Measure solutions to the conservative renewal equation

Pierre Gabriel

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Pierre GABRIEL
University of Versailles, France

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Introduction

In this mini-course we are interested in the conservative renewal equation

$$\begin{cases} \dfrac{\partial n}{\partial t}(t,a) + \dfrac{\partial n}{\partial a}(t,a) + \beta(a)n(t,a) = 0, \qquad t,a > 0, \\ n(t,0) = \displaystyle\int_0^\infty \beta(a)n(t,a)\,\mathrm{d}a, \qquad t > 0, \\ n(0,a) = n^{\mathrm{in}}(a), \qquad a \geqslant 0. \end{cases} \tag{1}$$

It is a standard model of population dynamics, sometimes referred to as the McKendrick-von Foerster model. The population is structured by an age variable $a \geqslant 0$ which grows at the same speed as time and is reset to zero according to the rate $\beta(a)$. It is used for instance as a model of cell division, the age of the cells being the time elapsed since the mitosis of their mother. Suppose we follow one cell in a cell line over time and whenever a division occurs we continue to follow only one of the two daughter cells. Equation (1) prescribes the time evolution of the probability distribution $n(t,a)$ of the cell to be at age a at time t, starting with an initial probability distribution n^{in}. Integrating (formally) the equation with respect to age we get the conservation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty n(t,a)\,\mathrm{d}a = 0$$

which ensures that, if n^{in} is a probability distribution (*i.e.* $\int_0^\infty n^{\mathrm{in}} = 1$), then $n(t,\cdot)$ is a probability distribution for any time $t \geqslant 0$. It is also worth noticing that Equation (1) admits stationary solutions which are explicitly given by

$$N(a) = N(0)\,\mathrm{e}^{-\int_0^a \beta(u)\mathrm{d}u}.$$

The problem of asymptotic behavior for Equation (1) consists in investigating the convergence of any solution to a stationary one when time goes to infinity.

The goal of the mini-course is to define measure solutions to Equation (1), prove existence and uniqueness of such solutions, and demonstrate their exponential convergence to the equilibrium. Considering measure solutions instead of L^1 solutions (*i.e.* probability density functions) presents the crucial advantage to authorize Dirac masses as initial data. This is very important for the biological problem since it corresponds to the case when the age of the cell at the initial time is known with accuracy. Additionally the method we use for the asymptotic behavior is based on a contraction argument which is proved in a first time for Dirac masses initial data before being extended to general initial data.

1 Some recalls about measure theory

We first recall some classical results about measure theory and more particularly about its functional point of view. We endow $\mathbb{R}_+$ with its standard topology and the associated Borel σ-algebra. We denote by $\mathcal{M}(\mathbb{R}_+)$ the set of signed Borel measures on $\mathbb{R}_+$.

Theorem (Jordan decomposition)**.** *Any $\mu \in \mathcal{M}(\mathbb{R}_+)$ admits a unique decomposition of the form $\mu = \mu_+ - \mu_-$ where μ_+ and μ_- are finite positive Borel measures which are mutually singular. The positive measure $|\mu| = \mu_+ + \mu_-$ is called the total variation measure of the measure μ. We call total variation of μ the (finite) quantity*

$$\|\mu\|_{TV} := |\mu|(\mathbb{R}_+) = \mu_+(\mathbb{R}_+) + \mu_-(\mathbb{R}_+).$$

We denote by $C_b(\mathbb{R}_+)$ the vector space of bounded continuous functions on $\mathbb{R}_+$. Endowed with the norm $\|f\|_\infty := \sup_{x\geqslant 0} |f(x)|$ it is a Banach space. We also consider the closed subspace $C_0(\mathbb{R}_+)$ of continuous functions which tend to zero at infinity. To any $\mu \in \mathcal{M}(\mathbb{R}_+)$ we can associate a continuous linear form $T_\mu \in C_b(\mathbb{R}_+)'$ defined by

