $\Gamma_n(A) = \#\{A \cap \Pi_n\}$. Show that Γ_n satisfies (4.14) (with the same n).

3. Let μ be a finite measure without atoms. We use the analysis of Questions 1 and 2 to build a Poisson point process of intensity μ on $\mathbb{R}^d$. Let ν be the probability measure defined by (4.16) and let $X_1, X_2, \dots,$ be some iid random variables of law ν. Let N be a Poisson distribution of parameter $\mu(\mathbb{R}^d)$ independent on $(X_n)_{n\geq 1}$. Let $\Pi = \{X_1, \dots, X_N\}$. Show that Π is a Poisson point process of intensity μ on $\mathbb{R}^d$.

4. Show the following result

   Theorem 4.2 (Superposition principle)**.** *Let $\Pi_1, \Pi_2, \dots$ be a countable collection of independent Poisson point processes on $\mathbb{R}^d$ with respective intensity measures $\mu_1, \mu_2, \dots$ Then*

$$\Pi = \bigcup_{n\geq 1} \Pi_n$$

   is a Poisson point process on $\mathbb{R}^d$ with intensity measure

$$\mu = \sum_{n\geq 1} \mu_n. \tag{4.17}$$

5. Let μ be a measure on $\mathbb{R}^d$ without atoms that can be written as (4.17) where each μ_n is finite. Show that there exists a Poisson point process on $\mathbb{R}^d$ with intensity measure μ.

6. Show that a σ-finite measure can be written as (4.17) where each μ_n is finite.

7. Let $\lambda > 0$. Consider the case where $d = 1$, and μ is λ times the restriction of the Lebesgue measure to $\mathbb{R}_+$. What is the process $N(t) = \Gamma([0, t])$ then?

The solution to Exercise 4.5 is here.

Exercise 4.6 (Poisson process - continued)**.** To complete the analysis of Exercise 4.3, it is sufficient (why?) to show that

$$\begin{aligned}\mathbb{P}(N(t_1) = n_1, \dots, N(t_k) = n_k) \\ = e^{-\lambda(t_1 - t_0)} \frac{(\lambda(t_1 - t_0))^{n_1 - n_0}}{(n_1 - n_0)!} \cdots e^{-\lambda(t_k - t_{k-1})} \frac{(\lambda(t_k - t_{k-1}))^{n_k - n_{k-1}}}{(n_k - n_{k-1})!},\end{aligned} \tag{4.18}$$

for all $0 \leq t_1 \leq \cdots \leq t_k$ and $n_1 \leq \cdots \leq n_k$. We will need the following tools. Let $t > 0$. Let $U_1, \dots, U_n$ be some independent uniform random variables on $[0, t]$. The *order statistics* of $(U_1, \dots, U_n)$ is the rearrangement $(U_{(1)}, \dots, U_{(n)})$ of the variables U_i in increasing order:

$$U_{(1)} < \cdots < U_{(n)}, \quad \{U_{(1)}, \dots, U_{(n)}\} = \{U_1, \dots, U_n\}.$$

Let Δ denote the subset $\{0 < u_1 < u_2 < \cdots < u_n < t\}$ of $[0, t]^n$. The variables U_i are exchangeable, so the law of $(U_{(1)}, \dots, U_{(n)})$ is given by

$$\mathbb{E}\left[\varphi(U_{(1)}, \dots, U_{(n)})\right] = n! \int \cdots \int_{\Delta} \varphi(u_1, \dots, u_n) du_1 \cdots du_n. \tag{4.19}$$

1. Show that, conditionally to $N_t = n$, $(S_1, \dots, S_n)$ has the law of $(U_{(1)}, \dots, U_{(n)})$.

   Hint: Express $\mathbb{E}\left[\varphi(S_1, \dots, S_n)\mathbf{1}_{S_n \leq t < S_{n+1}}\right]$ in terms of the variables $T_1, \dots, T_{n+1}$ and do the adequate changes of variables.

2. Conclude.

The solution to Exercise 4.6 is here.

4.2.2 From discrete-time to continuous-time Markov process

Proposition 4.3. *Let E be a Polish space. Let $(X_n)_{n\geq 0}$ be a discrete time-homogeneous Markov chain on E with transition operator P_n, $n \in \mathbb{N}$. Let $N(t)$ be a Poisson process of exponent $\lambda > 0$ independent on $(X_n)_{n\geq 0}$ and let $\xi_t = X_{N(t)}$. Let also $(\mathcal{F}_t) = (\mathcal{F}_t^{(\xi,N)})$ be the filtration generated by $(\xi_t, N(t))_{t\geq 0}$. Then $(\xi_t)_{t\geq 0}$ is a time-homogeneous Markov process with respect to $(\mathcal{F}_t)_{t\geq 0}$, with transition operator and infinitesimal generator given by*

$$\Pi_t = \exp\left(-\lambda t(\mathrm{Id} - P_1)\right), \quad \mathscr{L} = -\lambda(\mathrm{Id} - P_1), \tag{4.20}$$

respectively.

Proof of Proposition 4.3. Note first that $P_n = P_1^n$ for all $n \geq 0$. This is the semi-group property in discrete time. Then, we want to establish the following kind of Markov property: for all $A \in \mathcal{F}_t$, for all $\varphi \in \mathrm{BM}(E)$,

$$\mathbb{E}\left[\mathbf{1}_A \varphi(X_{n+N(t)})\right] = \mathbb{E}\left[\mathbf{1}_A P_n \varphi(X_{N(t)})\right]. \tag{4.21}$$

Indeed, (4.21) means that $\mathbb{E}\left[\varphi(X_{n+N(t)})|\mathcal{F}_t\right] = P_1^n \varphi(X_{N(t)})$. Assuming that (4.21) is satisfied for the moment, we use the decomposition

$$\mathbb{E}[\varphi(X_{N(t+s)})|\mathcal{F}_t] = \sum_{n=0}^{\infty} \mathbb{E}[\varphi(X_{N(t+s)})\mathbf{1}_{N(t+s)-N(t)=n}|\mathcal{F}_t].$$

By independence, this gives

$$\mathbb{E}[\varphi(X_{N(t+s)})|\mathcal{F}_t] = \sum_{n=0}^{\infty} \mathbb{P}(N(t+s) - N(t) = n)\mathbb{E}[\varphi(X_{N(t)+n})|\mathcal{F}_t].$$

In the last summand, we replace

$$\mathbb{P}(N(t+s) - N(t) = n) = e^{-\lambda s}\frac{(\lambda s)^n}{n!}, \quad \mathbb{E}[\varphi(X_{N(t)+n})|\mathcal{F}_t] = P_1^n \varphi(X_{N(t)}).$$

The summation over n gives $\mathbb{E}[\varphi(\xi_{t+s})|\mathcal{F}_t] = (\Pi_s\varphi)(\xi_t)$, where Π_t is defined by (4.20). It follows that $(\xi_t)_{t\geq 0}$ is a time-homogeneous Markov process with respect to $(\mathcal{F}_t)_{t\geq 0}$. It is also clear that $\mathscr{L} = -(\mathrm{Id} - P_1)$. To establish (4.21), we observe that each side of the equality defines a set-function, by dependance on A, which is a finite measure. By [3, Theorem 3.3], it is sufficient to prove (4.21) for all sets A in a class $\mathcal{M}$ which is a π-system generating $\mathcal{F}_t$, in the sense that $\sigma(\mathcal{M}) = \mathcal{F}_t$. To that effect, we consider the class $\mathcal{M}$ of sets of the form $B \cap D \cap \{N(t) = m\}$, where $m \in \mathbb{N}$, $B \in \mathcal{F}_m^X$, $D \in \mathcal{F}_t^N$. It is clear that $\mathcal{M}$ is a π-system. The σ-algebra $\mathcal{F}_t$ is generated by all the random variables $X_{N(t_1)}, \ldots, X_{N(t_j)}$ and $N(s_1), \ldots, N(s_k)$ for $j, k \in \mathbb{N}^*$ and times $t_i, s_i \leq t$. By considering all the possible values taken by $N(t_1), \ldots, N(t_j)$ and $N(t)$, the event

$$\left\{X_{N(t_1)} \in \Gamma_1, \ldots, X_{N(t_j)} \in \Gamma_j, N(s_1) \in E_1, \ldots, N(s_k) \in E_k\right\},$$

where $\Gamma_1, \ldots, \Gamma_j \in \mathcal{B}(E)$, $E_1, \ldots, E_k \subset \mathbb{N}$, can be written as a union over $m_1 \in \mathbb{N}, \ldots, m_j, m \in \mathbb{N}$ of the intersection $A := A_1 \cap A_2 \cap \{N(t) = m\}$ of the events $A_1 = \left\{X_{m_1} \in \Gamma_1, \ldots, X_{m_j} \in \Gamma_j\right\}$ with the events

$$A_2 = \{N(t_1) = m_1, \ldots, N(t_j) = m_j\} \cap \{N(s_1) \in E_1, \ldots, N(s_k) \in E_k\}.$$

Since N is non-decreasing and $t_i \leq t$, the set A is possibly non-empty only if the integers m_i are all smaller than m. In the latter case, we have $A \in \mathcal{M}$. We conclude that $\sigma(\mathcal{M}) = \mathcal{F}_t$. For $A = B \cap D \cap \{N(t) = m\} \in \mathcal{M}$, we have then

$$\mathbb{E}\left[\mathbf{1}_A \varphi(X_{n+N(t)})\right] = \mathbb{P}(D \cap \{N(t) = m\}) \mathbb{E}\left[\mathbf{1}_B \varphi(X_{n+m})\right]$$

by independence. By the Markov property, $\mathbb{E}\left[\mathbf{1}_B \varphi(X_{n+m})\right]$ is equal to $\mathbb{E}\left[\mathbf{1}_B P_n \varphi(X_m)\right]$. We use independence again to conclude. □

Exercise 4.7 (Poisson process as Markov process)**.** Let $(N(t))$ be a Poisson process of parameter $\lambda > 0$. Show that $(N(t))$ is a Markov process, give the transition semi-group and the generator. *The solution to Exercise 4.7 is here.*

4.2.3 Markov property for the symmetric simple exclusion process

The evolution of the symmetric simple exclusion process is described in Section 1.2. Recall that we are in the situation where $L = 2N$, and that, to a given configuration η of particles, is associated the function $H \in E_L$ given by

$$H(\mathtt{x}) = \sum_{\mathtt{y}=0}^{\mathtt{x}-1} (2\eta(\mathtt{y}) - 1). \tag{4.22}$$

The evolution of (H_t) can thus be described as follows

1. Let $X'_L := \{0, \dots, L-1\}$. Draw a family $(T_\mathtt{x})_{\mathtt{x} \in X'_L}$ of independent exponential variables of parameter 1.
2. Select the point $\mathtt{x}_*$ such that $T_{\mathtt{x}_*} = \inf_{\mathtt{y} \in X'_L} T_\mathtt{y}$.
3. Perform the transformation $H_{T_{\mathtt{x}_*}-} \to H_{T_{\mathtt{x}_*}}$ according to the rule of evolution of the symmetric simple exclusion process.
4. Start over.

Let us first give some precisions on step 3. Then we will discuss the steps 1-2 can be replaced by the following procedure. Let

$$E_L^{(0)} = \{H \in E_L; H(0) = 0\}$$

Introduce the discrete Laplace operator $\Delta_D \colon E_L^{(0)} \to E_L^{(0)}$ defined by $\Delta_D H(\mathtt{x}) = 0$ if $\mathtt{x} = 0$ or L and

$$\Delta_D H(\mathtt{x}) = H(\mathtt{x}+1) + H(\mathtt{x}-1) - 2H(\mathtt{x}), \quad \forall \mathtt{x} \in \{1, \dots, L-1\}, \tag{4.23}$$

The index D in Δ_D is for "Dirichlet", since Δ_D is actually the discrete Laplace operator with homogeneous Dirichlet boundary conditions. Consider the transformation

$$H \leftarrow H + \delta_\mathtt{x} \Delta_D H \tag{4.24}$$

where

$$\delta_\mathtt{x}(\mathtt{y}) = \mathbf{1}_{\mathtt{x}=\mathtt{y}}, \quad \mathtt{x} \in X_L. \tag{4.25}$$

We consider the graph of H (when H is extended as a piecewise affine function). Examining 4.25 shows that (4.24) is the transformation that flips a corner at $\mathtt{x}$ (local extremum) in a the graph of H into the opposite corner (nothing happens if H has no local extremum at $\mathtt{x}$). We also consider

the different possible configurations of particles, to observe that, when the site $\mathtt{x}$ is selected at time t, H_{t-} becomes $H_{t-} + \delta_{\mathtt{x}+k}\Delta_D H_{t-}$:

$$H_{t-} \leftarrow H_{t-} + \delta_{\mathtt{x}+k}\Delta_D H_{t-} \tag{4.26}$$

where k is a random variable (independent on the variables $T_{\mathtt{y}}$) with Bernoulli distribution $b(1/2)$: $\mathbb{P}(k=1) = \mathbb{P}(k=0) = 1/2$. Let us now discuss the steps 1-2. We assert that it can be replaced by the following procedure: draw a time $\bar{T}$ with exponential law of parameter L, select, independently a site $\mathtt{x} \in X'_L$ with uniform law (and then perform (4.26) at time $t = \bar{T}$). Indeed, if T and T' are two exponential independent random variables of parameters λ and λ', then $T \wedge T'$ is also an exponential random variable of parameter $\lambda + \lambda'$:

$$\mathbb{P}(T \wedge T' > t) = \mathbb{P}(\{T > t\} \cap \{T' > t\}) = \mathbb{P}(T > t)\mathbb{P}(T' > t) = e^{-\lambda t}e^{-\lambda' t} = e^{-(\lambda+\lambda')t}.$$

Let

$$p_{\mathtt{x}}(t) = \mathbb{P}\left[T_{\mathtt{x}} = \inf_{\mathtt{y}\in X'_L} T_{\mathtt{y}}; T_{\mathtt{x}} > t\right].$$

Clearly, $p_{\mathtt{x}}(t) = p_{\mathtt{y}}(t)$ for all $\mathtt{x}, \mathtt{y} \in X'_L$. Since $\bar{T} := \inf_{\mathtt{y}\in X'_L} T_{\mathtt{y}}$ is exponential of parameter L, this gives

$$p_{\mathtt{x}}(t) = \frac{1}{L}\sum_{\mathtt{y}\in X'_L} p_{\mathtt{y}}(t)\frac{1}{L}\mathbb{P}\left[\bigcup_{\mathtt{z}\in X'_L}\{T_{\mathtt{z}} = \inf_{\mathtt{y}\in X'_L} T_{\mathtt{y}}; T_{\mathtt{z}} > t\}\right],$$

which is simply

$$p_{\mathtt{x}}(t) = \frac{1}{L}\mathbb{P}(\bar{T} > t) = \mathbb{P}(\mathtt{Y} = \mathtt{x}, \bar{T} > t),$$

where $\mathtt{Y}$ is uniform in X'_L and independent on $\bar{T}$. With that approach, we see that $H_t = \mathtt{H}_{N(t)}$, where $(N(t))$ is a Poisson process of parameter L and $(\mathtt{H}_n)$ is an independent process that evolves in discrete time as follows

$$\mathtt{H}_{n+1} = \mathtt{H}_n + \delta_{\mathtt{Y}+k}\Delta_D \mathtt{H}_n$$

The process $(\mathtt{H}_n)$ is Markov and time-homogeneous with transition operator P_1 given by

$$\begin{aligned} P_1\varphi(H) = \mathbb{E}_H\varphi(\mathtt{H}_1) &= \frac{1}{L}\sum_{\mathtt{x}=0}^{L-1}\left[\frac{1}{2}\varphi(H + \delta_{\mathtt{x}}\Delta_D H) + \frac{1}{2}\varphi(H + \delta_{\mathtt{x}+1}\Delta_D H)\right] \\ &= \frac{1}{L}\sum_{\mathtt{x}=1}^{L-1}\varphi(H + \delta_{\mathtt{x}}\Delta_D H) + \frac{1}{2L}\varphi(H + \delta_0\Delta_D H) + \frac{1}{2L}\varphi(H + \delta_L\Delta_D H) \end{aligned} \tag{4.27}$$

From Proposition 4.3 and (4.27), we deduce the following result.

Theorem 4.4. *Let $E_L^{(1)}$ be the set of functions $H\colon X_L \to \mathbb{R}$ such that $H(0) = 0$ and $|H_{\mathtt{x}+1} - H_{\mathtt{x}}| = 1$ for all $\mathtt{x} \in X_L$ (in the case $\mathtt{x} = L-1$, we use the convention $H(L) = 0$). Let Δ_D and δ be defined by (4.23) and (4.25) respectively. The symmetric simple exclusion process (H_t) described in Section 1.2 is a Markov process with generator $\mathscr{L}$ given by*

$$\begin{aligned} \mathscr{L}\varphi(H) = &\sum_{\mathtt{x}=1}^{L-1}(\varphi(H + \delta_{\mathtt{x}}\Delta_D H) - \varphi(H)) \\ &+ \frac{1}{2L}(\varphi(H + \delta_0\Delta_D H) - \varphi(H)) + \frac{1}{2L}(\varphi(H + \delta_L\Delta_D H) - \varphi(H)), \end{aligned} \tag{4.28}$$

with domain the set of functions $\varphi\colon E_L^{(1)} \to \mathbb{R}$.

4.3 Deterministic limit

The result of Theorem (4.1) is a kind of law of large numbers (a "functional law of large numbers"). Indeed, let us introduce the average $\langle H_t \rangle = \mathbb{E}[H_t]$. The convergence (4.12) is a consequence of these two following facts:

1. after change of scale, the symmetric simple exclusion process is close to its average value with high probability: for all $\delta > 0$,
$$\lim_{L\to+\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \|\hat{H}_{L^2 t} - \langle \hat{H}_{L^2 t}\rangle\|_{L^2(0,1)} > \delta \right) = 0, \tag{4.29}$$
2. we have the deterministic convergence
$$\lim_{L\to+\infty} \sup_{t\in[0,T]} \|h(t,\cdot) - \langle \hat{H}_{L^2 t}\rangle\|_{L^2(0,1)} = 0, \tag{4.30}$$
where h is the solution to (4.9)-(4.10)-(4.11).

In this section, we will establish the convergence (4.30). Before we proceed, let us study $\langle H_t \rangle$ more closely. Given $\mathtt{x} \in \{1,\dots,L-1\}$, we consider the evaluation map $\pi_{\mathtt{x}} \colon H \mapsto H(\mathtt{x})$. We have $\langle H_t(\mathtt{x})\rangle = \mathbb{E}\pi_{\mathtt{x}}(H_t) = P_t\pi_{\mathtt{x}}(H)$. By definition of the generator $\mathscr{L}$ the derivative in time is $\partial_t\langle H_t(\mathtt{x})\rangle = P_t\mathscr{L}\pi_{\mathtt{x}}(H)$. The explicit formula (4.28) gives $\mathscr{L}\pi_{\mathtt{x}}(H) = \Delta_D H(\mathtt{x})$. By linearity of the operator Δ_D, we deduce that $\partial_t\langle H_t(\mathtt{x})\rangle = \Delta_D\langle H_t(\mathtt{x})\rangle$. After examination of the various boundary conditions, we conclude that $\langle H_t\rangle$ is solution to the following problem:

$$\partial_t\langle H_t\rangle - \Delta_D\langle H_t\rangle = 0 \text{ in } (0,+\infty)\times\{1,\dots,L-1\}, \tag{4.31}$$
$$\langle H_t(0)\rangle = 0 \text{ for all } t \in (0,+\infty), \tag{4.32}$$
$$\langle H_0(\mathtt{x})\rangle = H_{\mathrm{in}}(\mathtt{x}) \text{ for all } \mathtt{x} \in \{1,\dots,L-1\}. \tag{4.33}$$

Different approaches to the convergence result (4.30) are possible. Our proof will be based on a spectral decomposition that will be exploited also to establish the averaging property (4.29) in Section 4.4.

Proposition 4.5 (Spectral basis)**.** *The Laplace operator with homogeneous Dirchlet boundary conditions in dimension* 1*, which is the operator* $-\partial_x^2$*, with domain*

$$\mathcal{D}(-\partial_x^2) = \left\{h \in H^2(0,1); h(0) = h(1) = 0\right\}$$

admits a spectral basis $(a_k)_{k\in\mathbb{N}^*}$*, where* $a_k(x) = \sqrt{2}\sin(\pi k x)$*. This constitutes an orthonormal basis of* $L^2(0,1)$*. The eigenvalue associated to* a_k *is* $\mu_k = \pi^2 k^2$*.*
Let $E_L^{(0)}$ *be the subset of* E_L *constituted of the functions* H *such that* $H(0) = 0$*. The discrete Laplace operator* $-\Delta_D \colon E_L^{(0)} \to E_L^{(0)}$ *is self-adjoint and admits the spectral basis* $(\check{a}_k)_{1\le k\le L-1}$*. The eigenvalue associated to* $\check{a}_k$ *is* $\nu_k = 4\sin^2\left(\frac{\pi k}{2L}\right)$*.*

Proof of Proposition 4.5. We simply give the proof of some assertions about the discrete case. If we extend any $H \in E_L^{(0)}$ to the value L by setting $H(L) = 0$, then for H, G in $E_L^{(0)}$, we easily check that $-\Delta_D H = D_- \circ D_+ H$, where

$$D_+H(\mathtt{x}) = H(\mathtt{x}+1) - H(\mathtt{x}), \quad D_-H(\mathtt{x}) = H(\mathtt{x}) - H(\mathtt{x}-1).$$

Then we use the formula $\langle D_- H, G\rangle_{E_L} = -\langle H, D_+ G\rangle_{E_L}$ to get

$$\langle -\Delta_D H, G\rangle_{E_L} = \langle D_+H, D_+G\rangle_{E_L} = \langle H, -\Delta_D G\rangle_{E_L}. \tag{4.34}$$

This shows that $-\Delta_D$ is self-adjoint. We also have $-\Delta_D \check{a}_k = \nu_k \check{a}_k$, with

$$\nu_k = (e^{i\pi k/L} + e^{-i\pi k/L} - 2) = 4\sin^2\left(\frac{\pi k}{2L}\right). \tag{4.35}$$

The second identity in (4.35) uses the elementary trigonometry formula

$$1 - \cos(2a) = 2\sin^2(a) \tag{4.36}$$

Let $1 \leq k, l \leq L-1$. Using (4.34) and the fact that $\nu_k \neq \nu_l$ if $k \neq l$, we obtain the orthogonality relation $\langle \check{a}_k, \check{a}_l \rangle_{E_L} = 0$ when $k \neq l$. If $k = l$, the trigonometric identity (4.36) gives

$$\langle \check{a}_k, \check{a}_k \rangle_{E_L} = L^{-1} \sum_{\mathtt{x} \in X_L} (1 - \cos(2\pi k \mathtt{x}/L)) = 1 - L^{-1} \mathrm{Re}\left(\sum_{\mathtt{x}=0}^{L-1} e^{2i\pi k \mathtt{x}/L} \right) = 1.$$

The family $(\check{a}_k)_{1 \leq k \leq L-1}$ is free since $\langle \check{a}_k, \check{a}_l \rangle_{E_L} = \delta_{kl}$. It constitutes a basis of $E_L^{(0)}$ hence, since, clearly, $\dim(E_L^{(0)}) = L - 1$. This concludes the proof. □

It follows from Proposition 4.5 that the solution h to (4.9)-(4.10)-(4.11) is given by

$$h(t) = \sum_{k=1}^{\infty} e^{-\mu_k t} \langle h_{\mathrm{in}}, a^k \rangle_{L^2(0,1)}, \tag{4.37}$$

and that $\langle H_t \rangle$ is given by

$$\langle H_t \rangle = \sum_{k=1}^{L-1} e^{-\nu_k t} \langle \langle H_{\mathrm{in}} \rangle, \check{a}^k \rangle_{E_L}. \tag{4.38}$$

Regularity of functions can be expressed in terms of decay of the "Fourier" coefficients. This is what accounts for the following result.

Lemma 4.6. *Let h_{in} be a 1-Lipschitz continuous function on $[0,1]$ vanishing at 0 and 1. Let h be the unique solution to (4.9)-(4.10)-(4.11) (see (4.8)). Then*

$$\sup_{t \in (0,T)} |\langle h(t,\cdot), a_k \rangle_{L^2(0,1)}| \leq \frac{\sqrt{2}}{\pi k}, \tag{4.39}$$

for all $k \geq 1$, where $(a_k)_{k \in \mathbb{N}^*}$ is the orthonormal basis defined in Proposition 4.5.