$$T_\mu : f \mapsto \int_{\mathbb{R}_+} f \,\mathrm{d}\mu.$$

The continuity is ensured by the inequality

$$|T_\mu f| \leqslant \|\mu\|_{TV} \|f\|_\infty.$$

The following theorem ensures that the application $\mu \mapsto T_\mu$ is an isometry from $\mathcal{M}(\mathbb{R}_+)$ onto $C_0(\mathbb{R}_+)'$, where $C_0(\mathbb{R}_+)'$ is endowed with the dual norm

$$\|T\|_{C_0(\mathbb{R}_+)'} := \sup_{\|f\|_\infty \leqslant 1} |Tf|.$$

Theorem (Riesz representation)**.** *For any $T \in C_0(\mathbb{R}_+)'$ there exists a unique $\mu \in \mathcal{M}(\mathbb{R}_+)$ such that $T = T_\mu$. Additionally we have $\|T\|_{C_0(\mathbb{R}_+)'} = \|\mu\|_{TV}$.*

This theorem ensures that $(\mathcal{M}(\mathbb{R}_+), \|\cdot\|_{TV})$ is a Banach space. It also ensures the existence of an isometric inclusion $C_0(\mathbb{R}_+)' \subset C_b(\mathbb{R}_+)'$. Notice that this inclusion is strict: there exist nontrivial continuous linear forms on $C_b(\mathbb{R}_+)$ which are trivial on $C_0(\mathbb{R}_+)$. Such a continuous linear form can be built for instance by using the Hahn-Banach theorem to extend the application which associates, to continuous functions which have a finite limit at infinity, the value of this limit.

More precisely the mapping $\Psi : C_b(\mathbb{R}_+)' \to C_0(\mathbb{R}_+)'$ defined by $\Psi(T) = T$ is surjective (due to the Riesz representation theorem). So $C_0(\mathbb{R}_+)'$ is isomorphic to ${}^{C_b(\mathbb{R}_+)'}/_{\mathrm{Ker}\,\Psi}$, *i.e.* for all $T \in C_b(\mathbb{R}_+)'$ there exists a unique decomposition $T = \mu + L$ with $\mu \in C_0(\mathbb{R}_+)' = \mathcal{M}(\mathbb{R}_+)$ and $L \in \mathrm{Ker}\,\Psi$. Additionally $T \in C_0(\mathbb{R}_+)'$ if and only if for all $f \in C_b(\mathbb{R}_+)$ we have $Tf = \lim_{n\to\infty} T(f_n)$ where f_n is defined for $n \in \mathbb{N}$ by

$$f_n(x) = \begin{cases} f(x) & \text{if} \quad 0 \leqslant x \leqslant n, \\ (n+1-x)f(x) & \text{if} \quad n < x < n+1, \\ 0 & \text{if} \quad x \geqslant n+1. \end{cases} \tag{2}$$

Abusing notations, we will now denote for $f \in C_b(\mathbb{R}_+)$ and $\mu \in \mathcal{M}(\mathbb{R}_+)$

$$\mu f := T_\mu f = \int_{\mathbb{R}_+} f \,\mathrm{d}\mu.$$

We end by recalling two notions of convergence in $\mathcal{M}(\mathbb{R}_+)$ which are weaker than the convergence in norm.

Definition (Weak convergence)**.** *A sequence $(\mu^n)_{n\in\mathbb{N}} \subset \mathcal{M}(\mathbb{R}_+)$ converges narrowly (resp. weak*) to $\mu \in \mathcal{M}(\mathbb{R}_+)$ as $n \to \infty$ if*

$$\lim_{n\to\infty} \mu^n f = \mu f$$

for all $f \in C_b(\mathbb{R}_+)$ (resp. for all $f \in C_0(\mathbb{R}_+)$).