Proof of Lemma 4.6. At time t, we have $\langle h(t,\cdot), a^k \rangle_{L^2(0,1)} = e^{-\mu_k t} \langle h_{\mathrm{in}}, a^k \rangle_{L^2(0,1)}$. It is sufficient to consider the case $t = 0$ therefore. The estimate (4.39) then follows from the fact that the function h_{in} is 1-Lipschitz continuous. Indeed, integration by parts gives

$$\langle h_{\mathrm{in}}, a^k \rangle_{L^2(0,1)} = -\langle h_{\mathrm{in}}', A^k \rangle_{L^2(0,1)}, \quad A_k(x) = \int_0^x a_k(y)dy = \frac{\sqrt{2}}{\pi k}(1 - \cos(\pi k x)),$$

and then the bound $|\langle h_{\mathrm{in}}, a^k \rangle_{L^2(0,1)}| \leq \sqrt{2}/\pi k$. □

We need a result similar to Lemma 4.6 on functions of the discrete variable $\mathtt{x} \in X_L$.

Lemma 4.7. *There exists a constant $C \geq 0$ such that*

$$|\langle H, \check{a}_k \rangle_{E_L}| \leq \frac{C}{k}, \quad |\langle \hat{H}, a_k \rangle_{L^2(0,1)}| \leq \frac{C}{k}, \tag{4.40}$$

for all $H \in E_L^{(1)}$, for all $k \in \{1, \ldots, L-1\}$, where $(a_k)_{k \in \mathbb{N}^*}$ is the orthonormal basis defined in Proposition 4.5. One can take $C = \sqrt{2}$.

Proof of Lemma 4.7. This time we use a discrete integration by parts:

$$\langle H, \check{a}_k \rangle_{E_L} = \frac{1}{L^3} \sum_{\mathtt{x}=1}^{L-1} (H(\mathtt{x}) - H(\mathtt{x}-1))\, B_k(\mathtt{x}), \tag{4.41}$$

where $B_k(\mathtt{x}) := \sqrt{2}L \sum_{\mathtt{y}=\mathtt{x}}^{L-1} \sin(k\pi \mathtt{y}/L)$ satisfies $|B_k(\mathtt{x})| \le \sqrt{2}L^2 k^{-1}$ for all $\mathtt{x} \in X_L$. Indeed, we compute

$$|B_k(\mathtt{x})| = \sqrt{2}L \left| \mathrm{Im} \sum_{\mathtt{y}=\mathtt{x}}^{L-1} e^{i\pi k \mathtt{y}/L} \right| \le \sqrt{2}L \left| \frac{1 - e^{i\pi k(L-\mathtt{x})/L}}{1 - e^{i\pi k/L}} \right| \le \frac{2L\sqrt{2}}{|1 - e^{i\pi k/L}|}.$$

We have

$$|1 - e^{i\pi k/L}| = 2\sin(\pi k/(2L)) \ge 2k/L,$$

since $\sin(x) \ge (2/\pi)x$ if $x \in [0, \pi/2]$, which gives $|B_k(\mathtt{x})| \le \sqrt{2}L^2 k^{-1}$ as desired. The product $\langle \hat{H}, a_k \rangle_{L^2(0,1)}$ satisfies an identity similar to (4.41), with

$$B_k(\mathtt{x}) := L^2\sqrt{2} \sum_{\mathtt{y}=\mathtt{x}}^{L-1} \int_{\mathtt{y}/L}^{(\mathtt{y}+1)/L} \sin(\pi k z) dz.$$

Using the bound

$$|B_k(\mathtt{x})| = \frac{\sqrt{2}L^2}{\pi k} |\cos(\pi k) - \cos(\pi k \mathtt{x}/L)| \le \frac{2\sqrt{2}L^2}{\pi k},$$

we obtain the second estimate in (4.40). □

Proof of the convergence (4.30). Let $K(L)$ satisfy

$$\lim_{L \to +\infty} K(L) = +\infty, \quad K(L) = o(L^{2/5}).$$

By Lemma 4.6, Lemma 4.7 and the Parseval identity, we have

$$\sup_{t\in[0,T]} \|h(t,\cdot) - \langle \hat{H}_{L^2 t} \rangle\|^2_{L^2(0,1)} = \sup_{t\in[0,T]} \sum_{k=1}^{K(L)} \left| \langle h(t,\cdot) - \langle \hat{H}_{L^2 t} \rangle, a_k \rangle_{L^2(0,1)} \right|^2 + o(1) \tag{4.42}$$

when $L \to +\infty$. We will show that (4.42) can be approached, still with an $o(1)$ error, by

$$\sup_{t\in[0,T]} \sum_{k=1}^{K(L)} \left| \langle h(t,\cdot), a_k \rangle_{L^2(0,1)} - \langle \langle H_{L^2 t} \rangle, \check{a}_k \rangle_{E_L} \right|^2. \tag{4.43}$$

We use (4.5) and the fact that $H(\mathtt{x}) \le L$ for all $\mathtt{x} \in X_L$ when $H \in E_L^{(1)}$. Since $\mathrm{Lip}(a_k) = \mathcal{O}(k)$, we obtain

$$\sum_{k=1}^{K(L)} \left| \langle \langle \hat{H}_{L^2 t} \rangle, a_k \rangle_{L^2(0,1)} - \langle \langle H_{L^2 t} \rangle, \check{a}_k \rangle_{E_L} \right|^2 = \sum_{k=1}^{K(L)} \left| \mathcal{O}(L^{-1}k) \right|^2 = \mathcal{O}(L^{-2}K(L)^3) = o(1) \tag{4.44}$$

where the $\mathcal{O}$ and o are uniform in $t \in [0,T]$. Now we can use the spectral decompositions (4.37) and (4.38) to see that (4.43) is equal to

$$\sup_{t\in[0,T]} \sum_{k=1}^{K(L)} \left| e^{-\mu_k t} \langle h_{\mathrm{in}}, a_k \rangle_{L^2(0,1)} - e^{-L^2 \nu_k t} \langle H_{\mathrm{in}}, \check{a}_k \rangle_{E_L} \right|^2. \tag{4.45}$$

Let us compare $\langle h_{\rm in}, a_k\rangle_{L^2(0,1)}$ to $\langle H_{\rm in}, \check{a}_k\rangle_{E_L}$. By (4.6), we have

$$\sup_{x\in[0,1]} |\hat{\check{h}}_{\rm in}(x) - \hat{H}_{\rm in}(x)| \le L^{-1} \sup_{\mathtt{x}\in X_L} |\check{h}_{\rm in}(\mathtt{x}) - H_{\rm in}(\mathtt{x})| \le L^{-1}.$$

We have also

$$\sup_{x\in[0,1]} |\hat{\check{h}}_{\rm in}(x) - h_{\rm in}(x)| = \sup_{x\in[0,1]} |h_{\rm in}([xL]/L) - h_{\rm in}(x)| \le L^{-1},$$

since $h_{\rm in}$ is 1-Lipschitz continuous. Finally, we can estimate the L^2-norm by the L^∞-norm to obtain $\|h_{\rm in} - \hat{H}_{\rm in}\|_{L^2(0,1)} \le 2L^{-1}$ and

$$|\langle h_{\rm in}, a_k\rangle_{L^2(0,1)} - \langle \hat{H}_{\rm in}, a_k\rangle_{L^2(0,1)}| \le 2L^{-1}. \tag{4.46}$$

By (4.5), we also have $\langle H_{\rm in}, \check{a}_k\rangle_{E_L} = \langle \hat{H}_{\rm in}, a_k\rangle_{L^2(0,1)} + \mathcal{O}(L^{-1}k)$. An estimate similar to (4.44) then shows that (4.45) is equal (up to $o(1)$) to

$$\sup_{t\in[0,T]} \sum_{k=1}^{K(L)} \left| e^{-\mu_k t} - e^{-L^2\nu_k t}\right|^2 \langle h_{\rm in}, a_k\rangle^2_{L^2(0,1)}. \tag{4.47}$$

At that point, we need to compare the eigenvalues μ_k to the rescaled eigenvalues $L^2\nu_k$. The two standard inequalities $\frac{2}{\pi}x \le \sin(x) \le x$, $|\sin(x) - x| \le x^3$, for $0 \le x \le \frac{\pi}{2}$, have the consequence that there exists a constant $C \ge 0$ such that

$$\frac{4}{\pi^2}\mu_k \le L^2\nu_k \le \mu_k, \quad \mu_k - L^2\nu_k \le C\frac{k^3}{L}, \tag{4.48}$$

for all $k \in \{1, \cdots, L-1\}$. Using (4.39), we deduce that (4.47) is bounded from above by

$$C^2 T L^{-2} K(L)^5 \|h_{\rm in}\|^2_{L^2(0,1)},$$

which is o(1) since $K(L) = o(L^{2/5})$ by hypothesis. This concludes the proof. □

4.4 Averaging

In this section, we will establish the convergence (4.29). We use (4.3) and Proposition 4.5, which give

$$\|\hat{H}_{L^2 t} - \langle \hat{H}_{L^2 t}\rangle\|^2_{L^2(0,1)} = \sum_{k=1}^{L-1} |\langle H_{L^2 t} - \langle H_{L^2 t}\rangle, \check{a}_k\rangle_{E_L}|^2. \tag{4.49}$$

We need to analyze the behavior on $[0, L^2T]$ of the process $\langle H_t, \check{a}_k\rangle_{E_L}$, which is of the form $\varphi_k(H_t)$, with $\varphi_k(H) = \langle H, \check{a}_k\rangle_{E_L}$. The formula (4.28) for the generator $\mathscr{L}$ of (H_t) gives $\mathscr{L}\varphi_k(H) = \langle \Delta_D H, \check{a}_k\rangle_{E_L}$. By Proposition 4.5 and the fact that Δ_D is self-adjoint on $E_L^{(0)}$, we obtain $\mathscr{L}\varphi_k(H) = -\nu_k\varphi_k(H)$, when $H \in E_L^{(0)}$. Let us then apply Corollary 3.9 with $\psi(t, H) = e^{\nu_k t}\varphi_k(H)$. The quantity $(\partial_t + \mathscr{L})\psi$ vanishes and we obtain that

$$M_t^{(k)} := e^{\nu_k t}\langle H_t, \check{a}_k\rangle_{E_L} - \langle H_{\rm in}, \check{a}_k\rangle_{E_L} \tag{4.50}$$

and

$$Z_t^{(k)} := \left|M_t^{(k)}\right|^2 - \int_0^t e^{2\nu_k s}(\mathscr{L}|\varphi_k|^2 - 2\varphi_k\mathscr{L}\varphi_k)(H_s)ds \tag{4.51}$$

are both martingales. Since $t \mapsto \mathbb{E}\left[M_t^{(k)}\right]$ is constant, by the martingale property, and vanishes at $t = 0$, we have

$$0 = \mathbb{E}\left[M_t^{(k)}\right] = e^{\nu_k t}\langle\langle H_t\rangle, \check{a}_k\rangle_{E_L} - \langle H_{\text{in}}, \check{a}_k\rangle_{E_L}.$$

Consequently our quantity of interest is $\langle H_t - \langle H_t\rangle, \check{a}_k\rangle_{E_L} = e^{-\nu_k t} M_t^{(k)}$. We can use the Doob's martingale inequality (Theorem 2.2 with $p = 2$), and the trivial bound $e^{-\nu_k t} \leq 1$, to obtain the estimate

$$\mathbb{P}\left(\sup_{t\in[0,L^2T]} |\langle H_t - \langle H_t\rangle, \check{a}_k\rangle_{E_L}| > a\right) \leq \frac{4}{a^2}\mathbb{E}|M_{L^2T}^{(k)}|^2. \tag{4.52}$$

Since $\mathbb{E}\left[Z_t^{(k)}\right] = 0$, (4.51) gives us the bound from above

$$\mathbb{P}\left(\sup_{t\in[0,L^2T]} |\langle H_t - \langle H_t\rangle, \check{a}_k\rangle_{E_L}| > a\right) \leq \frac{4}{a^2}\int_0^{L^2T} e^{2\nu_k s}(\mathscr{L}|\varphi_k|^2 - 2\varphi_k\mathscr{L}\varphi_k)(H_s)ds. \tag{4.53}$$

We will compute the "carré du champ" $\mathscr{L}|\varphi_k|^2 - 2\varphi_k\mathscr{L}\varphi_k$ to understand better what gives (4.53). Before we start, let us pause a moment to consider the inequalities that we have used. We come back to (4.52) in particular, where we have discarded the term $e^{-\nu_k t}$. We may have lost something here. If k is $\mathcal{O}(1)$, then ν_k is of order L^{-2} for L large, and $t \mapsto e^{-\nu_k t}$ is not smaller than a given positive constant on the time interval $[0, L^2T]$. If k takes greater values, then things are different. However, as soon as $k \geq K(L)$, where $K(L)$ is a quantity that grows to $+\infty$ with L, but possibly very slowly, we can use the bound of Lemma 4.7 to get the estimate

$$\sum_{k=K(L)}^{L-1} |\langle H_{L^2t} - \langle H_{L^2t}\rangle, \check{a}_k\rangle_{E_L}|^2 \leq C^2 \sum_{k\geq K(L)} \frac{1}{k^2} \leq C^2 K(L)^{-1}. \tag{4.54}$$

We have only to consider the indexes $k \leq K(L)$ hence. If this is not exactly a bounded range of indexes, we will see that the loss of the $e^{-\nu_k t}$ factor in (4.52) is not a problem.
We go back to the computation of the carré du champ now. We can write φ_k as the sum over $\mathtt{x} \in \{1, \dots, L-1\}$ of $L^{-3}\check{a}_k(\mathtt{x})\pi_{\mathtt{x}}$, where $\pi_{\mathtt{x}}$ is the evaluation at $\mathtt{x}$. We need to compute $\mathscr{L}(\pi_{\mathtt{x}} \otimes \pi_{\mathtt{y}})$ therefore, where $\pi_{\mathtt{x}} \otimes \pi_{\mathtt{y}}(H) := H(\mathtt{x})H(\mathtt{y})$. By (4.28), this is

$$\mathscr{L}(\pi_{\mathtt{x}} \otimes \pi_{\mathtt{y}})(H) = \sum_{\mathtt{z}=0}^{L-1} \left[(H(\mathtt{x}) + \delta_{\mathtt{z}}(\mathtt{x})\Delta_D H(\mathtt{x}))(H(\mathtt{y}) + \delta_{\mathtt{z}}(\mathtt{y})\Delta_D H(\mathtt{y})) - H(\mathtt{x})H(\mathtt{y})\right],$$

which is equal to $H(\mathtt{y})\Delta_D H(\mathtt{x}) + H(\mathtt{x})\Delta_D H(\mathtt{y})$ if $\mathtt{x} \neq \mathtt{y}$, and to $2H(\mathtt{x})\Delta_D H(\mathtt{x}) + |\Delta_D H(\mathtt{x})|^2$ if $\mathtt{x} = \mathtt{y}$. We obtain

$$(\mathscr{L}|\varphi_k|^2 - 2\varphi_k\mathscr{L}\varphi_k)(H) = \frac{1}{L^6}\sum_{\mathtt{x}=1}^{L-1} |\check{a}_k(\mathtt{x})|^2 |\Delta_D H(\mathtt{x})|^2. \tag{4.55}$$

If $H \in E_L^{(1)}$, then $|\Delta_D H(\mathtt{x})| \leq 2$ for all $\mathtt{x}$. This shows that the right-hand side of (4.55) is bounded by $4L^{-3}\|\check{a}_k\|_{E_L}^2$. Since $\check{a}_k$ is normalized, we conclude finally that

$$0 \leq (\mathscr{L}|\varphi_k|^2 - 2\varphi_k\mathscr{L}\varphi_k)(H) \leq 4L^{-3}, \tag{4.56}$$

for all $H \in E_L^{(1)}$. Let $\theta \in (0, 1/2)$ be fixed and let A_L denote the event

$$A_L = \bigcap_{1\leq k<K(L)} \left\{\sup_{t\in[0,L^2T]} |\langle H_t - \langle H_t\rangle, \check{a}_k\rangle_{E_L}| \leq L^{-\theta}\right\}.$$

Let us choose $K(L) = (\log(L))^{1/3}$. We will show that we have then $\lim_{L\to+\infty} \mathbb{P}(A_L) = 1$. By (4.49) and (4.54) and, we see that

$$\sup_{t\in[0,T]} \|H_{L^2 t} - \langle H_{L^2 t}\rangle\|^2_{L^2(0,1)} \leq C^2(\log(L))^{-1/3} + (\log(L))^{1/3} L^{-2\theta},$$

when A_L is realized, so it is clearly sufficient to prove $\lim_{L\to+\infty} \mathbb{P}(A_L) = 1$ to get the desired result. The union bound gives

$$\mathbb{P}(A_L^c) \leq \sum_{1\leq k<K(L)} \mathbb{P}\left(\sup_{t\in[0,L^2T]} |\langle H_t - \langle H_t\rangle, \check{a}_k\rangle_{E_L}| > L^{-\theta}\right).$$

Using (4.53) and (4.56), we obtain

$$\mathbb{P}(A_L^c) \leq 16L^{2\theta} \sum_{1\leq k<K(L)} \int_0^{L^2T} e^{\nu_k s} L^{-3} ds \leq 16L^{2\theta-1} \sum_{1\leq k<K(L)} \frac{e^{\nu_k L^2 T}}{\nu_k L^2}. \tag{4.57}$$

From the inequality $\frac{2}{\pi}x \leq \sin(x) \leq x$ for $0 \leq x \leq \frac{\pi}{2}$, we infer that $\nu_k L^2$ is bounded between $16k^2$ and $4\pi^2 k^2$. We deduce then from (4.57) that

$$\mathbb{P}(A_L^c) \leq L^{2\theta-1} \sum_{1\leq k<K(L)} \frac{e^{4\pi^2 k^2 T}}{k^2} \leq SL^{2\theta-1} e^{4\pi^2 T(\log(L))^{2/3}},$$

where $S = \sum_{k\geq 1} k^{-2} = \pi^2/6$ is finite. This shows that $\lim_{L\to+\infty} \mathbb{P}(A_L) = 1$, as required.

5 Conservation laws and the Finite Volume method

5.1 Discrete conservation laws, continuous limit

We go back to Section 1.1 of the introductory part. We considered a discrete evolution equation

$$u_K^{n+1} = u_K^n + \frac{\Delta t_n}{|K|} \sum_{L\in\mathcal{N}(K)} |K|L|Q_{L\to K}^n. \tag{5.1}$$

The quantity u_K^n represents the density of a certain extensive quantity u in the space-time cell $K \times (t_n, t_{n+1})$. The time grid is constituted from the discrete times $t_0 < t_1 < \cdots < t_n < \cdots$, where $t_n = n\Delta t$, $n \in \mathbb{N}$ for a fixed time-step Δt. The space $\mathbb{R}^d$ is partitioned as follows: we are given a family $\mathcal{T}$ of disjoint open bounded sets such that:

- for all distinct $K, L \in \mathcal{T}$, the interface $\overline{K} \cap \overline{L}$ is contained in an hyperplane of $\mathbb{R}^d$,
- up to a negligible set for the d-dimensional Lebesgue measure, the union of the sets K in $\mathcal{T}$ is equal to $\mathbb{R}^d$.

We also use the following notations:

- $K|L$ is the intersection $\overline{K} \cap \overline{L}$,
- $|K|$ is the d-dimensional Lebesgue measure of K and $|K|L|$ is the $d-1$-dimensional Lebesgue measure of $K|L$,

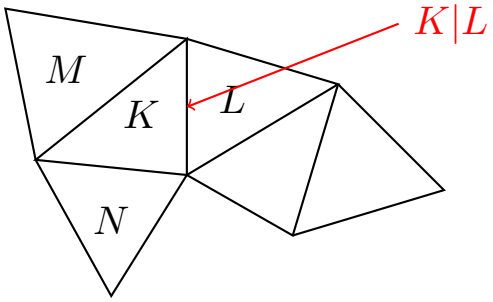


Figure 2: A mesh in $\mathbb{R}^2$

- $\mathcal{N}(K) = \{L \in \mathcal{T}; 0 < |K|L| < +\infty\}$ is the set of neighbors of K,
- when $K \in \mathcal{T}$ and $L \in \mathcal{N}(K)$, $n_{K\to L}$ is the outward unit normal to K along $K|L$ and $Q^n_{K\to L}$ is some numerical flux, $\Delta t\, Q^n_{K\to L}$ representing the amount of u that has passed from K to L trough the interface $K|L$ on the time interval (t_n, t_{n+1}).

In the introductory section 1.1, we also assumed that the condition

$$Q^n_{L\to K} = -Q^n_{K\to L}, \tag{5.2}$$

for all $n \in \mathbb{N}$, for all $K, L \in \mathcal{T}$ being neighbors, is satisfied. The condition (1.3) ensures that, in the time interval (t_n, t_{n+1}), the (algebraic) quantity of u transferred from the cell K to the cell L is the exact opposite of the quantity of u transferred from L to K: no loss or creation of u occurs at the interface $K|L$. Define

$$h = \sup_{K\in\mathcal{T}} \operatorname{diam}(K), \quad u_{h,\Delta t} = \sum_{n\in\mathbb{N}} \sum_{K\in\mathcal{T}} u^n_k \mathbf{1}_{K\times[t_n,t_{n+1})}. \tag{5.3}$$

Under some additional conditions on the the discrete fluxes $Q^n_{K\to L}$, we will study the limit when $h, \Delta t \to 0$ of $u_{h,k}$. We will show that we obtain in the limit a conservation law

$$\partial_t u + \operatorname{div}_x(Q) = 0, \tag{5.4}$$

where $Q = Q(x,t)$. There are various instances of such conservation laws. For example the heat equation $\partial_t - \operatorname{div}(K\nabla u) = 0$ or the diffusion equation $\partial_t - \operatorname{div}(D\nabla u) = 0$, the flux being then given by the the Fourier law, $Q = -K\nabla_x u$, or the Fick law, $Q = -D\nabla u$ respectively. An other example is the continuity equation

$$\partial_t u + \operatorname{div}_x(au) = 0, \tag{5.5}$$

where a is a vector-field over $\mathbb{R}^d$. The continuity equation can be rewritten

$$\partial_t u + a \cdot \nabla_x u + \operatorname{div}_x(a)u = 0, \tag{5.6}$$

and coincides with the transport equation $\partial u + a \cdot \nabla_x u = 0$ when a is divergence-free. We can also mention the Fokker-Planck equation of the kinetic theory of gases,

$$\partial_t f + v \cdot \nabla_x f + F(x) \cdot \nabla_v f = \operatorname{div}_v(\nabla_v f + vf), \tag{5.7}$$

which is of the form (1.4), or more precisely $\partial_t f + \mathrm{div}_{x,v}(Q) = 0$, with a flux

$$Q = \begin{pmatrix} vf \\ F(x)f - (\nabla_v f + vf) \end{pmatrix}.$$

In all these examples, the equations are linear. We can also consider the non-linear equations

$$\partial_t u - \Delta\phi(u) = 0, \tag{5.8}$$

or

$$\partial_t u + \mathrm{div}_x(A(u)) = 0, \tag{5.9}$$

where $A\colon \mathbb{R} \to \mathbb{R}^d$. The hydrodynamic limits of particles in stochastic interaction that we will consider later can be of very different types, including in particular (5.8) and (5.9). Although both (5.8) and (5.9) may be considered in our framework, we will restrict our attention to models with (5.9) as continuous limit. We refer to [11] for the derivation of (5.8).

5.2 Discrete fluxes

We will begin this section with a discussion on expected discrete fluxes in some specific situations, before giving the description of our general framework. First let us start from (5.9), and see how this can approximate by a discrete system of equations (this is the usual procedure in numerical analysis). Let us integrate (5.9) on a space-time cell $K \times (t_n, t_{n+1})$. We assume that u is smooth for simplicity (beware that this is typically not the case of the solutions to (5.9)). Using Stokes' formula, we obtain

$$\int_K u(t_{n+1}, x)dx - \int_K u(t_n, x)dx = -\int_{t_n}^{t_{n+1}} \int_{\partial K} A(u) \cdot n d\sigma dt. \tag{5.10}$$

We use the approximation

$$\int_K u(t_m, x)dx \sim |K| u_K^m$$

and develop

$$\int_{\partial K} A(u) \cdot n d\sigma = \sum_{L \in \mathcal{N}(K)} \int_{K|L} A(u) \cdot n_{K\to L} d\sigma. \tag{5.11}$$

This gives us the equation

$$u_K^{n+1} = u_K^n - \frac{1}{|K|} \sum_{L \in \mathcal{N}(K)} \int_{t_n}^{t_{n+1}} \int_{K|L} A(u) \cdot n_{K\to L} d\sigma dt. \tag{5.12}$$

We would like to use an approximation like

$$\int_{t_n}^{t_{n+1}} \int_{K|L} A(u) \cdot n_{K\to L} d\sigma dt \simeq \Delta t_n |K| L| A(u_M^n) \cdot n_{K\to L},$$

where M is either the cell K or the cell L, but, precisely, how to make this choice? The study of linear equations gives some insight on this problem.

5.3 Discrete fluxes for linear equations

Consider the continuity equation (5.5). Assume for simplicity that the vector field a is constant. Then (5.5) is equivalent to the transport equation (5.6). What one would observe by looking at the behavior of the solution to (5.6) on the interface $K|L$ between the times t_n and t_{n+1} is a flow of u across $K|L$ in the direction u. Let $n_{K\to L}$ denote the unit normal to K along $K|L$ in the direction of L. The value of $|a\cdot n_{K\to L}|$ ponders the amplitude of the flux across $K|L$, while the sign of $a\cdot n_{K\to L}$ determines the direction of the flow of u across $K|L$. It is quite natural then to set $Q^n_{K\to L}=a\cdot n_{K\to L}u^n_K$ if $a\cdot n_{K\to L}\geq 0$. The condition of conservation (5.2) will be satisfied then if we also set $Q^n_{K\to L}=a\cdot n_{K\to L}u^n_L$ when $a\cdot n_{K\to L}\leq 0$. This can be summed up in the formula

$$Q^n_{K\to L}=(a\cdot n_{K\to L})^+u^n_K-(a\cdot n_{K\to L})^-u^n_L. \tag{5.13}$$

A generalization of (5.14) in the case where a is a non-constant vector field is

$$Q^n_{K\to L}=a^+_{K\to L}u^n_K-a^-_{K\to L}u^n_L, \tag{5.14}$$

where

$$a_{K\to L}=\frac{1}{|K|L|}\int_{K|L}a(x)\cdot n_{K\to L}d\sigma(x). \tag{5.15}$$

A further generalization of (5.13) can be given in the case where the flux $A(u)$ in (5.9) actually depends on x also and is of the form $A(x,u)=f(u)a(x)$, where f is a *non-decreasing* locally Lipschitz function $\mathbb{R}\to\mathbb{R}$ and $a\colon\mathbb{R}^d\to\mathbb{R}^d$ is a divergence-free smooth vector field. Indeed, (5.9) can be rewritten as the non-linear transport equation $\partial_t u+f'(u)a\cdot\nabla_x u=0$ and, using the definition (5.15), the sign of $f'(u)a_{K\to L}$ is the sign of $a_{K\to L}$ since $f'(u)\geq 0$. In that situation, one can consider the flux

$$Q^n_{K\to L}=a^+_{K\to L}f(u^n_K)-a^-_{K\to L}f(u^n_L). \tag{5.16}$$

5.4 General monotone fluxes

Consider the case of a general flux A in (5.9). By general flux A, we mean any function $A\colon\mathbb{R}\to\mathbb{R}^d$ that is locally Lipschitz continuous. Sometimes, we also the consider the extension to some fluxes $A(x,u)$ depending also on the space variable. What kind of numerical flux may be compatible with such an expected limit as (5.9)? Inspired by the examples in Section 5.3, we look for some numerical fluxes $Q^n_{K\to L}$ given by a relation

$$Q^n_{K\to L}=A_{K\to L}(u^n_K,u^n_L), \tag{5.17}$$

where $A_{K\to L}$ is a function with the following properties:

1. compatibility with the flux A:

$$A_{K\to L}(v,v)=A(v)\cdot n_{K\to L}, \tag{5.18}$$

for all $v\in\mathbb{R}$,

2. regularity: the function $A_{K\to L}$ is locally Lipschitz continuous: for every $R>0$, there exists a constant $L_A(R)\geq 0$ such that

$$|A_{K\to L}(v,w)-A_{K\to L}(v',w')|\leq L_A(R)(|v-v'|+|w-w'|), \tag{5.19}$$

for all $v,v',w,w'\in[-R,R]$ and for all neighboring cells $K,L\in\mathcal{T}$,

3. monotony: for all $v, w \in \mathbb{R}$, the function $A_{K\to L}(v,\cdot)$ is non-increasing, while the function $A_{K\to L}(\cdot, w)$ is non-decreasing,

4. conservation property:
$$A_{K\to L}(v,w) = -A_{L\to K}(w,v), \tag{5.20}$$
for all $v, w \in \mathbb{R}$ and for all neighboring cells $K, L \in \mathcal{T}$.

If we choose the definition (5.17) of the flux, then (5.20) yields the conservation property (5.2). There is some redundancy in the properties required above: (5.20) and the single fact that $A_{K\to L}(v,\cdot)$ is non-increasing implies that $A_{K\to L}(\cdot, w)$ is non-decreasing for instance. In the next two paragraphs we infer some consequences on the discrete evolution equation (5.1) of (5.17), (5.18), (5.19), (5.20) and the monotony properties of $A_{K\to L}$.

Exercise 5.1 (Godunov flux, Engquist-Osher flux)**.** Define $A^G_{K\to L}(v,w)$ as follows: if $v \le w$, then $A^G_{K\to L}(v,w)$ is the minimum value of $u \mapsto A(u)\cdot n_{K\to L}$ on the interval $[v,w]$. If $w \le v$, then $A^G_{K\to L}(v,w)$ is the maximum value of $u \mapsto A(u)\cdot n_{K\to L}$ on the interval $[w,v]$. Define also $A^{EO}_{K\to L}(v,w)$ by the formula

$$A^{EO}_{K\to L}(v,w) = \int_0^v (a(\xi)\cdot n_{K\to L})^+ d\xi - \int_0^w (a(\xi)\cdot n_{K\to L})^- d\xi,$$

where $a(u) = A'(u)$. Show that $A^G_{K\to L}$ and $A^{EO}_{K\to L}$ have the required properties and show that they coincide with the upwind flux (5.13) in the linear case $A(u) = au$.
The solution to Exercise 5.1 is here.

5.5 Constants as solutions

Any constant function $u^n_K \equiv v$ is solution to (5.1). By (5.18), we have indeed

$$\sum_{L\in\mathcal{N}(K)} |K|L|Q^n_{K\to L} = \sum_{L\in\mathcal{N}(K)} |K|L|A(v)\cdot n_{K\to L} = \sum_{L\in\mathcal{N}(K)} \int_{K|L} A(v)\cdot n_{K\to L} d\sigma(x).$$

We use the Stokes formula

$$\sum_{L\in\mathcal{N}(K)} \int_{K|L} \Psi(x)\cdot n_{K\to L} d\sigma(x) = \int_K \operatorname{div}\Psi(x) dx, \tag{5.21}$$

to obtain

$$\sum_{L\in\mathcal{N}(K)} |K|L|Q^n_{K\to L} = 0,$$

as desired.

Exercise 5.2 (Spatially dependent flux)**.** Assume that $A(x,u)$ satisfies the divergence-free condition $(\operatorname{div}_x A)(x,u) = 0$ for all $u \in \mathbb{R}$. Assume also that $Q^n_{K\to L}$ is given by (5.17), where $A_{K\to L}$ satisfies the following generalized version of (5.18):

$$A_{K\to L}(v,v) = \frac{1}{|K|L|} \int_{K|L} A(x,v)\cdot n_{K\to L} d\sigma(x). \tag{5.22}$$

Show that constant are solutions.
The solution to Exercise 5.2 is here.

For later use, we record the identity

$$\sum_{L\in\mathcal{N}(K)} |K|L|A_{K\to L}(u_K^n, u_K^n) = 0, \tag{5.23}$$

valid for all neighboring cells $K, L \in \mathcal{T}$. We can use it to transform (5.1) into the identity

$$u_K^{n+1} = u_K^n + \frac{\Delta t_n}{|K|} \sum_{L\in\mathcal{N}(K)} |K|L|[A_{K\to L}(u_K^n, u_K^n) - A_{K\to L}(u_K^n, u_L^n)]. \tag{5.24}$$

On the formula (5.24), we can see the stabilizing effect of the monotony of the numerical flux. Imagine that $u_K^n > u_L^n$ for all neighboring cells L of K. Then

$$A_{K\to L}(u_K^n, u_K^n) - A_{K\to L}(u_K^n, u_L^n) \le 0$$

for all L since $A_{K\to L}$ is non-increasing in its second argument, which implies that $u_K^{n+1} \le u_K^n$. The estimates in the following two sections essentially use this.

5.6 Comparison principle

5.6.1 Periodic discrete conservation law

In all that follows we will consider for simplicity a periodic setting. We assume that the mesh $\mathcal{T}$ is periodic, in the sense that there exists a mesh $\mathcal{T}^\sharp$ of the hypercube $(0,1)^d$ such every $K \in \mathcal{T}$ is the translation of an element $K^\sharp$ of $\mathcal{T}^\sharp$ by a vector of $\mathbb{Z}^d$. We also assume that $K \mapsto u_K^0$ is periodic, in the sense that $K \sim L$ (where the relation of equivalence $K \sim L$ is defined by $K = \ell + L$, $\ell \in \mathbb{Z}^d$) implies $u_K^0 = u_L^0$. This will be the case if we assume, as will be done later, that

$$\forall K \in \mathcal{T}, u_K^0 = \frac{1}{|K|}\int_K u_0(x)dx, \tag{5.25}$$

where $u_0 \colon \mathbb{R}^d \to \mathbb{R}$ is $\mathbb{Z}^d$-periodic. We denote by $\mathbb{T}^d$ the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

5.6.2 Comparison principle and consequences

Remember the notation (5.3):

$$u_{h,\Delta t} = \sum_{n\in\mathbb{N}} \sum_{K\in\mathcal{T}} u_k^n \mathbf{1}_{K\times[t_n,t_{n+1})}.$$

Note in particular that, if $F \colon \mathbb{R} \to \mathbb{R}$ is continuous, then

$$\int_{\mathbb{T}^d} F(u_{h,\Delta t}(t_n, x))dx = \sum_{K\in\mathcal{T}^\sharp} |K|F(u_K^n). \tag{5.26}$$

Proposition 5.1 (L^1-contraction)**.** *Let $u_{h,\Delta t}$ and $v_{h,\Delta t}$ be two sequences defined by (5.1), with a flux given as in Section 5.4. Define*

$$R_K^n(u_{h,\Delta t}) = \max\left\{|u_L^n|; L \in \mathcal{N}(K) \cup \{K\}\right\}, \quad |\partial K| := \sum_{L\in\mathcal{N}(K)} |K|L|.$$

Let $n \in \mathbb{N}$ be fixed, and assume that the conditions

$$2\frac{\Delta t|\partial K|}{|K|} L_A(R_K^n(u_{h,\Delta t}))) \le 1, \quad 2\frac{\Delta t|\partial K|}{|K|} L_A(R_K^n(v_{h,\Delta t}))) \le 1 \tag{5.27}$$

are satisfied for all $K \in \mathcal{T}$, where L_A is defined in (5.19). We have then

$$\int_{\mathbb{T}^d} (u_{h,\Delta t}(t_{n+1}) - v_{h,\Delta t}(t_{n+1}))^+ dx \leq \int_{\mathbb{T}^d} (u_{h,\Delta t}(t_n) - v_{h,\Delta t}(t_n))^+ dx. \tag{5.28}$$

Remark 5.1 (CFL condition). Recall that h is defined in (5.3) by $h = \sup_{K\in\mathcal{T}} \mathrm{diam}(K)$. Suppose that there exists $\alpha > 0$ such that

$$\alpha h^d \leq |K|, \quad |\partial K| \leq \frac{1}{\alpha} h^{d-1}, \tag{5.29}$$

for all $K \in \mathcal{T}$. Then (5.27) is satisfied if

$$\Delta t \leq Ch, \tag{5.30}$$

where $C^{-1} = 2\alpha^{-2} L_A(R)$, and R is a bound for all the quantities $R_K^n(w)$, $w = u_{h,\Delta t}$ or $v_{h,\Delta t}$ (we will see soon how to ensure that R is finite). The condition (5.30) puts a constraint of the size of the time step, depending on the size of the space-step h. It is called a Courant-Friedrichs-Lewy (CFL) condition.

Exercise 5.3 (Spatially dependent flux)**.** Give some examples of meshes in dimension $d = 2$ which do not satisfy one of the two bounds in (5.29).
The solution to Exercise 5.3 is here.

Proof of Proposition 5.1. Here, and later in the analysis of (5.1), we will use the notation

$$a \wedge b = \min(a, b), \quad a \vee b = \max(a, b). \tag{5.31}$$

We have then the formula

$$(u - v)^+ = u \vee v - v, \tag{5.32}$$

for all $u, v \in \mathbb{R}$. Our first goal is to estimate $u_K^{n+1} \vee v_K^{n+1}$. Let us consider the right-hand side of (5.24). It is a non-decreasing function of the variables u_L^n, $L \in \mathcal{N}(K)$. With respect to the variable u_K^n, it can be written as a sum $\mathrm{Id} + f$, where f is a locally Lipschitz continuous function. On the domain where $\mathrm{Lip}(f) \leq 1$, it will be also an non-decreasing function of u_K^n. Actually, our function f here is has the form $F(u, u, u)$, where

$$F(u_1, u_2, u_3) = \frac{\Delta t_n}{|K|} \sum_{L\in\mathcal{N}(K)} |K|L|[A_{K\to L}(u_1, u_2) - A_{K\to L}(u_3, u_L^n)]$$

is a non-decreasing function of u_1. We are only interested in the Lipschitz dependency of this function with respect to u_2 and u_3, which, using (5.19), is bounded by the first term in (5.27). To sum up, as long as the first condition in (5.27) is satisfied, we have

$$u_K^{n+1} = H_K^n(u_K^n, u_L^n; L \in \mathcal{N}(K)), \tag{5.33}$$

where H_K^n is a non-decreasing function of its arguments. We deduce, under (5.27), that

$$u_K^{n+1} \vee v_K^{n+1} \leq H_K^n(u_K^n \vee v_K^n, u_L^n \vee v_L^n; L \in \mathcal{N}(K)), \tag{5.34}$$

for all $K \in \mathcal{T}$. Then we use (5.32) and (5.33) to obtain the inequality

$$(u_K^{n+1} - v_K^{n+1})^+ \leq H_K^n(u_K^n \vee v_K^n, u_L^n \vee v_L^n; L \in \mathcal{N}(K)) - H_K^n(v_K^n, v_L^n; L \in \mathcal{N}(K)). \tag{5.35}$$

We write (5.35) under the form

$$(u_K^{n+1} - v_K^{n+1})^+ \\ \leq (u_K^n - v_K^n)^+ + \frac{\Delta t_n}{|K|} \sum_{L \in \mathcal{N}(K)} |K|L|[\Phi_{K\to L}(u_K^n, u_K^n; v_K^n, v_K^n) - \Phi_{K\to L}(u_K^n, u_L^n; v_K^n, v_L^n)], \quad (5.36)$$

where

$$\Phi_{K\to L}(v, w; v', w') := A_{K\to L}(v \vee v', w \vee w') - A_{K\to L}(v', w'). \quad (5.37)$$

We multiply (5.36) by $|K|$ and sum over $K \in \mathcal{T}^\sharp$. It gives us the desired estimate (5.28), provided we can show that

$$\sum_{K\in\mathcal{T}^\sharp} \sum_{L\in\mathcal{N}(K)} |K|L|[\Phi_{K\to L}(u_K^n, u_K^n; v_K^n, v_K^n) - \Phi_{K\to L}(u_K^n, u_L^n; v_K^n, v_L^n)] = 0. \quad (5.38)$$

The cancellation property (5.38) follows from the two identities

$$\sum_{L\in\mathcal{N}(K)} |K|L|[\Phi_{K\to L}(u_K^n, u_K^n; v_K^n, v_K^n) = 0, \quad \sum_{K\in\mathcal{T}^\sharp} \sum_{L\in\mathcal{N}(K)} |K|L|\Phi_{K\to L}(u_K^n, u_L^n; v_K^n, v_L^n)] = 0. \quad (5.39)$$

The left identity in (5.39) follows from (5.23). The second identity in (5.39) is a consequence of (5.20) and of the formula

$$\sum_{K\in\mathcal{T}^\sharp} \sum_{L\in\mathcal{N}(K)} a(K, L) = \frac{1}{2} \sum_{K\in\mathcal{T}^\sharp} \sum_{L\in\mathcal{N}(K)} (a(K, L) + a(L, K)), \quad (5.40)$$

satisfied by any periodic function $a\colon \mathcal{T} \times \mathcal{T} \to \mathbb{R}$. Indeed, if $K_* \in \mathcal{T}^\sharp$ and $L_* \in \mathcal{N}(K)$, then the term $a(L_*, K_*)$ in the right-hand side of (5.40) will appear in the sum on the left when $K = L_*$ and $L = K_*$ (in the case where the interface $K_*|L_*$ is on the boundary of $(0, 1)^d$, we need to use the periodic character of a to complete this argument). □

From Proposition 5.1, we deduce first a comparison principle and an L^∞ estimate.

Proposition 5.2 (Comparison principle, L^1 estimate)**.** *Under the hypotheses of Proposition 5.1, we have*

$$\int_{\mathbb{T}^d} |u_{h,\Delta t}(t_{n+1}) - v_{h,\Delta t}(t_{n+1})|dx \leq \int_{\mathbb{T}^d} |u_{h,\Delta t}(t_n) - v_{h,\Delta t}(t_n)|dx. \quad (5.41)$$

Besides, if $v_{h,\Delta t}(t_n) \geq u_{h,\Delta t}(t_n)$ a.e. in $\mathbb{T}^d$, then $v_{h,\Delta t}(t_{n+1}) \geq u_{h,\Delta t}(t_{n+1})$ a.e. in $\mathbb{T}^d$.

Proposition 5.3 (Comparison principle, L^∞ estimate)**.** *Assume $|u_{h,\Delta t}(0)| \leq R$ a.e. in $\mathbb{T}^d$. Then, under the CFL condition*

$$\forall K \in \mathcal{T}, \forall n \geq 1, \quad 2\frac{\Delta t_n |\partial K|}{|K|} L_A(R) \leq 1, \quad (5.42)$$

we have the L^∞ bound $|u_{h,\Delta t}(t)| \leq R$ a.e. in $\mathbb{T}^d$, for a.e. $t \geq 0$.

Proof of Proposition 5.2 and Proposition 5.3. We exchange the roles of $u_{h,\Delta t}$ and $v_{h,\Delta t}$ to deduce the L^1-contraction (5.41) from (5.28) and we use the fact that $v_{h,\Delta t}(t_n) \geq u_{h,\Delta t}(t_n)$ a.e. in $\mathbb{T}^d$ if, and only if the integral over $\mathbb{T}^d$ of $(u_{h,\Delta t}(t_n) - v_{h,\Delta t}(t_n))^+$ vanishes to prove the comparison principle. The L^∞ bound $|u_{h,\Delta t}(t_n)| \leq R$ is proved by recursion on n, using the comparison principle and the fact that the constant functions R and $-R$ are solutions of (5.1). □

5.6.3 Asymptotic behavior

We consider the behavior of the numerical solution to (5.1) when the characteristic scales h and Δt get smaller and smaller. Let $(\Delta t^{(k)})$ be a sequence of positive reals that converge to 0, let $(\mathcal{T}_k)$ be a sequence of meshes that are $\mathbb{Z}^d$-periodic and such that $h_k := \sup_{K\in\mathcal{T}_k} \mathrm{diam}(K)$ tends to 0 when $k \to +\infty$. We assume that (5.29) is satisfied for all k, for all $K \in \mathcal{T}_k$, where α is independent on k. We also assume that (5.19) is satisfied with a Lipschitz constant $L_A(R)$ independent on k. Let $(u_K^0)_{K\in\mathcal{T}_k}$ be given by (5.25), where $u_0 \in L^\infty(\mathbb{T}^d)$. Consider the CFL condition

$$\sup_{n\geq 0} \Delta t_n^{(k)} \leq \Delta t^{(k)} L_A(R) \leq \alpha^2 h_k, \tag{5.43}$$

where $R \geq \|u_0\|_{L^\infty(\mathbb{T}^d)}$. Let $T > 0$ be fixed. By Proposition 5.3, the solution $u_{(k)} := u_{h_k,\Delta t^{(k)}}$ of (5.1) satisfies the bound $\|u_{(k)}\|_{L^\infty(\mathbb{T}^d\times(0,T))} \leq R$ for all k. Consequently, there is a subsequence still denoted $(u_{(k)})$ which converges to a certain function u in $L^\infty(\mathbb{T}^d \times (0,T))$ for the weak-$*$ topology. We would like to show that u is solution to the conservation law (5.9). In the case where A is not a linear function, there are two difficulties to establish this:

1. we use a weak mode of convergence (convergence in $L^\infty(\mathbb{T}^d \times (0,T))$ for the weak-$*$ topology), which is not sufficient in all generality to deal with the convergence of non-linear terms,
2. the theory of the Cauchy Problem for (5.9) in L^∞ requires a specific treatment, via the use of entropy solutions.

We will establish the convergence of $(u_{(k)})$ towards a solution of (5.9) in the linear case only, see Section 5.10. Some additional estimates on $u_{(k)}$ are necessary for this, and we will give them in the following section 5.7, for a general numerical fluxes, associated, via (5.18), to a not-necessarily linear flux A. We refer to [10, Chapter 6] for the proof of convergence of (5.1) in the general case.

5.7 Energy estimate

Consider the parabolic equation

$$u_t + \mathrm{div}(A(u)) - \eta\Delta u = 0 \text{ in } \mathbb{T}^d \times (0,+\infty). \tag{5.44}$$

Here $\eta > 0$ is supposed to be small. The flux in (5.44) is $A(u) - \eta\nabla u$. This is a perturbation of the flux $A(u)$. The addition of the term $-\eta\nabla u$ has a stabilizing effect, of the same nature as the stabilizing effect discussed at the end of Section 5.5, in relation with the monotony properties of the numerical fluxes. In (5.44), the additional term $-\eta\Delta u$ has a positive contribution in the energy estimate: if we multiply (5.44) by u (say, a smooth solution) and integrate over $\mathbb{T}^d \times (0,t)$, we obtain

$$\int_0^t \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d} u^2 dxds + \int_0^t \int_{\mathbb{T}^d} u\,\mathrm{div}(A(u))dxds - \eta\int_0^t \int_{\mathbb{T}^d} u\Delta u dxds = 0. \tag{5.45}$$

We develop the term

$$u\,\mathrm{div}(A(u)) = uA'(u)\cdot\nabla u = B'(u)\cdot\nabla u = \mathrm{div}(B(u)), \quad B'(u) := uA'(u),$$

and, using periodicity, we obtain

$$\frac{1}{2}\int_{\mathbb{T}^d} |u(x,t)|^2 dx + \eta\int_0^t\int_{\mathbb{T}^d} |\nabla u|^2 dxds \leq \frac{1}{2}\int_{\mathbb{T}^d} |u_0|^2 dx.$$

We will establish a similar result in the discrete case.

Proposition 5.4 (Energy estimate). *Let $u_0 \in L^\infty(\mathbb{T}^d)$ satisfy $\|u_0\|_{L^\infty(\mathbb{T}^d)} \le R$. Define*

$$\mathcal{D}(t_N) = \frac{1}{2}\sum_{n=0}^{N-1} \Delta t_n \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} |K|L| \int_{u_L^n}^{u_K^n} \{A_{K\to L}(u_K^n, u_L^n) - A_{K\to L}(z,z)\}dz.$$

Assume that the following CFL condition is satisfied: there exists $\xi \in]0,1]$ such that

$$2\Delta t_n \frac{|\partial K|}{|K|} L_A(R) \le 1-\xi, \quad \forall K \in \mathcal{T}, n \ge 1. \tag{5.46}$$

Then we have the energy estimate

$$\frac{1}{2}\|u_{h,\Delta t}(t_N)\|^2_{L^2(\mathbb{T}^d)} + \xi\mathcal{D}(t_N) \le \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{T}^d)}, \tag{5.47}$$

for all $N \ge 1$.

Remark 5.2. the term $\mathcal{D}(t_N)$ is non-negative. Indeed, using the monotony properties of $A_{K\to L}$, we have $A_{K\to L}(u_K^n, u_L^n) - A_{K\to L}(z,z) \ge 0$ if $u_L^n \le z \le u_K^n$. Similarly, $A_{K\to L}(u_K^n, u_L^n) - A_{K\to L}(z,z) \le 0$ if $u_K^n \le z \le u_L^n$.