2 Definition of a measure solution

Before giving the definition of a measure solution to Equation (1), we need to state the assumptions on the division rate β. We assume that β is a continuous function on $\mathbb{R}_+$ which satisfies

$$\exists\, a_*, \beta_{\min}, \beta_{\max} > 0,\ \forall a \geqslant 0, \qquad \beta_{\min}\, \mathbb{1}_{[a_*,\infty)}(a) \leqslant \beta(a) \leqslant \beta_{\max}. \tag{3}$$

We explain now how we extend the classical sense of Equation (1) to measures. Assume that $n(t,a) \in C^1(\mathbb{R}_+\times\mathbb{R}_+)\cap C(\mathbb{R}_+; L^1(\mathbb{R}_+))$ satisfies (1) in the classical sense, and let $f \in C_c^1(\mathbb{R}_+)$ the space of continuously differentiable functions with compact support on $\mathbb{R}_+$. Then we have after integration of (1) multiplied by f

$$\int_{\mathbb{R}_+} n(t,a)f(a)\,\mathrm{d}a = \int_{\mathbb{R}_+} n^{\mathrm{in}}(a)f(a)\,\mathrm{d}a + \int_0^t \int_{\mathbb{R}_+} n(s,a)\big(f'(a) - \beta(a)f(a) + \beta(a)f(0)\big)\,\mathrm{d}a\,\mathrm{d}s.$$

This motivates the definition of a measure solution to Equation (1). From now on we will denote by $\mathcal{A}$ the operator defined on $C_b^1(\mathbb{R}_+) = \{f \in C^1(\mathbb{R}_+) \,|\, f, f' \in C_b(\mathbb{R}_+)\}$ by

$$\mathcal{A}f(a) := f'(a) + \beta(a)(f(0) - f(a)).$$

Definition 1. *A family $(\mu_t)_{t\geqslant 0} \subset \mathcal{M}(\mathbb{R}_+)$ is called a measure solution to Equation (1) with initial data $n^{\mathrm{in}} = \mu^{\mathrm{in}} \in \mathcal{M}(\mathbb{R}_+)$ if the mapping $t \mapsto \mu_t$ is narrowly continuous and for all $f \in C_c^1(\mathbb{R}_+)$ and all $t \geqslant 0$,*

$$\mu_t f = \mu^{\mathrm{in}} f + \int_0^t \mu_s \mathcal{A}f\,\mathrm{d}s, \tag{4}$$

i.e.

$$\int_{\mathbb{R}_+} f(a)\,\mathrm{d}\mu_t(a) = \int_{\mathbb{R}_+} f(a)\,\mathrm{d}\mu^{\mathrm{in}}(a) + \int_0^t \left(\int_{\mathbb{R}_+} \left[f'(a) - \beta(a)f(a) + \beta(a)f(0)\right]\mathrm{d}\mu_s(a)\right)\mathrm{d}s.$$

Proposition 2. *If $t \mapsto \mu_t$ is a solution to Equation (1) in the sense of the definition above, then for all $f \in C_b^1(\mathbb{R}_+)$ the function $t \mapsto \mu_t f$ is of class C^1 and satisfies*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\mu_t f) = \mu_t \mathcal{A} f, & t \geqslant 0 \\ \mu_0 = \mu^{\mathrm{in}}. \end{cases} \tag{5}$$

Reciprocally any solution to (5) satisfies (4).

Proof. We start by checking that (4) is also satisfied for $f \in C_b^1(\mathbb{R}_+)$. Let $f \in C_b^1(\mathbb{R}_+)$, $t > 0$ and $\rho \in C^1(\mathbb{R})$ a nonincreasing function which satisfies $\rho(a) = 1$ for $a \leqslant 0$ and $\rho(a) = 0$ for $a \geqslant 1$. For all $n \in \mathbb{N}$ and $a \geqslant 0$ define $f_n(a) = \rho(a-n)f(a)$. For all $n \in \mathbb{N}$, $f_n \in C_c^1(\mathbb{R}_+)$ satisfies (4) and it remains to check that we can pass to the limit $n \to \infty$. By monotone convergence we have $\mu_t f_n \to \mu_t f$ and $\mu^{\mathrm{in}} f_n \to \mu^{\mathrm{in}} f$ when $n \to \infty$. Additionally $\|\mathcal{A}f_n\|_\infty \leqslant (\|\rho'\|_\infty + 2\beta_{\max})\|f\|_\infty + \|f'\|_\infty$ so by dominated convergence we have $\mu_s\mathcal{A}f_n \to \mu_s\mathcal{A}f$ when $n \to \infty$ for all $s \in [0,t]$, and then $\int_0^t \mu_s\mathcal{A}f_n\,\mathrm{d}s \to \int_0^t \mu_s\mathcal{A}f\,\mathrm{d}s$.