Proof of Proposition 5.4. Note first that

$$\|u_{h,\Delta t}(0)\|_{L^\infty(\mathbb{T}^d)} \le \|u_0\|_{L^\infty(\mathbb{T}^d)} \le R.$$

By Proposition 5.3, we deduce that $|u_K^n| \le R$ for all $K \in \mathcal{T}$, $n \ge 1$. To start with the energy estimate, we multiply the identity (5.24) by $|K|u_K^n$ and we sum the result over $K \in \mathcal{T}^\sharp$ and $n \in \{0,\dots,N-1\}$. We obtain an identity $\mathrm{J}_{\Delta t} + \mathrm{J}_{\Delta x} = 0$, where

$$\mathrm{J}_{\Delta t} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}^\sharp} |K|u_K^n(u_K^{n+1} - u_K^n), \tag{5.48}$$

and

$$\mathrm{J}_{\Delta x} = \sum_{n=0}^{N-1} \Delta t_n \sum_{K\in\mathcal{T}^\sharp} \sum_{L\in\mathcal{N}(K)} u_K^n\{A_{K\to L}(u_K^n, u_L^n) - A_{K\to L}(u_K^n, u_K^n)\}. \tag{5.49}$$

We use the formula $a(b-a) = \frac{b^2-a^2}{2} - \frac{(a-b)^2}{2}$, which is the "finite difference" version of the the continuous identity $u\partial_t u = \frac{1}{2}\partial_t u^2$. It gives

$$\mathrm{J}_{\Delta t} = \frac{1}{2}\|u_{h,\Delta t}(t_N)\|^2_{L^2(\mathbb{T}^d)} - \frac{1}{2}\|u_{h,\Delta t}(0)\|^2_{L^2(\mathbb{T}^d)} - \frac{1}{2}\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}^\sharp} |K||u_K^{n+1} - u_K^n|^2. \tag{5.50}$$

We leave as an exercise the proof that (5.25) implies $\|u_{h,\Delta t}(0)\|_{L^2(\mathbb{T}^d)} \le \|u_0\|_{L^2(\mathbb{T}^d)}$. From (5.50), we deduce that

$$\frac{1}{2}\|u_{h,\Delta t}(t_N)\|^2_{L^2(\mathbb{T}^d)} + \mathrm{J}_{\Delta x} \le \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{2}\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}^\sharp} |K||u_K^{n+1} - u_K^n|^2 \tag{5.51}$$

The remaining term in the right-hand side of (5.51) will be absorbed in $\mathrm{J}_{\Delta x}$, by means of the CFL condition. The summation formula (5.40) and the conservation property (5.20) give the following expression of $\mathrm{J}_{\Delta x}$:

$$\mathrm{J}_{\Delta x} = \frac{1}{2}\sum_{n=0}^{N-1}\Delta t_n \sum_{K\in\mathcal{T}^\sharp}\sum_{L\in\mathcal{N}(K)} u_K^n\{A_{K\to L}(u_K^n,u_L^n) - A_{K\to L}(u_K^n,u_K^n)\} \\ - u_L^n\{A_{K\to L}(u_K^n,u_L^n) - A_{K\to L}(u_L^n,u_L^n)\}. \quad (5.52)$$

Denote by $\psi_{K\to L}$ an anti-derivative of $z\mapsto z\frac{d}{dz}A_{K\to L}(z,z)$. Integration by parts shows that

$$\psi_{K\to L}(v) - \psi_{K\to L}(w) = v\{A_{K\to L}(v,v) - A_{K\to L}(v,w)\} \\ - w\{A_{K\to L}(w,w) - A_{K\to L}(v,w)\} + \int_w^v \{A_{K\to L}(v,w) - A_{K\to L}(z,z)\}dz. \quad (5.53)$$

Taking $w = u_L^n$, $v = u_K^n$ in (5.53) shows that

$$\mathrm{J}_{\Delta x} = \frac{1}{2}\sum_{n=0}^{N-1}\Delta t_n\left\{\sum_{K\in\mathcal{T}^\sharp}\sum_{L\in\mathcal{N}(K)}\int_{u_L^n}^{u_K^n}\{A_{K\to L}(u_K^n,u_L^n) - A_{K\to L}(z,z)\}dz + \mathrm{R}_{\Delta x}^n\right\}, \quad (5.54)$$

where the remainder term is

$$\mathrm{R}_{\Delta x}^n = -\frac{1}{2}\sum_{K\in\mathcal{T}^\sharp}\sum_{L\in\mathcal{N}(K)}\psi_{K\to L}(u_K^n) - \psi_{K\to L}(u_L^n).$$

The cancellation property (5.23) and (5.40) give $\mathrm{R}_{\Delta x}^n = 0$. We conclude that $\mathrm{J}_{\Delta x} = \mathcal{D}(t_N)$. The estimate (5.47) will be established (as a consequence of (5.51)) if we can prove that

$$\frac{1}{2}\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}^\sharp}|K||u_K^{n+1} - u_K^n|^2 \le (1-\xi)\mathrm{J}_{\Delta x}. \quad (5.55)$$

We use Equation (5.24) and the Cauchy-Schwarz inequality to get

$$\left|u_K^{n+1} - u_K^n\right|^2 \le \frac{(\Delta t)^2|\partial K|}{|K|^2}\sum_{L\in\mathcal{N}(K)}|K|L||A_{K\to L}(u_K^n,u_L^n) - A_{K\to L}(u_K^n,u_K^n)|^2.$$

The CFL condition (5.46) gives then (5.55) with a term $\mathrm{J}_{\Delta x}^*$ instead of $\mathrm{J}_{\Delta x}$, where

$$\mathrm{J}_{\Delta x}^* = \frac{1}{4L_A(R)}\sum_{n=0}^{N}\Delta t_n\sum_{K\in\mathcal{T}^\sharp}\sum_{L\in\mathcal{N}(K)}|K|L||A_{K\to L}(u_K^n,u_L^n) - A_{K\to L}(u_K^n,u_K^n)|^2.$$

To conclude, we show that $\mathrm{J}_{\Delta x}^* \le \mathrm{J}_{\Delta x} = \mathcal{D}(t_N)$. To that purpose, we use the following inequality:

$$\int_0^r B(z)dz \ge \frac{1}{2\,\mathrm{Lip}(B)}B(r)^2, \quad r\in[0,R] \quad (5.56)$$

valid for any non-decreasing Lipschitz continuous function B on $[0,R]$. To obtain (5.56), we simply use the formula

$$B(r)^2 = 2\int_0^r B(s)B'(s)ds,$$

and bound $B'(s)$ by $\mathrm{Lip}(B)$. Suppose $u_K^n \geq u_L^n$ for instance. Then (5.56) applied to $B(z) := A_{K\to L}(u_K^n, u_L^n) - A_{K\to L}(u_K^n, z + u_L^n)$ and $r = u_K^n - u_L^n$ will give

$$|A_{K\to L}(u_K^n, u_L^n) - A_{K\to L}(u_K^n, u_K^n)|^2 \leq 2L_A(R) \int_{u_L^n}^{u_K^n} \{A_{K\to L}(u_K^n, u_L^n) - A_{K\to L}(u_K^n, z)\} dz. \quad (5.57)$$

We use the fact that $A_{K\to L}(u_K^n, z) \geq A_{K\to L}(z, z)$ since $u_K^n \geq z$ to get the desired identity. The reasoning in the case $u_K^n \leq u_L^n$ is similar. □

Remark 5.3 (Discrete H^1-estimate in the time variable). Note that (5.55) and (5.47) give the estimate

$$\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}^\sharp} |K| \left|u_K^{n+1} - u_K^n\right|^2 \leq (1-\xi)\mathcal{D}(t_N) \leq \frac{1-\xi}{\xi} \|u_0\|_{L^2(\mathbb{T}^d)}^2, \quad (5.58)$$

for all $N \geq 1$. Note also that the inequality $\mathrm{J}_{\Delta x}^* \leq \mathcal{D}(t_N)$ in the proof above and (5.47) give the estimate

$$\sum_{n=0}^{N} \Delta t \sum_{K\in\mathcal{T}^\sharp} \sum_{L\in\mathcal{N}(K)} |K|L||A_{K\to L}(u_K^n, u_L^n) - A_{K\to L}(u_K^n, u_K^n)|^2 \leq \frac{4L_A(R)}{\xi} \|u_0\|_{L^2(\mathbb{T}^d)}^2, \quad (5.59)$$

for all $N \geq 1$.

5.8 Approximate weak solutions

In this section, we will prove that $u_{(k)}$ obtained in Section 5.6.3 is an approximate weak solution of (5.9).

Definition 5.4 (Weak solution)**.** Let $u_0 \in L^\infty(\mathbb{T}^d)$, assume that $A\colon \mathbb{R} \to \mathbb{R}^d$ is a locally Lipschitz continuous function. Let $T > 0$. A function $u \in L^\infty(\mathbb{T}^d \times (0,T))$ is said to be a weak solution to (5.9) on $(0,T)$ with initial datum u_0 if

$$\int_0^T \int_{\mathbb{T}^d} (u\varphi_t + A(u)\cdot\nabla_x\varphi) dx dt + \int_{\mathbb{T}^d} u_0(x)\varphi(x,0) dx = 0, \quad (5.60)$$

for all test-function $\varphi \in C_c^\infty(\mathbb{T}^d \times [0,T))$.

Notation: if $u\colon \mathbb{T}^d \to \mathbb{R}$ and $1 \leq p < +\infty$, we denote by $\omega_{L^p}(u;h)$ the modulus of continuity in $L^p(\mathbb{T}^d)$:

$$\omega_{L^p}(u;h) = \sup_{|z|\leq h} \|u - u(\cdot + z)\|_{L^p(\mathbb{T}^d)}. \quad (5.61)$$

Theorem 5.5 (Approximate weak solutions)**.** *Let $u_0 \in L^\infty(\mathbb{T}^d)$ and let $R \geq \|u_0\|_{L^\infty(\mathbb{T}^d)}$. Assume that the CFL condition (5.46) is satisfied for all $K \in \mathcal{T}^\sharp$. Then $u_{h,\Delta t}$ is an approximate weak solution to (5.9) on $(0,T)$ with initial datum u_0 in the sense that*

$$\left| \int_0^T \int_{\mathbb{T}^D} (u_{h,\Delta t}\varphi_t + A(u_{h,\Delta t})\cdot\nabla_x\varphi) dx dt + \int_{\mathbb{T}^d} u_0(x)\varphi(x,0) dx \right| \leq \langle \mu_{h,\Delta t}^0, |\varphi| \rangle + \langle \mu_{h,\Delta t}^1, |\partial_t\varphi| \rangle + \langle \mu_{h,\Delta t}^2, |\nabla_x\varphi| \rangle, \quad (5.62)$$

for all test-function $\varphi \in C_c^\infty(\mathbb{T}^d \times [0,T))$*, where* $\mu^i_{h,\Delta t}$*,* $i \in \{0,1,2)\}$ *are some non-negative measures on* $\mathbb{T}^d \times [0,T]$ *which satisfy the estimate*

$$\mu^i_{h,\Delta t}(\mathbb{T}^d \times [0,T]) \leq C(\Delta t^{1/2} + h^{1/2} + \omega_{L^1}(u_0;h)), \tag{5.63}$$

where C *is a constant depending only on the dimension* d*, on* T*, on the constant* α *in* (5.29)*, on* R*, on* $L_A(R)$ *(cf.* (5.19)*), and on the constant* ξ *in* (5.46)*.*

Remark 5.4 (Entropy solutions). When A is non-linear, weak solutions to (5.60) are non unique. The Cauchy Problem for (5.9) is solved in the class of weak entropy solutions. A function $u \in L^\infty(\mathbb{T}^d \times (0,T))$ is said to be a weak entropy solution to (5.9) on $(0,T)$ with initial datum u_0 if

$$\int_0^T \int_{\mathbb{T}^d} (\eta(u)\varphi_t + \Phi(u)\cdot\nabla_x\varphi)dxdt + \int_{\mathbb{T}^d} \eta(u_0(x))\varphi(x,0)dx \geq 0, \tag{5.64}$$

for all non-negative test-function $\varphi \in C_c^\infty(\mathbb{T}^d \times [0,T))$ and all entropy, entropy-flux pair (η,Φ). This means that η is of class C^2, convex, Φ is locally lipschitz continuous, $\Phi'(u) = \eta'(u)A'(u)$ for a.e. $u \in \mathbb{R}$. Actually, it is sufficient to establish (5.60) for a family of generating entropy, entropy-flux pairs. One generally considers the Kruzhkov entropies $\eta(u) = |u-r|$, where the parameter r runs in $\mathbb{R}$. Such a η is not of class C^2, but the associated flux is well defined. We can also work with the semi Kruzhkov entropies $\eta^\pm(u) = (u-r)^\pm$. The associated fluxes are

$$\Phi^+(u;r) = A(u\vee r) - A(r), \quad \Phi^-(u;r) = A(u) - A(u\wedge r). \tag{5.65}$$

We can see on the expressions (5.36) and (5.65) (we take $v_K^n \equiv r$ in (5.36)) that we have already established a discrete version of (5.64):

$$(u_K^{n+1} - r)^+ \leq (u_K^n - r)^+ + \frac{\Delta t_n}{|K|}\sum_{L\in\mathcal{N}(K)} |K|L|[\Phi_{K\to L}(u_K^n, u_K^n; r) - \Phi_{K\to L}(u_K^n, u_L^n; r)], \tag{5.66}$$

where $\Phi_{K\to L}(v,w;r) = A_{K\to L}(v\vee r, w\vee r) - A_{K\to L}(r,r)$. If we start from (5.66) and adapt in a suitable way the proof of Theorem 5.5, we can establish that $u_{h,\Delta t}$ is an approximate weak entropy solution to (5.9) on $(0,T)$ with initial datum u_0 in the sense that

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} (\eta^\pm(u_{h,\Delta t};r)\varphi_t + \Phi^\pm(u_{h,\Delta t};r)\cdot\nabla_x\varphi)dxdt + \int_{\mathbb{T}^d} \eta^\pm(u_0(x);r)\varphi(x,0)dx \\ \geq -\langle\mu^0_{h,\Delta t}, |\varphi|\rangle - \langle\mu^1_{h,\Delta t}, |\partial_t\varphi|\rangle - \langle\mu^2_{h,\Delta t}, |\nabla_x\varphi|\rangle,\end{aligned} \tag{5.67}$$

for all non-negative test-function $\varphi \in C_c^\infty(\mathbb{T}^d \times [0,T))$ and for all $r \in \mathbb{R}$, where $\mu_{h,\Delta t}$ satisfies an estimate similar to (5.63). See [10].

Proof of Theorem 5.5. Let $\varphi \in C_c^\infty(\mathbb{T}^d \times [0,T))$. We first look at the error done at initial time. Define the error $\varepsilon_0(\varphi)$ by the formula

$$\varepsilon_0(\varphi) = \int_{\mathbb{T}^d} (u_0(x) - u_{h,\Delta t}(x,0))\varphi(x,0)dx. \tag{5.68}$$

By decomposition of the integral in (5.68), we have

$$\varepsilon_0(\varphi) = \sum_{K\in\mathcal{T}^\sharp} \int_K (u_0(x) - u_K^0)\varphi(x,0)dx.$$

For $x \in K$, $u_0(x) - u_{h,\Delta t}(x,0)$ is the average over K of $y \mapsto u_0(x) - u_0(y)$. Using Fubini's theorem, this gives the inequality $|\varepsilon_0(\varphi)| \leq \mu^0_{h,\Delta t}(|\varphi|)$, where

$$\mu^0_{h,\Delta t}(\psi) = \sum_{K\in\mathcal{T}^\sharp} \int_K \frac{1}{|K|} \int_K |u_0(x) - u_0(y)|\psi(x,0)dxdy$$

In particular, we have

$$|\mu^0_{h,\Delta t}(\mathbb{T}^d \times [0,T])| \leq \sum_{K\in\mathcal{T}^\sharp} \int_K \frac{1}{|K|} \int_K |u_0(x) - u_0(y)|dxdy.$$

This can be written

$$|\mu^0_{h,\Delta t}(\mathbb{T}^d \times [0,T])| \leq \sum_{K\in\mathcal{T}^\sharp} \frac{1}{|K|} \int_{x\in\mathbb{T}^d} \int_{y\in K} \mathbf{1}_K(x)|u_0(x) - u_0(y)|dxdy.$$

We do the change of variable $y = x + z$ to obtain

$$|\mu^0_{h,\Delta t}(\mathbb{T}^d \times [0,T])| \leq \sum_{K\in\mathcal{T}^\sharp} \frac{1}{|K|} \int_{x\in\mathbb{T}^d} \int_{z\in K-x} \mathbf{1}_K(x)|u_0(x) - u_0(x+z)|dxdz,$$

and thus

$$|\mu^0_{h,\Delta t}(\mathbb{T}^d \times [0,T])| \leq \sum_{K\in\mathcal{T}^\sharp} \frac{1}{|K|} \int_{x\in\mathbb{T}^d} \int_{z\in B(0,h)} \mathbf{1}_K(x)|u_0(x) - u_0(x+z)|dxdz,$$

since $K - x \subset B(0,h)$ if $x \in K$. We use the first bound of (5.29) and the fact that the sum over K of $\mathbf{1}_K(x)$ is 1 for a.e. x to get

$$|\mu^0_{h,\Delta t}(\mathbb{T}^d \times [0,T])| \leq \frac{1}{\alpha h^d} \int_{x\in\mathbb{T}^d} \int_{z\in B(0,h)} |u_0(x) - u_0(x+z)|dxdz,$$

We can exchange the integrals in x and z then to obtain

$$|\mu^0_{h,\Delta t}(\mathbb{T}^d \times [0,T])| \leq \frac{|B(0,h)|}{\alpha h^d} \sup_{|z|\leq h} \int_{x\in\mathbb{T}^d} |u_0(x) - u_0(x+z)|dxdz.$$

This gives the first estimate

$$|\mu^0_{h,\Delta t}(\mathbb{T}^d \times [0,T])| \leq \alpha^{-1}|B(0,1)|\omega(u_0;h). \tag{5.69}$$

Let us now study the term

$$\mathrm{I}_t = \int_0^T \int_{\mathbb{T}^d} u_{h,\Delta t}\varphi_t dxdt + \int_{\mathbb{T}^d} u_{h,\Delta t}(x,0)\varphi(x,0)dx.$$

Let $N \in \mathbb{N}$ be such that $t_{N-1} < T \leq t_N$. Since φ is compactly supported in $\mathbb{T}^d \times [0,T)$, we can assume that $T = t_N$. We expand I_t as

$$\mathrm{I}_t = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}^\sharp} |K|u^n_K(\varphi_K(t_{n+1}) - \varphi_K(t_n)) + \sum_{K\in\mathcal{T}^\sharp} |K|u^0_K\varphi_K(0),$$

where $\varphi_K(t)$ is the average value of $\varphi(\cdot,t)$ on the cell K. A discrete integration by parts gives

$$\mathrm{I}_t = -\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}^\sharp} |K|(u_K^{n+1}-u_K^n)\varphi_K(t_{n+1}). \tag{5.70}$$

We proceed similarly with the term

$$\mathrm{I}_x = \int_0^T \int_{\mathbb{T}^d} A(u_{h,\Delta t})\cdot\nabla_x\varphi dxdt.$$

We expand I_x as

$$\mathrm{I}_x = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}^\sharp}\int_{t_n}^{t_{n+1}}\int_K A(u_K^n)\cdot\nabla_x\varphi dxdt.$$

By the Stokes formula, this gives

$$\mathrm{I}_x = \sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}^\sharp}\sum_{L\in\mathcal{N}(K)} |K|L|A(u_K^n)\cdot n_{K\to L}\varphi_{K|L}^n, \tag{5.71}$$

where $\varphi_{K|L}^n$ is the average of the function φ on $K|L\times(t_n,t_{n+1})$. We use the consistency property (5.18) to write $A(u_K^n)\cdot n_{K\to L} = A_{K\to L}(u_K^n,u_K^n)$. We also add a corrective term to the sum in (5.71) to obtain

$$\mathrm{I}_x = \sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}^\sharp}\sum_{L\in\mathcal{N}(K)} |K|L|(A_{K\to L}(u_K^n,u_K^n)-A_{K\to L}(u_K^n,u_L^n))\varphi_{K|L}^n. \tag{5.72}$$

By the anti-symmetry property of the term $A_{K\to L}(u_K^n,u_L^n))\varphi_{K|L}^n$ (*cf.* (5.20)) and the summation formula (5.40), (5.71) and (5.72) coincide indeed. Let us now denote by φ_K^n the average value of the function φ over $K\times(t_n,t_{n+1})$. If we replace the quantities $\varphi_K(t_{n+1})$ in (5.70) and $\varphi_{K|L}^n$ in (5.72) by φ_K^n, then we obtain $\mathrm{I}_t+\mathrm{I}_x=0$. This follows from (5.24). Consequently, we have $\mathrm{I}_t+\mathrm{I}_x=\varepsilon^1(\varphi)+\varepsilon^2(\varphi)$, where

$$\varepsilon^1(\varphi) = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}^\sharp}|K|(u_K^{n+1}-u_K^n)(\varphi_K^n-\varphi_K(t_{n+1})),$$

and

$$\varepsilon^2(\varphi) = \sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}^\sharp}\sum_{L\in\mathcal{N}(K)}|K|L|(A_{K\to L}(u_K^n,u_K^n)-A_{K\to L}(u_K^n,u_L^n))(\varphi_{K|L}^n-\varphi_K^n).$$

To conclude to (5.62), we need to examine the error terms $\varepsilon^1(\varphi)$ and $\varepsilon^2(\varphi)$. Since

$$\int_{t_n}^{t_{n+1}}\{\varphi(t_{n+1})-\varphi(t)\}dt = \int_{t_n}^{t_{n+1}}\int_t^{t_{n+1}}\varphi_t(s)dsdt = \int_{t_n}^{t_{n+1}}(t_{n+1}-s)\varphi_t(s)ds,$$

we have

$$|\varphi_K(t_{n+1})-\varphi_K^n| \le \frac{1}{|K|}\int_{t_n}^{t_{n+1}}\int_K|\varphi_t(x,t)|\,dxdt.$$

This gives $|\varepsilon^1(\varphi)| \le \langle \mu^1_{h,\Delta t}, |\partial_t \varphi| \rangle$, where

$$\langle \psi, \mu^1_{h,\Delta t} \rangle = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\sharp} |u_K^n - u_K^{n+1}| \int_{t_n}^{t_{n+1}} \int_K \psi(x,t) dx dt. \tag{5.73}$$

In particular, the total mass of $\mu^1_{h,\Delta t}$ is

$$\mu^1_{h,\Delta t}(\mathbb{T}^d \times [0,T]) = \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\sharp} |K| |u_K^n - u_K^{n+1}|. \tag{5.74}$$

By the Cauchy-Schwarz inequality and (5.58), we have

$$\left[\mu^1_{h,\Delta t}(\mathbb{T}^d \times [0,T])\right]^2 \le T \frac{1-\xi}{\xi} \|u_0\|^2_{L^2(\mathbb{T}^d)} \Delta t. \tag{5.75}$$

Similarly, we develop

$$\varphi_{K|L} - \varphi_K = \frac{1}{|K|L||K|} \int_{K|L} \int_K |\varphi(x) - \varphi(y)| dx d\sigma(y)$$

and use the development $\varphi(x) - \varphi(y) = \int_0^1 \nabla \varphi(ry + (1-r)x) \cdot (x-y) dr$ to obtain $|\varepsilon^2(\varphi)| \le \langle \mu^2_{h,\Delta t}, |\nabla \varphi| \rangle$, where

$$\begin{aligned} \langle \psi, \mu^2_{h,\Delta t} \rangle := \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} & |A_{K \to L}(u_K^n, u_K^n) - A_{K \to L}(u_K^n, u_L^n)| \\ & \times \frac{1}{|K|} \int_{t_n}^{t_{n+1}} \int_{K|L} \int_K \int_0^1 \psi(ry + (1-r)x, t) |x-y| dr dx d\sigma(y) dt. \end{aligned} \tag{5.76}$$

We have $|x-y| \le h$ when $x \in K$, $y \in K|L$, so

$$\mu^2_{h,\Delta t}(\mathbb{T}^d \times [0,T]) \le h \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} |K|L| |A_{K \to L}(u_K^n, u_K^n) - A_{K \to L}(u_K^n, u_L^n)|. \tag{5.77}$$

We use the Cauchy-Schwarz inequality and the estimate (5.59) to get the bound

$$\left[\mu^2_{h,\Delta t}(\mathbb{T}^d \times [0,T])\right]^2 \le h^2 \Gamma L_A(R) \|u_0\|^2_{L^2(\mathbb{T}^d)}.$$

The factor Γ is

$$\Gamma = \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} |K|L|| = T \sum_{K \in \mathcal{T}^\sharp} |\partial K|.$$

By (5.29), we have the bound $\Gamma \le T \alpha^{-2} h^{-1}$, which shows that

$$\left[\mu^2_{h,\Delta t}(\mathbb{T}^d \times [0,T])\right]^2 \le T \alpha^{-2} L_A(R) \|u_0\|^2_{L^2(\mathbb{T}^d)} h. \tag{5.78}$$

We can bound the L^2-norm of u_0 by R in (5.75) and (5.78). This gives the desired estimate (5.63). □

5.9 Convergence in the linear case

We restrict now our analysis to the case of a linear flux A: $A(u) = au$. In this context, we consider a possibly non-constant vector field a. More precisely, we will assume that $a \in C^1(\mathbb{T}^d; \mathbb{R}^d)$ and that a is divergence free: $\mathrm{div}(a(x)) = 0$ for all $x \in \mathbb{T}^d$. We consider then the scheme (5.1) with a numerical flux given by (5.14)-(5.15), which is called the upwind, or upstream, flux. We have then (5.17) with some numerical flux functions

$$A_{K\to L}(v, w) = a^+_{K\to L} v - a^-_{K\to L} w \tag{5.79}$$

which satisfies all the properties listed in Section 5.4, with $L_A(R) = \|a\|_{L^\infty(\mathbb{T}^d)}$. We will admit that Theorem 5.5 remains valid, in the sense that we have

$$\left| \int_0^T \int_{\mathbb{T}^D} (u_{h,\Delta t}\varphi_t + u_{h,\Delta t} a(x)\cdot\nabla_x\varphi) dx dt + \int_{\mathbb{T}^d} u_0(x)\varphi(x,0)dx \right| \\ \leq \langle \mu^0_{h,\Delta t}, |\varphi|\rangle + \langle \mu^1_{h,\Delta t}, |\partial_t\varphi|\rangle + \langle \mu^2_{h,\Delta t}, |\nabla_x\varphi|\rangle, \tag{5.80}$$

and (5.63). In the asymptotic situation $\Delta t \to 0$, $h \to 0$ described in Section 5.6.3, we can pass to the limit in (5.80). This shows that u is a weak solution to (5.5) on $(0,T)$ with initial datum u_0, in the following sense (similar to Def. 5.4).