For $f \in C_b^1(\mathbb{R}_+)$ we have $\mathcal{A}f \in C_b(\mathbb{R}_+)$ so $s \mapsto \mu_s\mathcal{A}f$ is continuous and using (4) we deduce that

$$\frac{\mu_{t+h}f - \mu_t f}{h} = \frac{1}{h}\int_t^{t+h} \mu_s\mathcal{A}f\,\mathrm{d}s \to \mu_t\mathcal{A}f \qquad \text{when } h \to 0.$$

□

We give now another equivalent notion of weak solutions to Equation (1), which will be useful to prove uniqueness. Compared to (4), it uses test functions which depend on both variables t and a.

Proposition 3. *A family $(\mu_t)_{t\geqslant 0} \subset \mathcal{M}(\mathbb{R}_+)$ is a solution to Equation (1) in the sense of Definition 1 if and only if the mapping $t \mapsto \mu_t f$ is narrowly continuous and for all $\varphi \in C_c^1(\mathbb{R}_+ \times \mathbb{R}_+)$*

$$\int_0^\infty \int_{\mathbb{R}_+} \big[\partial_t \varphi(t,a) + \partial_a \varphi(t,a) - \beta(a)\varphi(t,a) + \beta(a)\varphi(t,0)\big] \mathrm{d}\mu_t(a)\,\mathrm{d}t + \int_{\mathbb{R}_+} \varphi(0,a)\,\mathrm{d}\mu^{\mathrm{in}}(a) = 0. \qquad (6)$$

This is also true by replacing $\varphi \in C_c^1(\mathbb{R}_+ \times \mathbb{R}_+)$ by $\varphi \in C_b^1(\mathbb{R}_+ \times \mathbb{R}_+)$ with compact support in time.

Proof. Assume that $(\mu_t)_{t\geqslant 0}$ satisfies Definition (1) and let $\varphi \in C_b^1(\mathbb{R}_+ \times \mathbb{R}_+)$ compactly supported in time. As we have seen in the proof of Proposition 2, we can use $\partial_t \varphi(t,\cdot) \in C_b^1(\mathbb{R}_+)$ as a test function in (4). After integration in time we get

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}_+} \partial_t \varphi(t,a)\mathrm{d}\mu_t(a)\mathrm{d}t &= \int_0^\infty \int_{\mathbb{R}_+} \partial_t \varphi(t,a)\mathrm{d}\mu^{\mathrm{in}}(a)\mathrm{d}t \\
&\quad + \int_0^\infty \int_0^t \int_{\mathbb{R}_+} \big[\partial_a \partial_t \varphi(t,a) - \beta(a)\partial_t \varphi(t,a) + \beta(a)\partial_t\varphi(t,0)\big] \mathrm{d}\mu_s(a)\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_{\mathbb{R}_+} \Big(\int_0^\infty \partial_t \varphi(t,a) dt \Big) \mathrm{d}\mu^{\mathrm{in}}(a) \\
&\quad + \int_0^\infty \int_{\mathbb{R}_+} \Big(\int_s^\infty \big[\partial_t \partial_a \varphi(t,a) - \beta(a)\partial_t\varphi(t,a) + \beta(a)\partial_t \varphi(t,0)\big]\mathrm{d}t \Big) \mathrm{d}\mu_s(a)\,\mathrm{d}s \\
&= -\int_{\mathbb{R}_+} \varphi(0,a)\mathrm{d}\mu^{\mathrm{in}}(a) - \int_0^\infty \int_{\mathbb{R}_+} \big[\partial_a \varphi(s,a) - \beta(a)\varphi(s,a) + \beta(a)\varphi(s,0)\big] \mathrm{d}\mu_s(a)\mathrm{d}s.
\end{aligned}$$

Reciprocally let $T > 0$, $f \in C_c^1(\mathbb{R}_+)$, and assume that $t \mapsto \mu_t$ is narrowly continuous and satisfies (6). We use the function ρ defined in the proof of Proposition 2 to define for $n \in \mathbb{N}$ and $t \geqslant 0$, $\rho_n(t) = \rho\big(n(t-T)\big)$. It is a decreasing sequence of decreasing $C_c^1(\mathbb{R}_+)$ functions which converges pointwise to $\mathbb{1}_{[0,T]}(t)$, and ρ_n' (seen as an element of $\mathcal{M}(\mathbb{R}_+)$) converges narrowly to $-\delta_T$. Using $\varphi_n(t,a) = \rho_n(t) f(a)$ as a test function in (6) we get

$$\begin{aligned}
0 &= \int_0^\infty \int_{\mathbb{R}_+} \big[\rho_n'(t) f(a) + \rho_n(t)\mathcal{A}f(a)\big] \mathrm{d}\mu_t(a)\,\mathrm{d}t + \int_{\mathbb{R}_+} f(a)\,\mathrm{d}\mu^{\mathrm{in}}(a) \\
&= \int_0^\infty \rho_n'(t)\,(\mu_t f)\,\mathrm{d}t + \int_0^\infty \rho_n(t)\,(\mu_t \mathcal{A} f)\,\mathrm{d}t + \mu^{\mathrm{in}} f \xrightarrow[n\to\infty]{} -\mu_T f + \int_0^T \mu_t \mathcal{A} f\,\mathrm{d}t + \mu^{\mathrm{in}} f.
\end{aligned}$$

□

3 The dual renewal equation

To build a measure solution to Equation (1) we use a duality approach. This is well suited for such problems, and the method we use in Section 5 to study the asymptotic behavior crucially relies on this approach. The idea is to start with the dual problem: find a solution to the dual renewal equation

$$\partial_t f(t,a) - \partial_a f(t,a) + \beta(a) f(t,a) = \beta(a) f(t,0), \qquad f(0,a) = f_0(a). \qquad (7)$$

As for the direct problem, we first give a weak definition of solutions for (7) by using the method of characteristics. Assume that $f \in C^1(\mathbb{R}_+ \times \mathbb{R}_+)$ satisfies (7) in the classical sense. Then easy computations show that for all $a \geqslant 0$ the function $\psi(t) = f(t, a-t)$ is solution to the ordinary differential equation $\psi'(t) + \beta(a-t)\psi(t) = 0$. After integration we get that f satisfies

$$f(t, a-t) = f_0(a)\mathrm{e}^{-\int_0^t \beta(a-u)\mathrm{d}u} + \int_0^t \beta(a-\tau)\mathrm{e}^{-\int_\tau^t \beta(a-u)\mathrm{d}u} f(\tau,0)\,\mathrm{d}\tau$$

and the change of variable $a \leftarrow a+t$ leads to the following definition.

Definition 4. *We say that $f \in C_b(\mathbb{R}_+ \times \mathbb{R}_+)$ is a solution to (7) when, for all $t, a \geqslant 0$,*

$$f(t,a) = f_0(a+t)\mathrm{e}^{-\int_0^t \beta(a+u)\mathrm{d}u} + \int_0^t \mathrm{e}^{-\int_0^\tau \beta(a+u)\mathrm{d}u}\beta(a+\tau)f(t-\tau,0)\,\mathrm{d}\tau \tag{8}$$

$$= f_0(a+t)\mathrm{e}^{-\int_a^{a+t} \beta(u)\mathrm{d}u} + \int_a^{t+a} \mathrm{e}^{-\int_a^\tau \beta(u)\mathrm{d}u}\beta(\tau)f(a+t-\tau,0)\,\mathrm{d}\tau. \tag{9}$$

Formulations (8) and (9) are the same up to changes of variables, but both will be useful in the sequel.

Theorem 5. *For all $f_0 \in C_b(\mathbb{R}_+)$, there exists a unique $f \in C_b(\mathbb{R}_+ \times \mathbb{R}_+)$ solution to (7). Additionally*

for all $t \geqslant 0$, $\|f(t,\cdot)\|_\infty \leqslant \|f_0\|_\infty$,
if $f_0 \geqslant 0$ then for all $t \geqslant 0$, $f(t,\cdot) \geqslant 0$,
if $f_0 \in C_b^1(\mathbb{R}_+)$ then for all $T > 0$, $f \in C_b^1([0,T] \times \mathbb{R}_+)$.