Definition 5.5 (Weak solution)**.** Let $u_0 \in L^\infty(\mathbb{T}^d)$. A function $u \in L^\infty(\mathbb{T}^d \times (0,T))$ is said to be a weak solution to (5.5) on $(0,T)$ with initial datum u_0 if

$$\int_0^T \int_{\mathbb{T}^d} u(\varphi_t + a(x)\cdot\nabla_x\varphi) dx dt + \int_{\mathbb{T}^d} u_0(x)\varphi(x,0)dx = 0, \tag{5.81}$$

for all test-function $\varphi \in C^\infty_c(\mathbb{T}^d \times [0,T))$.

We use then the following theorem.

Theorem 5.6. *Let $u_0 \in L^\infty(\mathbb{T}^d)$ and $T > 0$. The continuity equation (5.5) admits a unique weak solution in $L^\infty(\mathbb{T}^d \times (0,T))$ with initial datum u_0. It is given explicitly by $u(x,t) = u_0 \circ \Phi^t(x)$, where (Φ_t) is the flow associated to the ODE $\dot{x} = a(x)$ and Φ^t is the inverse[4] of $x \mapsto \Phi_t(x)$.*

Exercise 5.6 (Uniqueness in transport equations)**.** Prove Theorem 5.6 (beware, this is not obvious).
The solution to Exercise 5.6 is here.

5.10 Error estimate in the linear case

Our aim in this section and the following ones is to establish the following result.

Theorem 5.7. *Let $u_0 \in L^\infty \cap \mathrm{BV}(\mathbb{T}^d)$ and $T > 0$. Let $A(x,u) = a(x)u$, where $a \in C^1(\mathbb{T}^d; \mathbb{R}^d)$ is divergence-free. Let $u_{h,\Delta t}$ be the solution of the upwind Finite Volume method (5.1) with fluxes given by (5.17)-(5.79)-(5.15). Let $u \in L^\infty(\mathbb{T}^d \times (0,T))$ be the weak solution to (5.5) on $(0,T)$ with initial datum u_0. Assume that (5.29) and (5.46) are satisfied. Assume also that $\Delta t \leq C_0 h$*

[4] an expression of Φ^t is $\Phi^t(x) = \Phi_{-t}(x)$, this is a consequence of the group property $\Phi_t \circ \Phi_s = \Phi_{t+s}$; when the sense of the time evolution does matter, for instance in the study of stochastic differential equations, it is important to define Φ^t as the inverse of $x \mapsto \Phi_t(x)$, not as $\Phi_{-t}(x)$

for a certain constant C_0. Then, there is a constant $c(d) > 0$ depending on d only such that, for $h, \Delta t \leq c(d)$, we have the error estimate

$$\|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)} \leq C|Du_0|(\mathbb{T}^d)h^{1/2}, \tag{5.82}$$

for all $t \in [0,T]$, where C is a constant depending only on the dimension d, on T, on C_0, on $\|a\|_{C^1(\mathbb{T}^d)}$, on the constant α in (5.29) and on the constant ξ in (5.46).

We will make some comments on Theorem 5.7, but first we need a brief remainder on the space BV.

5.10.1 Functions of bounded variations

Let U be an open subset of $\mathbb{R}^d$. If $\varphi \in C(U;\mathbb{R}^d)$, we denote by $\|\varphi\|_{C(U)}$ the sup over $x \in U$ of the euclidean norm $|\varphi(x)|$ of $\varphi(x)$.

Definition 5.7 (Functions of bounded variation)**.** Let U be an open subset of $\mathbb{R}^d$. A function $u \in L^1(U)$ is said to have bounded variation in U if

$$\sup\left\{\int_U u \operatorname{div}\varphi dx\right\} < +\infty \tag{5.83}$$

where the supremum is taken over all $\varphi \in C_c^1(U;\mathbb{R}^d)$ such that $\|\varphi\|_{C(U)} \leq 1$. We denote by $\mathrm{BV}(U)$ the space of functions of bounded variations.

We denote by $\mathrm{BV}_{\mathrm{loc}}(U)$ the space of functions having locally bounded variations, defined as the set of functions $u \in L^1_{\mathrm{loc}}(U)$ such that $u \in \mathrm{BV}(V)$ for all open subset $V \subset\subset U$ (this last notation means that there exists a compact K of $\mathbb{R}^d$ such that $V \subset K \subset U$).

Exercise 5.8 (Some functions of bounded variation)**.**

1. Let $U = (-1,1)$. Let $u\colon U \to \mathbb{R}$ be defined as the integral over $[0,x]$ of a function $f \in L^1_{\mathrm{loc}}(U)$. Show that $u \in \mathrm{BV}_{\mathrm{loc}}(U)$ and that $u \in \mathrm{BV}(U)$ if, and only if, $f \in L^1(U)$.
2. Let $U = (-1,1)$. Let $u\colon U \to \mathbb{R}$ be the Heavyside function: $u(x) = 0$ if $x < 0$, $u(x) = 1$ if $x > 0$. Show that $u \in \mathrm{BV}(U)$.
3. Let $U = B(0,1)$ in $\mathbb{R}^2$. Let u be the characteristic function of the disk $B(0,1/2)$. Show that $u \in \mathrm{BV}(U)$.

The solution to Exercise 5.8 is here.

To enunciate the following structure theorem for functions of bounded variations, let us recall the following facts about measures.

1. (See [20, Chapter 6]). Let $(X,\mathcal{A})$ be a measure space. A complex measure over $(X,\mathcal{A})$ is a set function $\mu\colon \mathcal{A} \to \mathbb{C}$ such that, for all $A \in \mathcal{A}$, one has

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i), \tag{5.84}$$

for all countable partition $(A_i)_{i\geq 1}$ of A, the sum in (5.84) being absolutely convergent. If μ is a complex measure, the formula

$$|\mu|(A) = \sup\left\{\sum_{i=1}^{\infty} |\mu(A_i)|\right\}, \tag{5.85}$$

where the supremum is taken over all countable partitions $(A_i)_{i\geq 1}$ of A, defines a non-negative finite measure $|\mu|$ on $\mathcal{A}$ called the *total variation* of μ.

2. (See [20, p. 130]). A complex measure μ defined on the Borel subsets of a topological Hausdorff space X is said to be regular if for all Borel set A,

$$|\mu|(A) = \sup\{|\mu|(K); K \text{ compact} \subset A\} = \inf\{|\mu|(V); V \text{ open} \supset A\}. \tag{5.86}$$

Theorem 5.8 (Structure theorem for functions of bounded variations)**.** *Let U be an open set in $\mathbb{R}^d$. Let $u \in L^1(U)$. Then $u \in \mathrm{BV}(U)$ if, and only if, there exists a non-negative regular finite measure κ on U and a Borel map $n\colon U \to \mathbb{R}^d$ such that $|n(x)| = 1$ for κ-a.e. $x \in U$ and*

$$\int_U u \operatorname{div} \varphi dx = -\int_U \varphi \cdot n d\kappa, \tag{5.87}$$

for all $\varphi \in C_c^1(U;\mathbb{R}^d)$. The sup in (5.83) is then equal to $\kappa(U)$.

Proof of Theorem 5.8. In essential, the proof is an application of the theorem of representation of Riesz. We take as a reference Theorem 6.19 in [20]. In [20], the result is given for a functional of complex-valued functions. Since we need to consider a functional of vector valued functions, we will come back on the main steps of the proof of Theorem 6.19 in [20]. For simplicity, we will use the same notations as Rudin, except that vector-valued function are denoted using bold fonts. Consider the functional

$$\mathbf{\Phi}(\mathbf{f}) = -\int_U u \operatorname{div}(\mathbf{f}) dx$$

It is defined for $\mathbf{f} \in C_c^1(U;\mathbb{R}^d)$. By (5.83), it can be extended to a linear continuous functional (still denoted $\mathbf{\Phi}$) on $C_0(U;\mathbb{R}^d)$. We consider then the further extension to $C_0(U;\mathbb{C}^d)$ defined by $\mathbf{\Phi}(\mathbf{f}) := \mathbf{\Phi}(\mathbf{f}_1) + i\mathbf{\Phi}(\mathbf{f}_2)$, where $\mathbf{f}_1$ is the real part of $\mathbf{f}$ and $\mathbf{f}_2$ the imaginary part of $\mathbf{f}$. Our aim is to prove that there exists a non-negative regular finite measure λ on U and a Borel map $\mathbf{g}\colon U \to \mathbb{R}^d$ such that $|\mathbf{g}(x)| = 1$ for λ-a.e. $x \in U$ and

$$\mathbf{\Phi}(\mathbf{f}) = \int_U \mathbf{f} \cdot \mathbf{g} d\lambda, \tag{5.88}$$

for all $\mathbf{f} \in C_0(U;\mathbb{R}^d)$, where $(\mathbf{f}\cdot\mathbf{g})(x) = \sum_{i=1}^d \mathbf{f}_i(x)\mathbf{g}_i(x)$. Let us focus on the "factor" λ in (5.88). If (5.88) is satisfied, then the functional

$$\varphi \mapsto \int_U \varphi d\lambda,$$

defined for $\varphi \in C_c(U;\mathbb{R})$, dominates $\mathbf{\Phi}$ in the sense that, if $\varphi \geq 0$, then

$$\sup\left\{|\mathbf{\Phi}(\mathbf{h})|; \mathbf{h} \in C_c(U;\mathbb{C}^d), |\mathbf{h}| \leq \varphi\right\} \leq \int_U \varphi d\lambda. \tag{5.89}$$

For $\varphi \in C_c(U;\mathbb{R}_+)$, we *define* $\Lambda(\varphi)$ as the left-hand side of (5.89):

$$\Lambda(f) = \sup\left\{|\mathbf{\Phi}(\mathbf{h})|; \mathbf{h} \in C_c(U;\mathbb{C}^d), |\mathbf{h}| \leq \varphi\right\}. \tag{5.90}$$

This can be seen as a functional analogue to (5.85). Our aim is to show that we have the representation

$$\Lambda(\varphi) = \int_U \varphi d\lambda. \tag{5.91}$$

For a general $\varphi \in C_c(U;\mathbb{R})$, we set $\Lambda(\varphi) = \Lambda(\varphi^+) - \Lambda(\varphi^-)$. It is easy to see that this defines a continuous functional on $C_c(U;\mathbb{R})$ which is positive. We will show that Λ is actually a *linear* functional (see below). By the representation theorem of Riesz, [20, Theorem 2.14], there exists a non-negative regular finite measure λ on U such that (5.91) is satisfied for all $\varphi \in C_c(U;\mathbb{R})$. Next, we use the following representation result.

Proposition 5.9. *Every continuous linear form $\mathbf{\Psi}$ on $E := L^1(U, \lambda; \mathbb{C}^d)$ admits a representation*

$$\mathbf{\Psi}(\mathbf{f}) = \int_U \mathbf{f} \cdot \mathbf{g} d\lambda, \quad \mathbf{g} \in L^\infty(U; \mathbb{C}^d). \tag{5.92}$$

We have $\|\mathbf{\Psi}\|_{E'} = \|\mathbf{g}\|_{L^\infty(U)}$ in this correspondence, where $\|\mathbf{g}\|_{L^\infty(U)}$ is the the essential supremum over $x \in U$ of the euclidean norm $|\mathbf{g}(x)|$ of $\mathbf{g}(x)$ and $\|\mathbf{\Psi}\|_{E'}$ is the norm of the linear form $\mathbf{\Psi}$.

Proposition 5.9 is an extension of [20, Theorem 6.16] to the vector valued case. We admit this result, which can be proved by a systematic examination of the proof of [20, Theorem 6.16]. Let us apply Proposition 5.9 to $\mathbf{\Phi}$. The functional $\mathbf{\Phi}$ satisfies the hypotheses of the proposition: (5.89) with $\varphi = |\mathbf{h}|$ shows that

$$|\mathbf{\Phi}(h)| \leq \int_U |\mathbf{h}| d\lambda, \tag{5.93}$$

for all $\mathbf{h} \in C_c(U; \mathbb{C}^d)$. Since $C_c(U; \mathbb{C}^d)$ is dense, we can extend $\mathbf{\Phi}$ as a continuous linear form on E with norm $\|\mathbf{\Phi}\|_{E'} \leq 1$. This gives us the representation (5.88) with $\|\mathbf{g}\|_{L^\infty(U)} \leq 1$. To conclude, there remains to show that $|\mathbf{g}(x)| = 1$ for λ-a.e. $x \in U$. By (5.88) and the Cauchy-Schwarz inequality $|\mathbf{f} \cdot \mathbf{g}| \leq |\mathbf{f}||\mathbf{g}|$, we have

$$|\mathbf{\Phi}(\mathbf{f})| \leq \int_U |\mathbf{g}| d\lambda, \quad \mathbf{f} \in C_c(U; \mathbb{C}^d), \quad |\mathbf{f}(x)| \leq 1.$$

Taking the sup over $\mathbf{f}$ in the previous inequality gives $\Lambda(1) \leq \int_U |\mathbf{g}| d\lambda$. Since $\Lambda(1) = \lambda(U)$, $|\mathbf{g}|$ is equal to 1 λ-a.e. To finish the proof, let us show that the map Λ defined by (5.90) is linear. Let $f, g \in C_c(U; \mathbb{R}_+)$, $\varepsilon > 0$ and $\mathbf{h}_1, \mathbf{h}_2 \in C_c(U; \mathbb{C}^d)$ such that

$$\Lambda(f) \leq |\mathbf{\Phi}(\mathbf{h}_1)| + \varepsilon, \quad \Lambda(g) \leq |\mathbf{\Phi}(\mathbf{h}_2)| + \varepsilon.$$

There are some complex numbers $\alpha_1, \alpha_2 \in \mathbb{C}$ of modulus 1 such that $|\mathbf{\Phi}(\mathbf{h}_i)| = \alpha_i \mathbf{\Phi}(\mathbf{h}_i)$. Then the sum $\Lambda(f) + \Lambda(g)$ is bounded by

$$\alpha_1 \mathbf{\Phi}(\mathbf{h}_1) + \alpha_2 \mathbf{\Phi}(\mathbf{h}_2) + 2\varepsilon = \mathbf{\Phi}(\alpha_1 \mathbf{h}_1 + \alpha_2 \mathbf{h}_2) + 2\varepsilon \leq \Lambda(f + g) + 2\varepsilon,$$

which shows that $\Lambda(f) + \Lambda(g) \leq \Lambda(f + g)$. To prove the converse inequality, consider $\mathbf{h} \in C_c(U; \mathbb{C}^d)$ satisfying the constraint $|\mathbf{h}| \leq f + g$ and set $V = \{f + g > 0\}$ and

$$\mathbf{h}_1 = \frac{f}{f + g} \mathbf{1}_V \mathbf{h}, \quad \mathbf{h}_2 = \frac{g}{f + g} \mathbf{1}_V \mathbf{h}.$$

Then $\mathbf{h}_1, \mathbf{h}_2 \in C_c(U; \mathbb{C}^d)$ (why?), $\mathbf{h}_1 + \mathbf{h}_2 = \mathbf{h}$, $|\mathbf{h}_1| \leq f$, $|\mathbf{h}_2| \leq g$, which shows that $|\mathbf{\Phi}(\mathbf{h})| \leq \Lambda(f) + \Lambda(g)$. Taking the sup over $\mathbf{h}$ gives the desired result. □

Notation: if $u \in \mathrm{BV}(U)$, we denote by Du the (vector-valued) complex measure $n\kappa$ in (5.87) and by $|Du|$ the measure κ. The norm $\|u\|_{\mathrm{BV}(U)}$ of u is defined as

$$\|u\|_{\mathrm{BV}(U)} = \|u\|_{L^1(U)} + |Du|(U). \tag{5.94}$$

Exercise 5.9 (Some functions of bounded variation)**.** Compute Du, $|Du|$ and $\|u\|_{\mathrm{BV}(U)}$ for the functions u considered in the exercise 5.8.
The solution to Exercise 5.9 is here.

Definition 5.10 (Set of finite perimeter)**.** 1. A Lebesgue measurable set E of $\mathbb{R}^d$ is said to have finite perimeter in U if $\mathbf{1}_E \in \mathrm{BV}(\mathbb{R}^d)$. In that case, we set $P(E) = |D\mathbf{1}_E|(\mathbb{R}^d)$.

2. Let U be an open subset of $\mathbb{R}^d$. A Lebesgue measurable set E of $\mathbb{R}^d$ is said to have finite perimeter in U if $\mathbf{1}_E \in \mathrm{BV}(U)$. In that case, we set $P(E;U) = |D\mathbf{1}_E|(U)$.

We now state without proof the following results.

Theorem 5.10 (Lower semicontinuity of the total variation)**.** *Let U be an open set in $\mathbb{R}^d$. Let (u_n) be a sequence of functions of $\mathrm{BV}(U)$ which converges in $L^1_{\mathrm{loc}}(U)$ to a function u. Then*

$$|Du|(U) \leq \liminf_{n\to+\infty} |Du_n|(U). \tag{5.95}$$

Theorem 5.11 (Local approximation by smooth functions)**.** *Let U be an open set in $\mathbb{R}^d$. Let $u \in \mathrm{BV}(U)$. There exists a sequence of functions u_k in $\mathrm{BV}(U) \cap C^\infty(U)$ such that*

1. *$u_k \to u$ in $L^1(U)$, and*
2. *$|Du_k|(U) \to |Du|(U)$.*

Remark 5.5. Note that if $u \in \mathrm{BV}(U) \cap C^\infty(U)$ then $u \in W^{1,1}(U)$ and

$$|Du|(U) = \int_U |\nabla u(x)| dx. \tag{5.96}$$

Theorem 5.12 (Trace of functions of bounded variations)**.** *Let U be an open bounded set in $\mathbb{R}^d$, with ∂U Lipschitz continuous. Let σ denote the surface measure on ∂U and n the outward unit normal to U on ∂U. There exists a bounded linear application*

$$\gamma \colon \mathrm{BV}(U) \to L^1(\partial U, \sigma),$$

such that

$$\int_U u \operatorname{div} \varphi dx = -\int_U \varphi \cdot dDu + \int_{\partial U} (\gamma u)\varphi \cdot n d\sigma, \tag{5.97}$$

for all $\varphi \in C^1(\mathbb{R}^d;\mathbb{R}^d)$.

Theorem 5.13 (Patch of functions of bounded variations)**.** *Let U be an open bounded set in $\mathbb{R}^d$, with ∂U Lipschitz continuous. Let σ denote the surface measure on ∂U and n the outward unit normal to U on ∂U. Let $v \in \mathrm{BV}(U)$, $w \in \mathrm{BV}(\mathbb{R}^d \setminus \bar{U})$ and let $u \in L^1(\mathbb{R}^d)$ be the function defined as $u = v\mathbf{1}_U + w\mathbf{1}_{\mathbb{R}^d\setminus\bar{U}}$. Then $u \in \mathrm{BV}(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d} \varphi \cdot dDu = \int_U \varphi \cdot dDv + \int_{\mathbb{R}^d\setminus\bar{U}} \varphi \cdot dDw + \int_{\partial U} (\gamma v - \gamma w)\varphi \cdot n d\sigma, \tag{5.98}$$

for all $\varphi \in C_c(\mathbb{R}^d;\mathbb{R}^d)$. The BV *norm of u is*

$$\|u\|_{\mathrm{BV}(\mathbb{R}^d)} = \|v\|_{\mathrm{BV}(U)} + \|w\|_{\mathrm{BV}(\mathbb{R}^d\setminus\bar{U})} + \int_{\partial U} |\gamma v - \gamma w| d\sigma. \tag{5.99}$$

Theorem 5.14 (Co-area formula for functions of bounded variations)**.** *Let U be an open set in $\mathbb{R}^d$. Let $u \in L^1(U)$ be a non-negative function. For $t \in \mathbb{R}$, we denote by E_t the super-level set $\{u > t\}$. Then, for a.e. $t \in \mathbb{R}$, E_t has finite perimeter in U, and we have*

$$\|u\|_{L^1(U)} = \int_0^\infty \|\mathbf{1}_{E_t}\|_{L^1(U)} dt, \quad |Du|(U) = \int_0^\infty |D\mathbf{1}_{E_t}|(U) dt. \tag{5.100}$$

See [9, Theorem 5.2] for the proof of Theorem 5.10, [9, Theorem 5.3] for the proof of Theorem 5.11, [9, Theorem 5.6] for the proof of Theorem 5.12, [9, Theorem 5.8] for the proof of Theorem 5.13 and [9, Theorem 5.9] for the proof of Theorem 5.14. To complete this section, let us give the definition of the norm $\|u\|_{\mathrm{BV}(\mathbb{T}^d)}$ of a $\mathbb{Z}^d$-periodic function $u\colon \mathbb{R}^d \to \mathbb{R}$. First, by $\mathrm{BV}(\mathbb{T}^d)$ we denote the set of $\mathbb{Z}^d$-periodic functions $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$. The measure $|Du|$ is then a regular measure on $\mathbb{R}^d$, with $|Du|(G)$ finite for every compact $G \subset \mathbb{R}^d$. Let Q denote the unit cube $(0,1)^d$ and let $Q_1 = [0,1)^d$. We define then

$$|Du|(\mathbb{T}^d) = |Du|(Q_1), \quad \|u\|_{\mathrm{BV}(\mathbb{T}^d)} = \|u\|_{L^1(Q)} + |Du|(\mathbb{T}^d). \tag{5.101}$$

We take the measure $|Du|$ of Q_1, not Q, in (5.101). This makes a difference if $|Du|$ has a singular part with respect to the Lebesgue measure. This singular part may be due to some jumps of u, which is the case if we consider piecewise constant functions. Let $u \in L^1(\mathbb{T}^d)$, denote by u_h the piecewise constant function defined by

$$u_h(x) = u_K := \frac{1}{|K|}\int_K u(x)dx, \quad x \in K. \tag{5.102}$$

Then

$$Du_h(A) = \frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}(K)}(u_K - u_L)n_{K\to L}\mathcal{H}^{d-1}(A\cap K|L), \tag{5.103}$$

and we have

$$|Du_h|(\mathbb{T}^d) = \frac{1}{2}\sum_{K\in\mathcal{T}^\sharp}\sum_{L\in\mathcal{N}(K)}|K|L||u_K - u_L|. \tag{5.104}$$

Since $K|L$ is included in an hyperplane H, by hypothesis, the Hausdorff measure $\mathcal{H}^{d-1}(A\cap K|L)$ in (5.103) can simply be rewritten $\lambda_H(A\cap K|L)$, where λ_H is the $(d-1)$-dimensional Lebesgue measure on H.