Proof. To prove the existence and uniqueness of a solution we use the Banach fixed point theorem. For $f_0 \in C_b(\mathbb{R}_+)$ we define the operator $\Gamma : C_b([0,T] \times \mathbb{R}_+) \to C_b([0,T] \times \mathbb{R}_+)$ by

$$\Gamma f(t,a) = f_0(t+a)\mathrm{e}^{-\int_0^t \beta(a+u)\mathrm{d}u} + \int_0^t \mathrm{e}^{-\int_0^\tau \beta(a+u)\mathrm{d}u}\beta(a+\tau)f(t-\tau,0)\,\mathrm{d}\tau.$$

We easily have

$$\|\Gamma f - \Gamma g\|_\infty \leqslant T\|\beta\|_\infty \|f-g\|_\infty,$$

so Γ is a contraction if $T < \|\beta\|_\infty$ and there is a unique fixed point in $C_b([0,T] \times \mathbb{R}_+)$. Additionally, since $\|f\|_\infty \leqslant \|f_0\|_\infty$ implies $\|\Gamma f\|_\infty \leqslant \|f_0\|_\infty$, the closed ball of radius $\|f_0\|_\infty$ is invariant under Γ and the unique fixed point necessarily belongs to this ball. Iterating on $[T,2T]$, $[2T,3T]$, ..., we obtain a unique global solution in $C_b(\mathbb{R}_+ \times \mathbb{R}_+)$, and this solution satisfies $\|f\|_\infty \leqslant \|f_0\|_\infty$. Since Γ preserves non-negativity if $f_0 \geqslant 0$, we get similarly that f is nonnegative when f_0 is nonnegative.

If $f_0 \in C_b^1(\mathbb{R}_+)$ we can do the fixed point in $C_b^1([0,T] \times \mathbb{R}_+)$ endowed with the norm

$$\|f\|_{C_b^1} := \|f\|_\infty + \|\partial_t f\|_\infty + \|\partial_a f\|_\infty.$$

We have

$$\Gamma f(t,a) = f_0(t+a)\mathrm{e}^{-\int_a^{t+a} \beta(u)\mathrm{d}u} + \int_a^{t+a} \mathrm{e}^{-\int_a^\tau \beta(u)\mathrm{d}u}\beta(\tau)f(t-\tau+a,0)\,\mathrm{d}\tau$$

so by differentiation we get

$$\begin{aligned}\partial_t\Gamma f(t,a) &= [f_0'(t+a) - \beta(t+a)f_0(t+a)]\mathrm{e}^{-\int_a^{t+a} \beta(u)\mathrm{d}u}\\ &\quad + \mathrm{e}^{-\int_a^{t+a} \beta(u)\mathrm{d}u}\beta(t+a)f_0(0) + \int_a^{t+a} \mathrm{e}^{-\int_a^\tau \beta(u)\mathrm{d}u}\beta(\tau)\partial_t f(t-\tau+a,0)\,\mathrm{d}\tau\\ &= \mathcal{A}f_0(t+a)\mathrm{e}^{-\int_a^{t+a} \beta(u)\mathrm{d}u} + \int_a^{t+a} \mathrm{e}^{-\int_a^\tau \beta(u)\mathrm{d}u}\beta(\tau)\partial_t f(t-\tau+a,0)\,\mathrm{d}\tau. \end{aligned} \tag{10}$$