5.10.2 Comments on the error estimate

If $1 < p < +\infty$, one can establish the error estimate

$$\|u_{h,\Delta t}(t) - u(t)\|_{L^p(\mathbb{T}^d)} \le C\|u_0\|_{W^{1,p}(\mathbb{T}^d)}h^{1/2}. \tag{5.105}$$

See [17]. The estimate (5.105) cannot be generalized when the flux in the conservation law (5.9) is non-linear, for the reason that $W^{1,p}(\mathbb{T}^d)$ is not stable in the evolution: if $u_0 \in W^{1,p}(\mathbb{T}^d)$, there may be some time $t > 0$ such that the (entropy) solution u of (5.9) starting from u_0 loses the $W^{1,p}(\mathbb{T}^d)$ regularity at time t. This is a consequence of the apparition of discontinuities and is already clear in dimension $d = 1$. On the contrary, the space $\mathrm{BV}(\mathbb{T}^d)$ is stable in the evolution by (5.9). For general fluxes A, the error estimate (5.82) is observed in numerical practice, but has not been established yet, except when the mesh is a cartesian mesh, *i.e.* each cell is a product of one-dimensional cells of a one-dimensional mesh.

5.11 Error estimate in the linear case: proof

The following proof of the error estimate (5.82) is taken from [18]. A different proof, using probabilistic tools, has been given in [5].

5.11.1 Reduction of the problem

Projection on piecewise constant functions and BV-norm. We will use several times the following result.

Proposition 5.15. *Consider the map $u \mapsto u_h$ defined by (5.102). There exists a constant $C \geq 0$ only depending on d and on the constant α in (5.29) such that, if $u \in \mathrm{BV}(\mathbb{T}^d)$, then*

$$|Du_h|(\mathbb{T}^d) \leq C|Du|(\mathbb{T}^d) \text{ and } \|u_h - u\|_{L^1(\mathbb{T}^d)} \leq C|Du|(\mathbb{T}^d)h. \tag{5.106}$$

Proof of Proposition 5.15. Let $K \in \mathcal{T}^\sharp$ and let $L \in \mathcal{N}(K)$. We will establish first the estimate

$$|u_K - u_L| \leq \frac{2^{d+1}\max(|K|,|L|)h}{|K|\,|L|}|Du|(B(x_K,2h)), \tag{5.107}$$

where $x_K := \frac{1}{|K|}\int_K x dx$ is the center of gravity of K. Note that $K, L \subset B(x_K, 2h)$. Using Theorem 5.10 and Theorem 5.11 with $U = B(x_K, 2h)$, we may suppose that $u \in \mathrm{BV} \cap C^1(B(x_K,2h))$, in which case (*cf.* (5.96))

$$|Du|(B(x_K,2h)) = \int_{B(x_K,2h)} |\nabla u(z)|dz.$$

Since $|x-y| \leq 2h$ for every $(x,y) \in K \times L$, we have then

$$\begin{aligned}|u_K - u_L| &\leq \frac{1}{|K|\,|L|}\int_K\int_L |u(x)-u(y)|dxdy \\ &\leq \frac{2h}{|K|\,|L|}\int_K\int_L\int_0^1 |\nabla u((1-r)x+ry)|drdxdy.\end{aligned}$$

Now we perform the change of variables $(x,y,r) \mapsto (w = x-y, z = (1-r)x+ry, r = r)$, of Jacobian determinant equal to 1, and of inverse $(w,z,r) \mapsto (z+rw, z-(1-r)w, r)$. This gives

$$|u_K - u_L| \leq \frac{2h}{|K|\,|L|}\int_{B(x_K,2h)} |\nabla u(z)| \left(\int_0^1\int_{\mathbb{R}^d} g(w,z,r)dwdr\right)dz,$$

where g is defined by $g(w,z,r) = 1$ if $z+rw \in K$ and $z-(1-r)w \in L$, and $g(w,z,r) = 0$ otherwise. We remark that, for $(z,r) \in B(x_K,2h)\times[0,1]$, we have

$$\int_{\mathbb{R}^d} g(w,z,r)dw \leq |r^{-1}(K-z)| \leq 2^d|K|,$$

if $r \geq 1/2$ and

$$\int_{\mathbb{R}^d} g(w,z,r)dw \leq 2^d|L|,$$

if $r < 1/2$. The estimate (5.107) follows. Using (5.29), we deduce from (5.107) that, for all $K \in \mathcal{T}^\sharp$,

$$\sum_{L\in\mathcal{N}(K)} |K|L||u_K - u_L| \leq 2^{d+1}\alpha^{-2}|Du|(B(x_K,2h)).$$

Summing on $K \in \mathcal{T}^\sharp$, we get

$$\sum_{K\in\mathcal{T}^\sharp}\sum_{L\in\mathcal{N}(K)} |K|L||u_K - u_L| \leq 2^{d+1}\alpha^{-2}\int_{\mathbb{R}^d}\sum_{K\in\mathcal{T}^\sharp}\mathbf{1}_{B(x_K,2h)}(z)d|Du|(z).$$

Let us set $\chi(z) = \sum_{K\in\mathcal{T}^\sharp} \mathbf{1}_{B(x_K,2h)}(z)$. We have $d(x_K, K) \le h$, so $\chi(z) = 0$ if z is at a distance superior to $3h$ of Q. We may assume that $3h < 1$, and then $\chi(z) = 0$ if $z \notin Q'$, where Q' is the cube $[-1, 2)^d$ - which is contained in 3^d translates of Q_1. By (5.29), we also have

$$\alpha h^d \chi(z) \le \sum_{K\in\mathcal{T}^\sharp} \mathbf{1}_{B(x_K,2h)}(z)|K| \le \sum_{K:d(z,K)<3h} |K|.$$

Indeed, $|z - x_K| < 2h$ implies $d(z, K) < 3h$. Since the cells in $\mathcal{T}^\sharp$ are disjoint, we have

$$\sum_{K:d(z,K)<3h} |K| = \left| \bigcup_{K:d(z,K)<3h} K \right| \le |B(z, 4h)| = 4^d |B(0, 1)|h^d.$$

It follows that $\chi \le 4^d|B(0, 1)|\alpha^{-1}\mathbf{1}_{Q'}$, which gives us

$$\frac{1}{2} \sum_{K\in\mathcal{T}^\sharp} \sum_{L\in\mathcal{N}(K)} |K|L||u_K - u_L| \le 8^d\alpha^{-3}|B(0, 1)||Du|(Q') \le 24^d\alpha^{-3}|B(0, 1)||Du|(\mathbb{T}^d).$$

Let $K \in \mathcal{T}$. Similarly, we have

$$\int_K |u_h(x) - u(x)|dx \le \frac{1}{|K|} \int_{K\times K} |u(x) - u(y)|dxdy \le 2^d h|Du|(B(x_K, h)).$$

Summing on $K \in \mathcal{T}$ and using the fact that the cardinal of the set $\{K : d(K, z) \le h\}$ is bounded by $C\alpha^{-1}$, we get the second estimate of (5.106). □

Exercise 5.11 (Modulus of continuity of functions of bounded variation)**.** Show that

$$\omega_{L^1}(u; h) \le C|Du|(\mathbb{T}^d)h, \tag{5.108}$$

for all $u \in \mathrm{BV}(\mathbb{T}^d)$, for all $0 \le h \le 1$, where $\omega_{L^1}(u; h)$ is the modulus of continuity defined by (5.61) and where C is a constant depending on the dimension d only.
The solution to Exercise 5.11 is here.

Contraction in L^1. We will also need the following proposition.

Proposition 5.16 (L^p-conservation)**.** *Let $u, v \in L^\infty(\mathbb{T}^d \times (0, T))$ be some weak solutions to (5.5) on $(0, T)$ with respective initial data u_0, $v_0 \in L^\infty(\mathbb{T}^d)$. Assume that a is divergence free. Then, for every $p \in [1, +\infty]$, we have*

$$\|u(t) - v(t)\|_{L^p(\mathbb{T}^d)} = \|u_0 - v_0\|_{L^p(\mathbb{T}^d)}, \tag{5.109}$$

for all $t \in (0, T)$.

Proof of Proposition 5.16. We use Theorem 5.6. By linearity, we can assume $v \equiv 0$. We have $u(x, t) = u_0 \circ \Phi^t(x)$. This gives (5.109) since Φ^t is a bijection of $\mathbb{T}^d$ (case $p = +\infty$) and preserves the measure (case $p \in [1, +\infty)$), since a is divergence free. □

A trivial consequence of (5.109) is that

$$\|u(t) - v(t)\|_{L^1(\mathbb{T}^d)} \le \|u_0 - v_0\|_{L^1(\mathbb{T}^d)}. \tag{5.110}$$

We will also use Proposition 5.2, which gives (with obvious notations)

$$\|u_{h,\Delta t}(t) - v_{h,\Delta t}(t)\|_{L^1(\mathbb{T}^d)} \le \|u_0 - v_0\|_{L^1(\mathbb{T}^d)}, \tag{5.111}$$

for all $t \ge 0$.

Reduction 1. Discrete time. Let $t \in [0,T]$. There is a unique $n \geq 0$ such that $t_n \leq t < t_{n-1}$. We have then $u_{h,\Delta t}(t) = u_{h,\Delta t}(t_n)$ and

$$\|u(t) - u(t_n)\|_{L^1(\mathbb{T}^d)} = \|u_0 \circ \Phi^t - u_0 \circ \Phi^{t_n}\|_{L^1(\mathbb{T}^d)}$$

Since a is divergence free, Φ^t and ϕ_t preserve the Lebesgue measure, so

$$\|u(t) - u(t_n)\|_{L^1(\mathbb{T}^d)} = \|u_0 \circ \Phi^t \circ \Phi_{t_n} - u_0\|_{L^1(\mathbb{T}^d)} \leq C|Du_0|(\mathbb{T}^d)\|\Phi^t \circ \Phi_{t_n} - \mathrm{Id}\|_{C(\mathbb{T}^d)}$$

by (5.108). Here,

$$\|\Phi^t \circ \Phi_{t_n} - \mathrm{Id}\|_{C(\mathbb{T}^d)} = \sup_{x \in \bar{Q}} |\Phi^t \circ \Phi_{t_n}(x) - x|.$$

The group property of the flow gives

$$\Phi^t \circ \Phi_{t_n}(x) - x = \Phi_{t_n - t}(x) - x = \int_{t_n - t}^0 a(\Phi_s(x))ds,$$

so

$$\|u(t) - u(t_n)\|_{L^1(\mathbb{T}^d)} \leq C|Du_0|(\mathbb{T}^d)\Delta t \leq C|Du_0|(\mathbb{T}^d)h,$$

where C depends on d, C_0 and $\|a\|_{L^\infty(\mathbb{T}^d)}$. This shows that it is sufficient to establish (5.82) for a time t in the discrete grid $\{t_n; n \geq 0\}$. We proceed to this reduction to extend the analysis done in the proof of Theorem 5.5. Indeed, in the proof of Theorem 5.5, it was assumed that the test function φ was compactly supported in $\mathbb{T}^d \times [0,T)$. It is easy however to extend our proof to the case where $T = t_N$ and $\varphi \in C^1(\mathbb{T}^d \times [0,T])$. We have then an additional term for $t = T$ to take into account, and (5.62) will be replaced by the inequality

$$\left| \int_0^T \int_{\mathbb{T}^d} u_{h,\Delta t}(\varphi_t + a \cdot \nabla_x \varphi) dx dt + \int_{\mathbb{T}^d} u_0(x)\varphi(x,0)dx - \int_{\mathbb{T}^d} u_{h,\Delta t}(x,T)\varphi(x,T)dx \right| \leq \langle \mu^0_{h,\Delta t}, |\varphi| \rangle + \langle \mu^1_{h,\Delta t}, |\partial_t \varphi| \rangle + \langle \mu^2_{h,\Delta t}, |\nabla_x \varphi| \rangle. \quad (5.112)$$

Reduction 2. Non-negative functions. Since constants are solutions to (5.1) and (5.5) and since the addition of a constant to a function $u \in \mathrm{BV}(\mathbb{T}^d)$ does not modify the quantity $|Du|(\mathbb{T}^d)$, we may replace u_0 by $u_0 + \|u_0\|_{L^\infty(\mathbb{T}^d)}$, which allows us to work with non-negative functions only. This reduction step is not fundamental actually. The co-area formula for BV function, Theorem 5.14, has been stated for non-negative functions for simplicity; this is what accounts for the present reduction step.

Reduction 3. Projection on a cartesian grid. Let $L \in \mathbb{N}$, $L \sim h^{-1/2}$, for example $L = [h^{-1/2}]$. Let v_0 be the L^2-projection (see (5.102)) of u_0 on the functions which are piecewise constant with respect to the periodic mesh $\mathcal{T}_0 = L^{-1}(Q + \mathbb{Z}^d)$. This mesh satisfies (5.29) with $h_0 = L^{-1}$ and $\alpha = (2d)^{-1}$ since

$$|K_0| = h_0^d, \quad |\partial K_0| = 2dh_0^{d-1},$$

for all $K_0 \in \mathcal{T}_0^\sharp$. By Proposition 5.15, we have

$$|Dv_0|(\mathbb{T}^d) \leq C|Du_0|(\mathbb{T}^d), \quad \|u_0 - v_0\|_{L^1(\mathbb{T}^d)} \leq C|Du_0|(\mathbb{T}^d)h^{1/2},$$

where C depends on the dimension d only. In view of (5.110)-(5.111), of the second estimate in (5.106) and of (5.108), we can replace u_0 by v_0 to establish (5.82). Consequently, we may

assume without loss of generality that u_0 is piecewise constant with respect to $\mathcal{T}_0$. We use this first reduction step for the following reason: let $t > 0$ and let $A = \{u_0 > t\}$ be a super-level set of u_0. Then A is an union of given cells of $\mathcal{T}_0$. Let $\langle \partial A \rangle_{L^{-1}}$ denote the L^{-1} neighbourhood of ∂A:

$$\langle \partial A \rangle_{L^{-1}} = \{x \in \mathbb{R}^d; d(x, \partial A) < L^{-1}\}.$$

We want to prove that the volume $|\langle \partial A \rangle_{L^{-1}}|$ of $\langle \partial A \rangle_{L^{-1}}$ satisfies the estimate

$$|\langle \partial A \rangle_{L^{-1}}| \leq C|\partial A|h^{1/2}, \tag{5.113}$$

where the constant C depends on d only. To establish (5.113), denote by $\mathcal{T}_0[\partial A]$ the set of cells $K \subset A$ such that $\bar{K} \cap \partial A$ is non-empty. We have

$$\langle \partial A \rangle_{L^{-1}} \subset \bigcup_{K \in \mathcal{T}_0[\partial A]} \langle \partial K \rangle_{L^{-1}}.$$

Since $\langle \partial K \rangle_{L^{-1}}$ is included in the closure of the union of K and its $2d$ neighbouring cells, we obtain

$$|\langle \partial A \rangle_{L^{-1}}| \leq \sum_{K \in \mathcal{T}_0[\partial A]} (1 + 2d)L^{-d}. \tag{5.114}$$

On the other hand, each $K \in \mathcal{T}_0[\partial A]$ has at least one face of size $L^{-(d-1)}$ contributing to $|\partial A|$, so

$$|\partial A| \geq \sum_{K \in \mathcal{T}_0[\partial A]} L^{-(d-1)}. \tag{5.115}$$

The two estimates (5.114) and (5.115) give (5.113).

Reduction 4. Co-area formula. We apply Theorem 5.14. The equations we consider are linear: they satisfy a superposition principle. By (5.100), we may replace u_0 by the characteristic function of a super-level set A with finite perimeter. The advantage of this manipulation is the following one. Since $0 \leq u_{h,\Delta t} \leq 1$ by the comparison principle (Proposition 5.2), and $u(t) = u_0 \circ \Phi^t = \mathbf{1}_{A(t)}$, $A(t) := \Phi_t(A)$, we have

$$|u_{h,\Delta t}(x,t) - u(x,t)| = (u(x,t) - u_{h,\Delta t}(x,t))\varphi^\sharp(x,t), \quad \varphi^\sharp(t) := (\mathbf{1}_{A(t)} - \mathbf{1}_{A(t)^c}).$$

Note that $\varphi^\sharp(t) = \varphi^\sharp(0) \circ \Phi^t$, so $(\partial_t + a \cdot \nabla_x)\varphi^\sharp = 0$ in a weak sense. If we could use this $\varphi^\sharp$ as a test function in (5.112), we would get (taking $T = t = t_n$) the estimate

$$\begin{aligned} &\|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)} \\ &\qquad \leq \|u_{h,\Delta t}(0) - u(0)\|_{L^1(\mathbb{T}^d)} + \langle \mu^0_{h,\Delta t}, |\varphi^\sharp| \rangle + \langle \mu^1_{h,\Delta t}, |\partial_t \varphi^\sharp| \rangle + \langle \mu^2_{h,\Delta t}, |\nabla_x \varphi^\sharp| \rangle, \end{aligned} \tag{5.116}$$

and then would have to work on the error terms. Since $\varphi^\sharp$ is not sufficiently regular to justify (5.116), we proceed differently and consider a regularized version of $\varphi^\sharp$. Let T_1 denote the function

$$T_1(s) = \min(1, \max(-1, s)),$$

which truncates s when $|s| > 1$. Let δ denote the signed distance function

$$\delta(x) = d(x, \partial A)\mathbf{1}_A - d(x, \partial A)\mathbf{1}_{A^c},$$

where d is the euclidean distance. We set

$$\varphi_0(x) = T_1(L^{-1}\delta(x)), \quad \varphi(x,t) = \varphi_0 \circ \Phi^t(x). \tag{5.117}$$

The functions T_1 and δ are Lipschitz continuous[5], so φ_0 as well. This regularity is sufficient to justify, after a preliminary regularization procedure, that (5.112) is valid with φ as a test-function. We obtain

$$\left|\int_{\mathbb{T}^d}(u_{h,\Delta t}(t)-u(t))\varphi(x,t)dx\right| \\ \leq \|u_{h,\Delta t}(0)-u(0)\|_{L^1(\mathbb{T}^d)}+\langle\mu^0_{h,\Delta t},|\varphi|\rangle+\langle\mu^1_{h,\Delta t},|\partial_t\varphi|\rangle+\langle\mu^2_{h,\Delta t},|\nabla_x\varphi|\rangle, \quad (5.119)$$

instead of (5.116). In the next section, we will explain how to exploit (5.119) to prove (5.82).

Remark 5.6. The step consisting in Reduction 3 is necessary in our method of proof. We can illustrate this in dimension $d=2$. Indeed, assume from the start that u_0 is the characteristic function of a set A of finite perimeter, in which case the "Reduction 4" step is irrelevant. We consider $A_n=K_n\times[0,\eta]$, where $K_0=[0,1]$, $K_1=[0,1/3]\cup[2/3,1],\dots$ is the standard sequence used to define the triadic Cantor set and $\eta>0$ will tend to 0. We take $\eta>0$ only to get a non-trivial boundary ∂A_n. To simplify the argument, let us work directly with K_n. For $\varepsilon=3^{-N}$, $N\geq 1$ and $n\geq N$, we have $\langle K_n\rangle_\varepsilon=K_{N-1}$ and $|K_n|=(2/3)^n$, so the inequality

$$\langle K_n\rangle_\varepsilon\leq C|K_n|\varepsilon,$$

where C is an absolute constant, cannot be satisfied when n is too large.

5.11.2 Error estimate

We examine first the integral in the left-hand side of (5.119), that we would like to compare to the exact L^1-norm $\|u_{h,\Delta t}(t)-u(t)\|_{L^1(\mathbb{T}^d)}$. Since $\|u_{h,\Delta t}(t)-u(t)\|_{L^\infty(\mathbb{T}^d)}\leq 1$, we have

$$\|u_{h,\Delta t}(t)-u(t)\|_{L^1(\mathbb{T}^d)}\leq\left|\int_{\mathbb{T}^d}(u_{h,\Delta t}(t)-u(t))\varphi(x,t)dx\right|+\|\varphi^\sharp(t)-\varphi(t)\|_{L^1(\mathbb{T}^d)}.$$

By the conservation property (5.109) for $p=1$,

$$\|\varphi^\sharp(t)-\varphi(t)\|_{L^1(\mathbb{T}^d)}=\|\varphi^\sharp(0)-\varphi(0)\|_{L^1(\mathbb{T}^d)}\leq|\langle\partial A\rangle_{L^{-1}}|.$$

We use the estimate (5.113), and the fact that $|\partial A|=|Du_0|(\mathbb{T}^d)$, to obtain

$$\|u_{h,\Delta t}(t)-u(t)\|_{L^1(\mathbb{T}^d)}\leq\left|\int_{\mathbb{T}^d}(u_{h,\Delta t}(t)-u(t))\varphi(x,t)dx\right|+C|Du_0|(\mathbb{T}^d)h^{1/2}. \quad (5.120)$$

The first term $\|u_{h,\Delta t}(0)-u(0)\|_{L^1(\mathbb{T}^d)}$ in the right-hand side of (5.119) is bounded by $C|Du_0|(\mathbb{T}^d)h$ as a consequence of Proposition 5.15. By (the proof of) Theorem 5.5 and (5.108), we have

$$\langle\mu^0_{h,\Delta t},|\varphi|\rangle\leq\|\varphi_0\|_{L^\infty(\mathbb{T}^d)}\omega_{L^1}(u_0;h)\leq C|Du_0|(\mathbb{T}^d)h.$$

[5] if $x,y\in A$ and $z\in\partial A$, then

$$d(x,\partial A)\leq d(x,z)\leq d(x,y)+d(y,z).$$

Taking the inf on $z\in\partial A$, we obtain $d(x,\partial A)\leq d(x,y)+d(y,\partial A)$. By symmetry, we also have $d(y,\partial A)\leq d(x,y)+d(x,\partial A)$, hence

$$|\delta(x)-\delta(y)|\leq d(x,y),\quad x,y\in A. \quad (5.118)$$

Replacing A by A^c shows that (5.118) holds true when $x,y\in A^c$. If $x\in A$, $y\in A^c$, then the segment $[x,y]$ intersects ∂A at least at the point z_τ defined by

$$\tau=\sup\{t\in[0,1];[x,z_t]\subset A\},\quad z_t:=(1-t)x+ty.$$

Then

$$|\delta(x)-\delta(y)|=d(x,\partial A)+d(y,\partial A)\leq d(x,z_\tau)+d(y,z_\tau)=d(x,y).$$

We can now begin the study of the two most important terms in (5.119): $\langle \mu^1_{h,\Delta t}, |\partial_t \varphi| \rangle$ and $\langle \mu^2_{h,\Delta t}, |\nabla_x \varphi| \rangle$. To that purpose, we need to come back to the definition of $\mu^1_{h,\Delta t}$ and $\mu^2_{h,\Delta t}$ in the proof of Theorem 5.5, *cf.* (5.73) and (5.76):

$$\langle \mu^1_{h,\Delta t}, \psi \rangle = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\sharp} |u_K^n - u_K^{n+1}| \int_{t_n}^{t_{n+1}} \int_K \psi(x,t) dx dt, \tag{5.121}$$

and (taking into account the expression (5.14) of the numerical flux in (5.76)):

$$\begin{aligned} \langle \mu^2_{h,\Delta t}, \psi \rangle := \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} a^-_{K \to L} |u_K^n - u_L^n| \\ \times \frac{1}{|K|} \int_{t_n}^{t_{n+1}} \int_{K|L} \int_K \int_0^1 \psi(ry + (1-r)x, t) |x-y| dr dx d\sigma(y) dt. \end{aligned} \tag{5.122}$$

The norm of the gradient $\nabla_x \varphi(x,t) = (\nabla \Phi^t(x))^* (\nabla_x \varphi_0) \circ \Phi^t(x)$ is bounded by

$$\|\nabla \Phi^t\|_{L^\infty(\mathbb{T}^d)} \|\nabla_x \varphi_0\|_{L^\infty(\mathbb{T}^d)}.$$

We have $\|\nabla_x \varphi_0\|_{L^\infty(\mathbb{T}^d)} \le L \le h^{-1/2}$ and $\|\nabla \Phi^t\|_{L^\infty(\mathbb{T}^d)} \le e^{t\|a\|_{C^1(\mathbb{T}^d;\mathbb{R}^d)}}$. This last bound comes from the identities (where $\nabla \Phi = (\partial_i \Phi_j)_{i,j}$)

$$\nabla \Phi_t = \exp\left(\int_0^t \nabla a \circ \Phi_s ds \right), \quad \mathrm{I}_d = \nabla \Phi_t(x) (\nabla \Phi^t)(\Phi_t(x)),$$

for all $x \in \mathbb{T}^d$. Using the transport equation $\partial_t \varphi = -a \cdot \nabla_x \varphi$, we deduce from these estimates that

$$\|\nabla_{(t,x)} \varphi\|_{L^\infty(\mathbb{T}^d \times [0,T])} \le C h^{-1/2}, \tag{5.123}$$

where C depends on $\|a\|_{C^1(\mathbb{T}^d;\mathbb{R}^d)}$ and T only. We also remark that the derivatives $\nabla_{(t,x)} \varphi$ are supported in the "streak"

$$\mathcal{S} = \bigcup_{0 \le t \le T} \Phi_t(\langle \partial A \rangle_{L^{-1}}) \times \{t\}.$$

This has the consequence that

$$\langle \mu^1_{h,\Delta t}, |\partial_t \varphi| \rangle \le C h^{-1/2} \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\sharp} |K| |u_K^n - u_K^{n+1}| \chi(K \times (t_n, t_{n+1})),$$

where $\chi(K \times (t_n, t_{n+1})) = 1$ if $K \times (t_n, t_{n+1})$ intersects the set $\mathcal{S}$, and 0 otherwise. By the Cauchy-Schwarz inequality and (5.58), we obtain

$$|\langle \mu^1_{h,\Delta t}, |\partial_t \varphi| \rangle|^2 \le C h^{-1} \Delta t \mathcal{D}(t_N) \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\sharp} |K| \chi(K \times (t_n, t_{n+1})). \tag{5.124}$$

To estimate the term

$$\mathbb{S} := \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\sharp} |K| \chi(K \times (t_n, t_{n+1}))$$

in (5.124), let us fix $n \in \{0, \dots, N-1\}$. We have

$$\sum_{K \in \mathcal{T}^\sharp} |K| \chi(K \times (t_n, t_{n+1})) = |E_n|$$

where E_n is the union of the cells K such that $\chi(K \times (t_n, t_{n+1})) = 1$. If $x \in K \subset E_n$, then there exists $s \in (t_n, t_{n+1})$, $z \in \langle \partial A \rangle_{L^{-1}}$ such that $\Phi_s(z) \in K$. We have then

$$d(x, \Phi_{t_n}(z)) \leq d(x, \Phi_s(z)) + d(\Phi_s(z), \Phi_{t_n}(z)) \leq h + C\Delta t,$$

hence

$$d(\Phi^{t_n}(x), z) \leq C d(x, \Phi_{t_n}(z)) \leq Ch, \tag{5.125}$$

under the CFL condition $\Delta t \leq Ch$. The estimate (5.125) shows that $\Phi^{t_n}(x) \in \langle \partial A \rangle_{L^{-1}+Ch}$, and

$$E_n \subset \Phi_{t_n}(\langle \partial A \rangle_{L^{-1}+Ch}).$$

Since Φ_{t_n} preserves the Lebesgue measure, we obtain the estimate

$$|E_n| \leq |\langle \partial A \rangle_{L^{-1}+Ch}| \leq C|\partial A| h^{1/2}. \tag{5.126}$$

To get (5.126), we have used the estimate $|\partial A\langle_{L^{-1}+Ch}| \leq C|\partial A|(L^{-1} + h)$, which is a slight generalization of (5.113). It follows from (5.126) that $\mathbb{S}_{\langle h^{1/2} \rangle} \leq C|\partial A| h^{1/2}$. We report this estimate in (5.124) (and use the bound $\Delta t \leq C_0 h$) to conclude that

$$|\langle \mu^1_{h,\Delta t}, |\partial_t \varphi| \rangle|^2 \leq C\mathcal{D}(t_N) h^{1/2}. \tag{5.127}$$

By similar arguments, we obtain the analogous estimate

$$|\langle \mu^2_{h,\Delta t}, |\nabla_x \varphi| \rangle|^2 \leq C|Du_0|(\mathbb{T}^d)\mathcal{D}(t_N) h^{1/2}. \tag{5.128}$$

We have, indeed, by (5.122) and the bounds on $\nabla_x \varphi$,

$$|\langle \mu^2_{h,\Delta t}, |\nabla_x \varphi| \rangle| \leq Ch^{1/2} \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} |K|L| a^-_{K \to L} |u^n_K - u^n_L| \chi(\bar{K} \times [t_n, t_{n+1}]).$$

The first inequality in (5.59) reads

$$\sum_{n=0}^{N} \Delta t \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} |K|L| [a^-_{K \to L} |u^n_K - u^n_L|]^2 \leq 4\|a\|_{C(\mathbb{T}^d; \mathbb{R}^d)} \mathcal{D}(t_N).$$

By the Cauchy-Schwarz inequality, we obtain

$$|\langle \mu^2_{h,\Delta t}, |\nabla_x \varphi| \rangle|^2 \leq Ch\mathcal{D}(t_N) \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} |K|L| \chi(\bar{K} \times [t_n, t_{n+1}]).$$

Since

$$h \sum_{L \in \mathcal{N}(K)} |K|L| = h|\partial K| \leq \alpha^{-2}|K|,$$

by (5.29), we see that $|\langle \mu^2_{h,\Delta t}, |\nabla_x \varphi| \rangle|^2 \leq Ch\mathcal{D}(t_N)\mathbb{S}$ and the estimate on $\mathbb{S}$ given above yields (5.128). To sum up, we have shown that

$$\|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)} \leq C \left\{ |Du_0|(\mathbb{T}^d)\mathcal{D}(t_N) h^{1/2} \right\}^{1/2} + C|Du_0|(\mathbb{T}^d) h^{1/2}. \tag{5.129}$$

We see here that, simply estimating $\mathcal{D}(t_N)$ from above by $\|u_0\|^2_{L^2(\mathbb{T}^d)}$ will not be enough to conclude. Instead, the energy estimate (5.47) must be fully exploited. It gives, indeed (recall that $t \in [t_N, t_{N+1}]$) a bound on the quantity $2\xi\mathcal{D}(t_N)$ by the difference $\|u_0\|^2_{L^2(\mathbb{T}^d)} - \|u_{h,\Delta t}(t)\|^2_{L^2(\mathbb{T}^d)}$. By the conservation of the L^p-norms in the continuity equation(5.5), $\|u_0\|^2_{L^2(\mathbb{T}^d)} = \|u(t)\|^2_{L^2(\mathbb{T}^d)}$. Since $u(t)$ and $u_{h,\Delta t}(t)$ are bounded by 1 in $L^\infty(\mathbb{T}^d)$, we obtain

$$\xi\mathcal{D}(t_N) \leq \|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)}. \tag{5.130}$$

We report the estimate (5.130) and use the inequality $2ab \leq \eta a^2 + \eta^{-1}b^2$ with a parameter η small enough (with respect to the constant C) to obtain

$$\|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)} \leq \frac{1}{2}\|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)} + C|Du_0|(\mathbb{T}^d)h^{1/2}.$$

The error estimate (5.82) follows.

6 Solution to the exercises

Solution to Exercise 3.4. Since the stochastic continuity of (P_t) at s is equivalent to the weak convergence of $P_t^*\mu$ to $P_s^*\mu$ for all μ, we can use the Portmanteau Theorem and consider simply a function φ which is bounded and uniformly continuous. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|\varphi(x) - \varphi(y)| < \varepsilon$. We decompose then the difference

$$\mathbb{E}[\varphi(X_t)] - \mathbb{E}[\varphi(X_s)].$$

into two pieces. The first one is

$$\mathbb{E}\left[(\varphi(X_t) - \varphi(X_s))\mathbf{1}_{d(X_t,X_s)<\delta}\right],$$

which is bounded by ε. The second piece is

$$\mathbb{E}\left[(\varphi(X_t) - \varphi(X_s))\mathbf{1}_{d(X_t,X_s)\geq\delta}\right],$$

which can be bounded by $2\|\varphi\|_{\mathrm{BC}(E)}\mathbb{P}(d(X_t, X_s) \geq \delta)$, which is smaller than ε for t close enough to s.

Back to Exercise 3.4.

Solution to Exercise 3.6.

1. That $\mu_0 = \delta_0$ means that X_0 always take the value 0 (X_0 is deterministic). We have then $X_1 = \pm 1$ with equi-probability, so

$$\mu_1 = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1},$$

which is an example of Bernoulli's Law $b(\frac{1}{2})$. We have then

$$\mathbb{P}(X_2 = -2) = \frac{1}{4}, \quad \mathbb{P}(X_2 = 0) = \frac{1}{2}, \quad \mathbb{P}(X_2 = +2) = \frac{1}{4}.$$

The law of X_2 is therefore

$$\mu_2 = \frac{1}{4}\left[\delta_{-3/2} + \delta_{-1/2} + \delta_{1/2} + \delta_{3/2}\right].$$

2. The law μ_N is

$$\mu_N = \frac{1}{2^{N+1}}\delta_{-2} + \sum_{-2^{N-1}<k<2^{N-1}} \frac{1}{2^N}\delta_{\frac{k}{2^{N-2}}} + \frac{1}{2^{N+1}}\delta_{-2}. \tag{6.1}$$

3. The answer is that μ_0 is the uniform law on $[-2,2]$:

$$\mu_0(A) = \frac{1}{4}|A \cap [-2,2]|,$$

where $|A|$ is the Lebesgue measure of a Lebesgue set $A \subset \mathbb{R}$ (see the proof below for μ_∞). This answer can be simply guessed by examination of the evolution of the process (X_n). An other way to find the right μ_0 is to look at μ_N for large N. Indeed, a usual way to find an equilibrium for a system in evolution is to look as the behavior for large times: if there is convergence to a limit object, this will most probably be an equilibrium of the system. Here, for example, one can look at the evolution starting from the binomial $b(1/2)$ with values in $\{-2,+2\}$, as in Question 2. If $\varphi \in \mathrm{BC}(\mathbb{R})$, then

$$\begin{aligned}\int_{\mathbb{R}} \varphi d\mu_N &= \sum_{-2^{N-1}<k<2^{N-1}} \frac{1}{2^N}\varphi\left(\frac{k}{2^{N-2}}\right) + o(1)\\ &= \frac{1}{4}\sum_{-2^{N-1}<k<2^{N-1}} \frac{1}{2^{N-2}}\varphi\left(\frac{k}{2^{N-2}}\right) + o(1).\end{aligned}$$

We recognize a Riemann sum, which converges to

$$\int_{\mathbb{R}} \varphi d\mu_\infty := \frac{1}{4}\int_{-2}^{2} \varphi(x)dx.$$

The limit law μ_∞ is an invariant measure for good. Indeed, if $X_0 \sim \mu_\infty$, then, by the formula of total probability,

$$\begin{aligned}\mathbb{P}(X_1 \in A) &= \mathbb{P}(X_1 \in A|Z_1 = -1)\mathbb{P}(Z_1 = -1) + \mathbb{P}(X_1 \in A|Z_1 = +1)\mathbb{P}(Z_1 = +1)\\ &= \frac{1}{2}\mathbb{P}(X_0/2 \in A+1) + \frac{1}{2}\mathbb{P}(X_0/2 \in A-1),\end{aligned}$$

for any Borel subsets A of $\mathbb{R}$. This gives

$$8\mathbb{P}(X_1 \in A) = |A_+ \cap [-2,2]| + |A_- \cap [-2,2]|, \quad A_\pm := 2A \pm 2.$$

We compute, by the invariance by translation of the Lebesgue measure and the change of variable formula,

$$|A_+ \cap [-2,2]| = |2A \cap [-4,0]| = 2|A \cap [-2,0]|, \quad |A_- \cap [-2,2]| = 2|A \cap [0,2]|.$$

If follows that $\mathbb{P}(X_1 \in A) = \frac{1}{4}|A \cap [-2,2]| = \mu_\infty(A)$: X_1 has law μ_∞.

Back to Exercise 3.6.

Solution to Exercise 3.7. We will use the following result.

Lemma 6.1. *Let (E,d) be a complete, separable metric space. Then* $\mathrm{BC}(E)$ *is a* separating class*: if two probability measures ν_1 and ν_2 satisfy $\langle \nu_1, \varphi\rangle = \langle \nu_2, \varphi\rangle$ for all $\varphi \in \mathrm{BC}(E)$, then $\nu_1 = \nu_2$.*

Proof of Lemma 6.1. The set of closed subsets of E is a π-system. By [3, Theorem 3.3], it is sufficient to show that $\nu_1(A) = \nu_2(A)$ for all closed sets A. This follows from the pointwise monotone convergence $\varphi_n \downarrow \mathbf{1}_A$, where the function

$$\varphi_n(x) = 1 - \min(1, nd(x, A))$$

is continuous. □

Let $\varphi \in \mathrm{BC}(E)$. By Lemma 6.1, it is sufficient to show that $\langle P_t^*\nu, \varphi\rangle = \langle \nu, \varphi\rangle$. We write

$$\langle P_t^*\bar{\mu}_T, \varphi\rangle = \frac{1}{T}\int_0^T \langle P_{t+s}^*\mu, \varphi\rangle ds, \tag{6.2}$$

(we will justify later this commutation relation). A change of variable gives then

$$\langle P_t^*\bar{\mu}_T, \varphi\rangle = \frac{1}{T}\int_t^{T+t} \langle P_s^*\mu, \varphi\rangle ds = \frac{T+t}{T}\langle \bar{\mu}_T, \varphi\rangle - \frac{t}{T}\langle \bar{\mu}_t, \varphi\rangle. \tag{6.3}$$

Using the Feller property of (P_t) and the convergence $\bar{\mu}_T \to \nu$, we can pass to the limit in (6.3) to obtain the desired identity $\langle P_t^*\nu, \varphi\rangle = \langle \nu, \varphi\rangle$. There remains to justify (6.2). By continuity of $t \mapsto \langle P_t^*\mu, \varphi\rangle$, we have the following convergence of Riemann sums

$$\frac{1}{N}\sum_{n=0}^{N-1} P_{s_n}^*\mu \to \bar{\mu}_T, \quad s_n = \frac{nT}{N}.$$

We apply P_t^* to each member (the convergence holds true owing to the Feller property of (P_t)). By linearity of P_t^*, we get

$$\frac{1}{N}\sum_{n=0}^{N-1} P_t^*(P_{s_n}^*\mu) \to P_t^*\bar{\mu}_T. \tag{6.4}$$

The semi-group property of P_t implies $P_t^*(P_s^*\mu) = P_{t+s}^*\mu$, therefore the left-hand side of (6.4) is again a Riemann sum, which converges to the right-hand side of (6.2). This gives the desired result.

Back to Exercise 3.7.

Solution to Exercise 3.8.

1. Quite clear.
2. Quite clear also !
3. (a) Let $(p(t), q(t)) = \Phi_t(p, q)$. We compute the time derivative of $H(p(t), q(t))$:

$$\frac{d}{dt}H(p(t), q(t)) = D_pH(p, q)\dot{p} + D_qH(p, q)\dot{q} = 0.$$

   (b) We have

$$\langle P_t^*\mu_\beta, \varphi\rangle = \frac{1}{Z(\beta)}\int_{\mathbb{R}^d} \varphi \circ \Phi_t(x)e^{-\beta H(x)}dx.$$

   The change of variable $y = \Phi_t(x)$ has the inverse $x = \Phi_{-t}(y)$ since the system is autonomous, and has Jacobian 1 since

$$\mathrm{div}_{(p,q)}(D_qH, -D_pH) = \sum_{i=1}^n \partial^2 p_i q_i H - \partial^2 p_i q_i H = 0.$$

Using the identity $H \circ \Phi_{-t} = H$, we obtain

$$\int_{\mathbb{R}^d} \varphi \circ \Phi_t(x) e^{-\beta H(x)} dx = \int_{\mathbb{R}^d} \varphi e^{-\beta H(x)} dx.$$

Back to Exercise 3.8.

Solution to Exercise 3.9. Let us first prove that $\Gamma(\varphi) \geq 0$. Let $\varphi \in D(\mathscr{L})$ be such that $\varphi^2 \in D(\mathscr{L})$. Then $2\Gamma(\varphi)$ is the limit for b.p. convergence when $t \to 0+$ of the quantity

$$\frac{P_t[\varphi^2] - \varphi^2}{t} - 2\varphi \frac{P_t\varphi - \varphi}{t}. \tag{6.5}$$

Since P_t is given by

$$P_t\varphi(x) = \int_E \varphi(y) Q(t, x, dy),$$

where Q is a probability kernel, we can apply the Jensen inequality to bound (6.5) from below by

$$\frac{[P_t\varphi]^2 - \varphi^2}{t} - 2\varphi \frac{P_t\varphi - \varphi}{t}. \tag{6.6}$$

Rearranging the expression, we see that (6.6) is equal to $t^{-1}(P_t\varphi - \varphi)^2 \geq 0$.
Let us now consider the case of an ODE: $\dot{X}_t = F(X_t)$. Let Φ_t denote the associate flow, so that $X_t^x = \Phi_t(x)$. There is no randomness here, so we may consider that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F} = \{\emptyset, \Omega\}$ the trivial σ-algebra. However, it is sometimes relevant to put randomness in the initial datum only. In that configuration, we consider a non-trivial σ-algebra $\mathcal{F}$ and a trivial filtration $\mathcal{F}_t = \mathcal{F}$, for all t. In any case, we obtain a Markov process with transition operator $P_t\varphi = \varphi \circ \Phi_t$. Then we compute $\mathscr{L}\varphi(x) = F(x) \cdot \nabla\varphi(x)$ when $\varphi \in C_b^1(\mathbb{R}^d)$ and

$$2\Gamma(\varphi) = F \cdot \nabla(\varphi^2) - 2\varphi F \cdot \nabla\varphi = 0.$$

In the case of the SDE (3.39), showing that (X_t) is a Markov process is not immediate, see, *e.g.*, [2, p.313]. Define the non-negative matrix $a = \sigma^*\sigma$. By the Itô formula, we have, for $\varphi \in C_b^2(\mathbb{R}^d)$,

$$\mathbb{E}\left[\varphi(X_t)\right] = \mathbb{E}\left[\varphi(X_0)\right] + \int_0^t \mathbb{E}\left[\mathscr{L}\varphi(X_s)\right] ds, \tag{6.7}$$

where $\mathscr{L}$ is given by

$$\mathscr{L}\varphi(x) = F(x) \cdot \nabla\varphi + \frac{1}{2} a(x) : D^2\varphi(x). \tag{6.8}$$

In (6.8), we use the following notations: $D^2\varphi$ is the Hessian Matrix with ij-components $\partial^2_{x_i x_j}\varphi$; $A : B$ is the scalar product of $d \times d$ matrices:

$$A : B = \sum_{i,j=1}^{d} A_{ij} B_{ij}.$$

It follows from (6.7) and the continuity properties of the solution to (3.39) that the generator of (X_t) is indeed the operator $\mathscr{L}$ of (6.8). A simple computation gives then

$$\Gamma(\varphi)(x) = a(x) : \nabla\varphi(x) \otimes \nabla\varphi(x). \tag{6.9}$$

In (6.9), we use the following notation: given $u, v \in \mathbb{R}^d$, $u \otimes v$ is the (rank-1) matrix with ij-element $u_i v_j$. Then $A : u \otimes v = Av \cdot u$, scalar product of Av with u. This gives us the alternative expression

$$\Gamma(\varphi)(x) = a(x)\nabla\varphi(x) \cdot \nabla\varphi(x) = |\sigma(x)\nabla\varphi(x)|^2.$$

In the particular case $\sigma(x) = \mathrm{I}_d$, we obtain $\Gamma(\varphi) = |\nabla\varphi|^2$.

Back to Exercise 3.9.

Solution to Exercise 3.10. We have

$$P_t\varphi(x_i) = \mathbb{E}_{x_i}\varphi(X_t) = \sum_{j=1}^{L} \mathbb{E}_{x_i}\left[\mathbf{1}_{X(t)=x_j}\varphi(x_j)\right] = \sum_{j=1}^{L} \mathbb{P}_{x_i}(X(t) = x_j)\varphi(x_j) = \sum_{j=1}^{L} a_{ij}(t)\varphi(x_j),$$

which gives $P_t\varphi = A(t)\varphi$. With the conventions that are used, we observe that $\langle\varphi, \mu\rangle = (\varphi, \mu)$, where $(\cdot, \cdot)$ is the canonical scalar product in $\mathbb{R}^L$. Consequently,

$$\langle\varphi, P_t^*\mu\rangle = \langle P_t\varphi, \mu\rangle = (A(t)\varphi, \mu) = (\varphi, A(t)^*\mu),$$

and we obtain $P_t^*\mu = A(t)^*\mu$, where $A(t)^*$ is the adjoint of the matrix $A(t)$. The semi-group property reads $A(t+s) = A(t)A(s)$. It follows that

$$\frac{A(t+s) - A(s)}{t} = A(s)\frac{A(t) - \mathrm{I}_L}{t} = \frac{A(t) - \mathrm{I}_L}{t}A(s).$$

By letting $t \to 0$, we deduce that A satisfies the ODE $A'(t) = \mathscr{L}A(t) = A(t)\mathscr{L}$, which implies $A(t) = e^{t\mathscr{L}}$ since $A(0) = \mathrm{I}_L$. The equation satisfied by an invariant measure is $A(t)^*\mu = \mu$ for all $t \geq 0$. By differentiation, we obtain $\mathscr{L}^*\mu = 0$. Of course the latter equation implies $(\mathscr{L}^*)^n\mu = 0$ for all $n \geq 1$, and thus

$$A(t)^*\mu = e^{t\mathscr{L}^*}\mu = \sum_{n\geq 0} \frac{(\mathscr{L}^*)^n}{n!}\mu = \mu.$$

Consequently, there is strict equivalence between $A(t)^*\mu = \mu$ for all $t \geq 0$, and $\mathscr{L}^*\mu = 0$.

Back to Exercise 3.10.

Solution to Exercise 3.11. Assume $E = \{x_1, \ldots, x_L\}$ as in Exercise 3.10. Let A denote the matrix $A(1)$: $a_{ij} = \mathbb{P}_{x_i}(X_1 = x_j)$. We still have $P_n\varphi = A(n)\varphi$ and $P_n^*\mu = A(n)^*\mu$. By the semi-group property, we have $A(n) = A^n$ for all $n \geq 0$. The equation satisfied by the invariant measure is $(A^* - \mathrm{Id})\mu = 0$ (the equivalent to $\mathscr{L}$ here is $A - \mathrm{Id}$). Let us come back to the case of a general state space E (a Polish space in our framework). Let us first prove that (M_n) is a martingale. We can use the tower property (2.2) to show that it is sufficient to establish the identity $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$ for all $n \geq 0$. By the Markov property, we obain

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = P_1\varphi(X_n) - \varphi(X_0) - \sum_{k=0}^{n} \mathscr{L}\varphi(X_k).$$

Since $\mathscr{L} = P_1 - \mathrm{Id}$, this is precisely the desired identity $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$. Let us look at (3.43) now. Again, we want to prove that $\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = Z_n$. We write

$$M_{n+1} = \varphi(X_{n+1}) - Y_n, \quad Y_n = \varphi(X_0) + \sum_{k=0}^{n} \mathscr{L}\varphi(X_k),$$

where Y_n is $\mathcal{F}_n$-measurable. This gives

$$\begin{aligned}\mathbb{E}\left[|M_{n+1}|^2|\mathcal{F}_n\right] &= \mathbb{E}\left[|\varphi(X_{n+1})|^2|\mathcal{F}_n\right] - 2Y_n\mathbb{E}\left[\varphi(X_{n+1})|\mathcal{F}_n\right] + |Y_n|^2\\ &= P_1|\varphi|^2(X_n) - 2Y_nP_1\varphi(X_n) + |Y_n|^2\\ &= P_1|\varphi|^2(X_n) + Y_n(Y_n - 2P_1\varphi(X_n))\end{aligned}$$

We have also $Y_n = \varphi(X_n) + \mathscr{L}\varphi(X_n) - M_n = P_1\varphi(X_n) - M_n$, hence

$$\mathbb{E}\left[|M_{n+1}|^2|\mathcal{F}_n\right] = P_1|\varphi|^2(X_n) - (P_1\varphi(X_n) - M_n)(P_1\varphi(X_n) + M_n).$$

and

$$\mathbb{E}\left[|M_{n+1}|^2|\mathcal{F}_n\right] - |M_n|^2 = P_1|\varphi|^2(X_n) - |P_1\varphi(X_n)|^2.$$

We obtain then (3.43) by using the definition $\Gamma[\varphi] = P_1|\varphi|^2 - |P_1\varphi|^2$. The Jensen inequality applied to

$$P_1\varphi(x) = \int_E \varphi(y)Q(1,x,dy)$$

shows that $\Gamma[\varphi] \geq 0$.

Back to Exercise 3.11.

Solution to Exercise 4.3. Each T_n has density $f\colon t \mapsto \lambda e^{-\lambda t}\mathbf{1}_{\mathbb{R}_+}(t)$ with respect to the Lebesgue measure on $\mathbb{R}$. By independence, S_n has the law $f * \cdots * f$ (n times). We compute

$$f * f(t) = \int_{\mathbb{R}} \lambda^2 e^{-\lambda s}e^{-\lambda(t-s)}\mathbf{1}_{\mathbb{R}_+}(s)\mathbf{1}_{\mathbb{R}_+}(t-s)ds = \int_0^t \lambda^2 e^{-\lambda t}ds\mathbf{1}_{\mathbb{R}_+}(t) = \lambda^2 te^{-\lambda t}\mathbf{1}_{\mathbb{R}_+}(t),$$

and, by recursion on n, $f * \cdots * f(t) = \lambda^n \frac{t^{n-1}}{(n-1)!}e^{-\lambda t}\mathbf{1}_{\mathbb{R}_+}(t)$. We compute then

$$\begin{aligned}\mathbb{P}(N(t) = n) = \mathbb{P}(S_n \leq t < S_{n+1}) &= \mathbb{P}(S_n \leq t < S_n + T_{n+1})\\ &= \mathbb{E}[\mathbf{1}_{S_n \leq t < S_n + T_{n+1}}] = \int_{s=0}^t \int_{\tau=t-s}^\infty d\mu_{(S_n,T_{n+1})}(s,\tau).\end{aligned}$$

By independence, $\mu_{(S_n,T_{n+1})} = \mu_{S_n} \otimes \mu_{T_{n+1}}$, so

$$\mathbb{P}(N(t) = n) = \int_{s=0}^t \int_{\tau=t-s}^\infty \lambda^n \frac{s^{n-1}}{(n-1)!}e^{-\lambda s}ds\lambda e^{-\lambda\tau}d\tau = e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

The assertion that $N(t)$ is càdlàg is a deterministic statement, it comes from the fact that Γ is a measure: indeed, we note that, whatever the Radon measure μ on $\mathbb{R}_+$, the map $t \mapsto \mu([0,t])$ is càdlàg. It is clear that $N(0) = 0$ a.s. and that $(N(t))$ has jumps of amplitude $+1$.

Back to Exercise 4.3.

Solution to Exercise 4.5.

1. (a) Assume by contradiction $\mu(\{x_0\}) > 0$. For $A = \{x_0\}$, we have then $\mathbb{P}(\Gamma(A) \geq 2) > 0$, which is absurd since A cannot contain more than one point.

(b) The left-hand side of (4.14) is

$$\frac{\mathbb{P}(\Gamma(A_1)=n_1,\dots,\Gamma(A_k)=n_k,\Gamma(A_0)=n_0)}{\mathbb{P}(\Gamma(\mathbb{R}^d)=n)}=\frac{\prod_{i=0}^k e^{-\mu(A_i)}\frac{\mu(A_i)^{n_i}}{n_i!}}{e^{-\mu(\mathbb{R}^d)}\frac{\mu(\mathbb{R}^d)^n}{n!}}.$$

By rearrangement, we obtain (4.14).

2. (a) For $i\neq j$, using independence, the event $X_i=X_j$ has probability

$$\mathbb{P}(X_i=X_j)=\iint_{x=y} d\mu_{(X_i,X_j)}(x,y)=\int_{\mathbb{R}}\int_{\{x\}} d\nu(y)d\nu(x)=0.$$

This shows that $\#\Pi_n<n$ with probability 0.

(b) Here, the event $\Gamma_n(\mathbb{R}^d)=n$ has probability 1, so

$$\mathbb{P}(\Gamma_n(A_1)=n_1,\dots,\Gamma_n(A_k)=n_k|\Gamma_n(\mathbb{R}^d)=n)=\mathbb{P}(\Gamma_n(A_1)=n_1,\dots,\Gamma_n(A_k)=n_k).$$

We consider the realization of the event

$$\{\Gamma_n(A_1)=n_1,\dots,\Gamma_n(A_k)=n_k\}=\{\Gamma_n(A_0)=n_0,\Gamma_n(A_1)=n_1,\dots,\Gamma_n(A_k)=n_k\}.$$

Drawing each random variable X_i successively gives us n independent trials, where we have to test the $(k+1)$ outcomes $X_i\in A_j$, each having probability $p_j=\nu(A_j)$: this is precisely the situation of a generalized Bernoulli test, described by the multinomial distribution. To be complete, let us prove this result. We use a recursion on n, starting from the trivial case $n=1$. Without loss of generality, we assume that all n_j, $j=0,\dots,k$ are strictly positive. We condition to the location of the first variable X_1 to obtain

$$\begin{aligned}&\mathbb{P}(\Gamma_n(A_0)=n_0,\Gamma_n(A_1)=n_1,\dots,\Gamma_n(A_k)=n_k)\\&\quad=\sum_{j=0}^k\mathbb{P}(\Gamma_n(A_0)=n_0,\Gamma_n(A_1)=n_1,\dots,\Gamma_n(A_k)=n_k|X_1\in A_j)\mathbb{P}(X_1\in A_j)\end{aligned}$$

We have $\mathbb{P}(X_1\in A_j)=p_j$ and

$$\begin{aligned}&\mathbb{P}(\Gamma_n(A_0)=n_0,\Gamma_n(A_1)=n_1,\dots,\Gamma_n(A_k)=n_k|X_1\in A_j)\\&\quad=\mathbb{P}(\Gamma_{n-1}(A_0)=n_0,\Gamma_{n-1}(A_1)=n_1,\dots,\Gamma_{n-1}(A_j)=n_j-1,\dots\Gamma_{n-1}(A_k)=n_k)\\&\quad=\frac{1}{p_j}\frac{n_j}{n}\frac{n!}{n_0!n_1!\cdots n_k!}p_0^{n_0}\cdots p_k^{n_k},\end{aligned}$$

which gives the desired result by summation over j.

3. For each Borel subset A of $\mathbb{R}^d$, $\Gamma(A)$ is equal to the sum

$$\sum_{i=1}^\infty \mathbf{1}_A(X_i)\mathbf{1}_{[i,+\infty)}(N),$$

so $\Gamma(A)$ is a random variable. Using the notations of the previous questions, we have

$$\begin{aligned}&\mathbb{P}(\Gamma(A_1)=n_1,\dots,\Gamma(A_k)=n_k)\\&\quad=\sum_{n=n_1+\cdots+n_k}^\infty\mathbb{P}(\Gamma(A_1)=n_1,\dots,\Gamma(A_k)=n_k|N=n)\mathbb{P}(N=n)\\&\quad=\sum_{n_0=0}^\infty\mathbb{P}(\Gamma_n(A_0)=n_0,\Gamma_n(A_1)=n_1,\dots,\Gamma_n(A_k)=n_k)\mathbb{P}(N=n),\end{aligned}$$

which gives

$$\begin{aligned}
&\mathbb{P}(\Gamma(A_1) = n_1, \dots, \Gamma(A_k) = n_k) \\
&\quad = \sum_{n_0=0}^{\infty} \frac{n!}{n_0! n_1! \cdots n_k!} [\nu(A_0)]^{n_0} \cdots [\nu(A_k)]^{n_k} e^{-\mu(\mathbb{R}^d)} \frac{\mu(\mathbb{R}^d)^n}{n!} \\
&\quad = \sum_{n_0=0}^{\infty} \frac{1}{n_0! n_1! \cdots n_k!} [\mu(A_0)]^{n_0} \cdots [\mu(A_k)]^{n_k} e^{-\mu(\mathbb{R}^d)}.
\end{aligned}$$

There is no more n in this last expression. We explicit the summation over n_0 to obtain

$$\begin{aligned}
\mathbb{P}(\Gamma(A_1) = n_1, \dots, \Gamma(A_k) = n_k) &= \frac{1}{n_1! \cdots n_k!} [\mu(A_1)]^{n_1} \cdots [\mu(A_k)]^{n_k} e^{-\mu(\mathbb{R}^d)+\mu(A_0)} \\
&= e^{-\mu(A_1)} \frac{[\mu(A_1)]^{n_1}}{n_1!} \cdots e^{-\mu(A_k)} \frac{[\mu(A_k)]^{n_k}}{n_k!}. \qquad (6.10)
\end{aligned}$$

Taking $k = 1$ in (6.10) shows that $\Gamma(A)$ has a Poisson distribution of parameter $\mu(A)$, then (6.10) for general k shows that $\Gamma(A_1), \dots, \Gamma(A_k)$ have the desired independence property.

4. Let $X_1, X_2, \dots$ be independent random variables with Poisson distributions of respective parameter $\lambda_n \in [0, +\infty]$. We know that $X_1 + X_2$ then follows a Poisson distribution of parameter $\lambda_1 + \lambda_2$ (this is standard when $\lambda_1, \lambda_2 < +\infty$, but the case where one of the λ_i is $+\infty$ is trivial). By iteration, any finite sum $\sum_{n \in S} X_n$ follows a Poisson distribution of parameter $\sum_{n \in S} \lambda_n$:

$$\mathbb{P}\left[\sum_{n \in S} X_n = k\right] = e^{-\lambda} \frac{\lambda^k}{k!}. \qquad (6.11)$$

We can pass to the limit in (6.11) (using monotone convergence) to extend the identity to the case where S is countable (again, discussing the case where all parameters λ_n, $n \in S$ are finite, or one is infinite). This yields the Superposition Principle.

5. For each n, we can construct a Poisson process Π_n with intensity μ_n using some iid random variables $(X_{n,m})_{m \geq 1}$ and some independent Poisson variable N_n of parameter $\mu_n(\mathbb{R}^d)$. It is always possible to ensure that the family

$$\{X_{n,m}, N_n; n, m \geq 1\}$$

is independent. Then we obtain independent Poisson point processes with intensity μ_n. The Superposition Principle gives the result.

6. Suppose that

$$\mathbb{R}^d = \bigcup_{n \in \mathbb{N}} A_n, \quad \mu(A_n) < +\infty.$$

We can assume that the sets A_n are disjoint, otherwise, we consider

$$B_1 = A_1, \ B_2 = (A_1 \cup A_2) \setminus B_1, \dots, B_n = (A_1 \cup \cdots \cup A_n) \setminus B_{n-1}, \dots$$

Then (4.17) is realized with μ_n =restriction of μ to A_n.

7. When $d = 1$, $\mu = \lambda\times$restriction of the Lebesgue measure to $\mathbb{R}_+$, the process $N(t) = \Gamma([0,t])$ is a counting process with Poisson's distribution of parameter λt. If $0 \le t, s$ and $t_1 \le \cdots \le t_k \le t$, then

$$\mathbb{P}(N(t_1) = n_1, \dots, N(t_k) = n_k, N(t+s) - N(t) = m) \tag{6.12}$$
$$= \mathbb{P}(\Gamma(A_1) = n_1, \Gamma(A_2) = n_2 - n_1 \dots, \Gamma(A_k) = n_k - n_{k-1}, \Gamma(A) = m), \tag{6.13}$$

where $A_1 = [0, t_1]$, $A_2 = (t_1, t_2], \dots, A_k = (t_{k-1}, t_k]$, $A = (t, t+s]$. We can always assume that $t_k = t$. Using the independence properties of the Poisson point process, we obtain then, setting $t_0 = 0$ and $n_0 = 0$ and $t_{k+1} = t + s$, $n_{k+1} = m + n_k$,

$$\begin{aligned}\mathbb{P}(N(t_1) = n_1, \dots, N(t_k) = n_k, N(t+s) - N(t) = m)& \\ = e^{-\lambda(t_1 - t_0)} \frac{(\lambda(t_1 - t_0))^{n_1 - n_0}}{(n_1 - n_0)!} \cdots e^{-\lambda(t_{k+1} - t_k)} \frac{(\lambda(t_{k+1} - t_k))^{n_{k+1} - n_k}}{(n_{k+1} - n_k)!}& \\ = \mathbb{P}(N(t_1) = n_1, \dots, N(t_k) = n_k)\mathbb{P}(N(t+s) - N(t) = m)& \end{aligned} \tag{6.14}$$

This shows that $N(t+s) - N(t)$ is independent on $\mathcal{F}_t^N$ and follows a Poisson distribution of parameter λs. Therefore, $(N(t))$ is a Poisson process, as defined in Definition 4.2.

Back to Exercise 4.5.

Solution to Exercise 4.6.

1. By independence, the quantity $\mathbb{E}\left[\varphi(S_1, \dots, S_n)\mathbf{1}_{S_n \le t < S_{n+1}}\right]$ is equal to

$$\begin{aligned}\mathbb{E}\left[\varphi(T_1, T_1 + T_2 \dots, T_1 + \cdots + T_n)\mathbf{1}_{T_1 + \cdots + T_n \le t < T_1 + \cdots + T_n + T_{n+1}}\right]& \\ = \int_{t_1=0}^{\infty} \cdots \int_{t_{n+1}=0}^{\infty} \varphi(t_1, t_1 + t_2, \dots, t_1 + \cdots + t_n)\mathbf{1}_{t_1 + \cdots + t_n \le t < t_1 + \cdots + t_n + t_{n+1}}& \\ \times \lambda^{n+1} e^{-\lambda(t_1 + \cdots + t_{n+1})} dt_1 \cdots dt_{n+1}.&\end{aligned}$$

We do the change of variable (of Jacobian 1)

$$u_1 = t_1, u_2 = t_1 + t_2, \dots, u_{n+1} = t_1 + \cdots + t_{n+1}$$

to get the expression

$$\begin{aligned}\int_{u_1=0}^{t} \int_{u_2=u_1}^{t} \cdots \int_{u_n=u_{n-1}}^{t} \int_{u_{n+1}=t}^{\infty} \varphi(u_1, u_2, \dots, u_n)\lambda^{n+1} e^{-\lambda u_{n+1}} du_1 \cdots du_{n+1}& \\ = \lambda^n e^{-\lambda t} \int_{u_1=0}^{t} \int_{u_2=u_1}^{t} \cdots \int_{u_n=u_{n-1}}^{t} \varphi(u_1, u_2, \dots, u_n) du_1 \cdots du_n& \\ = \frac{\lambda^n}{n!} e^{-\lambda t} \mathbb{E}[\varphi(U_{(1)}, \dots, U_{(n)})]&\end{aligned}$$

This gives us the identity

$$\mathbb{E}\left[\varphi(S_1, \dots, S_n)\mathbf{1}_{N(t)=n}\right] = \mathbb{E}[\varphi(U_{(1)}, \dots, U_{(n)})]\mathbb{P}(N(t) = n),$$

as desired.

2. We use the same kind of transformation as in (6.12) to see that

$$\begin{aligned}\mathbb{P}(N(t_1)=n_1,\dots,N(t_k)=n_k)\\ =\mathbb{P}(\Gamma(A_1)=n_1,\Gamma(A_2)=n_2-n_1\dots,\Gamma(A_k)=n_k-n_{k-1},N(t_k)=n_k),\end{aligned}$$

where $A_1=[0,t_1]$, $A_2=(t_1,t_2],\dots,A_k=(t_{k-1},t_k]$. Let $m_1=n_1$ and $m_j=n_j-n_{j-1}$ for $j>1$. The result of the previous question shows that, conditionally to $N(t_k)=n_k$, the event

$$\Gamma(A_1)=m_1,\Gamma(A_2)=m_2\dots,\Gamma(A_k)=m_k$$

corresponds to the arrangement of m_j among n_k independent uniform variables U_i on $[0,t_k]$ in the set A_j, for all j. This is the multinomial distribution (already discussed in Exercise 4.5) that gives therefore the probability:

$$\begin{aligned}\mathbb{P}(N(t_1)=n_1,\dots,N(t_k)=n_k)\\ =\frac{n_k!}{m_1!\cdots m_k!}\left(\frac{|A_1|}{t_k}\right)^{m_1}\cdots\left(\frac{|A_k|}{t_k}\right)^{m_k}\mathbb{P}(N(t_k)=n_k).\end{aligned}\quad(6.15)$$

A simple computation then gives (4.18).

Back to Exercise 4.6.

Solution to Exercise 4.7. Either adapt the proof of Proposition 4.3, either apply directly this Proposition taking $E=\mathbb{N}$ and $X_n=X_0+n$. We have then $P_1\varphi(n)=\varphi(n+1)$, therefore $(N(t))$ has the generator

$$\mathscr{L}\varphi(n)=-\lambda(\varphi(n+1)-\varphi(n)),$$

with domain the whole set of bounded functions $\mathbb{N}\to\mathbb{R}$ and transition semigroup $P_t=e^{t\mathscr{L}}$.

Back to Exercise 4.7.

Solution to Exercise 5.1. Clearly, the properties (5.18), (5.20) and the monotony property are satisfied. To establish the regularity property (5.19), we use the fact that A is locally Lipschitz continuous.

Back to Exercise 5.1.

Solution to Exercise 5.2. Same proof as in the case $A=A(v)$. This times we use the divergence-free condition $(\mathrm{div}_x A)(x,v)=0$.

Back to Exercise 5.2.

Solution to Exercise 5.3. We suppose that α is fixed of course. Consider a mesh with triangles only. If one triangle as a basis of length $\sim h$, but a height that is almost 0, *i.e.* if there is an almost flat triangle in the mesh, then the first condition in (5.29) may not be satisfied.
If we consider triangles only then $|\partial K|\le 3\,\mathrm{diam}(K)\le 3h$ for any K. Now, consider a triangle with a basis of length ~ 1, and a height $\sim h$. Then fold the "arrow" of this triangle to form a polygonal set of diameter $\mathcal{O}(h)$ and perimeter ~ 1. If $\mathcal{T}$ contains such kind of set, then the second condition in (5.29) will not be satisfied.

Back to Exercise 5.3.

Solution to Exercise 5.6. We only give the sketch of the proof. By linearity, it is sufficient to consider the case $u_0 = 0$, in which case we want to prove $u \equiv 0$. If u is smooth, then $\partial_t u + a \cdot \nabla u = 0$ (recall that a is divergence free). By the usual chain-rule formula, it follows that $\partial_t \beta(u) + a \cdot \nabla \beta(u) = 0$ for any function β of class C^1. By integration, we obtain

$$\int_{\mathbb{T}^d} \beta(u(x,t))dx = \int_{\mathbb{T}^d} \beta(u(x,0))dx = \beta(0). \tag{6.16}$$

It is sufficient to apply (6.16) with a non-negative function β such that $\beta(s) = 0$ if, and only if, $s = 0$, for example $\beta(s) = s^2$, to conclude. In this special case $\beta(s) = s^2$, we can reformulate things as follows: our aim is to justify the "energy estimate"

$$\partial_t u^2 + a \cdot \nabla_x (u^2) = 0$$

for a weak solution u. This is a standard problem. It is discussed for example in [16, Section III.2.] for parabolic equations, or [23, Appendix A.20] for the kinetic Fokker-Planck equation. For transport equation, specifically, this problem is treated in [6]. Actually, [6] deals with less regular fields a, which, instead of being Lipschitz continuous, have a mere Sobolev regularity. See Section II.2 in [6].

To show that $(x,t) \mapsto u_0 \circ \Phi^t(x)$ is a weak solution, do the change of variable $x' = \Phi^t(x)$ in the weak formulation. *Back to Exercise 5.6.*

Solution to Exercise 5.8.

1. Let $\varphi \in C^1_c(U)$. We have

$$\int_{-1}^{1} u(x)\varphi'(x)dx = -\int_{-1}^{1} u'(x)\varphi(x)dx = -\int_{-1}^{1} f(x)\varphi(x)dx. \tag{6.17}$$

   If φ is supported in $(-r,r)$ with $r < 1$, then (6.17) is bounded by $\|f\|_{L^1(-r,r)}\|\varphi\|_{C(-1,1)}$. We have $u \in \mathrm{BV}(U)$ if, and only if there is a finite constant C such that $|\int_{-1}^{1} f(x)\varphi(x)dx| \leq C\|\varphi\|_{C(-1,1)}$ for all $\varphi \in C^1_c(U)$. Clearly, $f \in L^1(U)$ implies $u \in \mathrm{BV}(U)$. Conversely, if $u \in \mathrm{BV}(U)$, let us consider, for $\varepsilon > 0$, χ_ε the characteristic function of the interval $(-1+\varepsilon, 1-\varepsilon)$ and (ρ_ε), an approximation of the unit with ρ_ε supported in $(-\varepsilon,\varepsilon)$. Let also ψ be a function in $C^1_c(U)$. We have then

$$\left|\int_{-1}^{1} f\,\mathrm{sign}(f)_\varepsilon \psi dx\right| \leq C, \quad \varphi_\varepsilon := (\varphi\chi_\varepsilon) * \rho_\varepsilon. \tag{6.18}$$

   Taking the limit $\varepsilon \to 0$ in (6.18) gives

$$\left|\int_{-1}^{1} |f|\psi dx\right| \leq C. \tag{6.19}$$

   We consider then a non-decreasing sequence of functions $\psi \in C^1_c(U)$ which converges pointwise to the constant function 1. By monotone convergence, (6.19) gives $f \in L^1(U)$.

2. Let $\varphi \in C^1_c(U)$. We have

$$\int_{-1}^{1} u(x)\varphi'(x)dx = \int_{0}^{1} \varphi'(x)dx = -\varphi(0) \leq \|\varphi\|_{C(-1,1)}, \tag{6.20}$$

   hence $u \in \mathrm{BV}(U)$.

3. By the Stokes' formula, we have, for $\varphi \in C^1_c(U)$,

$$\int_U u \operatorname{div} \varphi dx = \int_{B(0,1/2)} \operatorname{div} \varphi dx = \int_{\partial B(0,1/2)} \varphi(x) \cdot n(x) d\sigma(x) \leq \pi \|\varphi\|_{C(-1,1)}, \qquad (6.21)$$

hence $u \in \mathrm{BV}(U)$.

Back to Exercise 5.8.

Solution to Exercise 5.9. In the first case, we assume $f \in L^1(-1,1)$. Then (6.17) shows that $Du = f\lambda$, where λ is the Lebesgue measure on $(-1,1)$. By [20, Theorem 6.13], we have $|Du| = |f|\lambda$ then and $\|u\|_{\mathrm{BV}(U)} = \|u\|_{L^1(U)} + \|f\|_{L^1(U)}$. In the second case, (6.20) shows that $Du = \delta_0$, the Dirac mass at 0. Then $|Du| = \delta_0$ also and $\|u\|_{\mathrm{BV}(U)} = \|u\|_{L^1(U)} + 1 = 2$. In the third case, (6.21) shows that $Du = n\sigma$, where n is the outward unit normal to U on ∂U and σ the surface measure. By [20, Theorem 6.13] again, $|Du| = \sigma$. We compute then $\|u\|_{\mathrm{BV}(U)} = \pi/4 + \pi = 5\pi/4$.
Back to Exercise 5.9.

Solution to Exercise 5.11. Assume first that u is of class C^1. Let $h \in [0,1]$. For $x \in Q = (0,1)^d$ and $z \in \mathbb{R}^d$ with $|z| \leq h$, we have

$$|u(x+z) - u(x)| = \left| \int_0^1 (\nabla u)(x+rz) \cdot z dr \right| \leq h \int_0^1 |\nabla u|(x+rz) dr.$$

We do the change of variable $(x', r') = (x + rz, r)$ of Jacobian determinant 1 to obtain

$$\int_Q |u(x+z) - u(x)| dx \leq h \int_0^1 \int_{Q+rz} |\nabla u(x)| dx dr \leq h \int_{Q'} |\nabla u(x)| dx = h|Du|(Q'),$$

where $Q' = (-1,2)^d$. This gives $\omega_{L^1}(u;h) \leq |Du|(Q')h$. This estimate remains true in the general case by Theorem 5.11 applied on $U = Q'$. Since $|Du|(Q') \leq 3^d |Du|(\mathbb{T}^d)$, we obtain the desired result with $C = 3^d$.
Back to Exercise 5.11.

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