and

$$\begin{aligned}\partial_a\Gamma f(t,a) &= \big[f_0'(t+a) + (\beta(a)-\beta(t+a))f_0(t+a)\big]\mathrm{e}^{-\int_a^{t+a} \beta(u)\mathrm{d}u} + \mathrm{e}^{-\int_a^{t+a} \beta(u)\mathrm{d}u}\beta(t+a)f_0(0) - \beta(a)f(t,0)\\ &\quad + \int_a^{t+a} \beta(a)\mathrm{e}^{-\int_a^\tau \beta(u)\mathrm{d}u}\beta(\tau)f(t-\tau+a,0)\,\mathrm{d}\tau + \int_a^{t+a} \mathrm{e}^{-\int_a^\tau \beta(u)\mathrm{d}u}\beta(\tau)\partial_t f(t-\tau+a,0)\,\mathrm{d}\tau\\ &= \big[f_0'(t+a) + (\beta(a)-\beta(t+a))(f_0(t+a)-f_0(0))\big]\mathrm{e}^{-\int_a^{t+a} \beta(u)\mathrm{d}u}\\ &\quad + \int_a^{t+a} \mathrm{e}^{-\int_a^\tau \beta(u)\mathrm{d}u}(\beta(\tau)-\beta(a))\partial_t f(t-\tau+a,0)\,\mathrm{d}\tau.\end{aligned}$$

Finally we get for $h = f - g$, which satisfies $h(0,\cdot) \equiv 0$,

$$\|\Gamma h\|_{C_b^1} \leqslant T\|\beta\|_\infty \left[\|h\|_\infty + 2\|\partial_t h\|_\infty\right] \leqslant 2T\|\beta\|_\infty \|h\|_{C_b^1}.$$

We conclude that if $f_0 \in C_b^1(\mathbb{R}_+)$ then the unique solution belongs to $C_b^1([0,T] \times \mathbb{R}_+)$ for all $T > 0$, and satisfies (7) in the classical sense. As a consequence it also satisfies

$$f(t,a) = f_0(a) + \int_0^t \left[\partial_a f(s,a) + \beta(a)(f(s,0) - f(s,a))\right] \mathrm{d}s = f_0(a) + \int_0^t \mathcal{A}f(s,a)\,\mathrm{d}s. \tag{11}$$

□

Lemma 6. *Let $f_0, g_0 \in C_b(\mathbb{R}_+)$ such that*

$$\exists A > 0,\ \forall a \in [0,A],\ f_0(a) = g_0(a)$$

and let f and g the solutions to Equation (7) with initial distributions f_0 and g_0 respectively. Then for all $T \in (0,A)$ we have

$$\forall t \in [0,T],\ \forall a \in [0, A-T], \qquad f(t,a) = g(t,a).$$

Proof. The closed subspace $\{f \in C_b([0,T] \times \mathbb{R}_+),\ \forall t \in [0,T], \forall a \in [0,A-T],\ f(t,a) = g(t,a)\}$ is invariant under Γ if f_0 satisfies $\forall a \in [0,A],\ f_0(a) = g_0(a)$. □

Proposition 7. *The family of operators $(M_t)_{t\geqslant 0}$ defined on $C_b(\mathbb{R}_+)$ by $M_t f_0 = f(t,\cdot)$ is a semigroup, i.e.*

$$M_0 f = f \qquad \text{and} \qquad M_{t+s} f = M_t(M_s f).$$

Additionally it is a positive and conservative contraction, i.e for all $t \geqslant 0$ we have

$$\begin{aligned}
&f \geqslant 0 \implies M_t f \geqslant 0,\\
&M_t 1 = 1,\\
&\forall f \in C_b(\mathbb{R}_+), \quad \|M_t f\|_\infty \leqslant \|f\|_\infty,
\end{aligned}$$

and it commutes with the differential operator $\mathcal{A}$, i.e.

$$\forall f \in C_b^1(\mathbb{R}_+), \quad M_t \mathcal{A} f = \mathcal{A} M_t f.$$

Proof. Let $f \in C_b(\mathbb{R}_+)$, fix $s \geqslant 0$, and define $g(t,a) = M_t(M_s f)(a)$ and $h(t,a) = M_{t+s}f(a)$. We have $g(0,a) = h(0,a)$ and

$$g(t,a) = M_s f(t+a)\mathrm{e}^{-\int_0^t \beta(a+u)\mathrm{d}u} + \int_0^t \mathrm{e}^{-\int_0^\tau \beta(a+u)\mathrm{d}u} \beta(a+\tau) g(t-\tau,0)\,\mathrm{d}\tau$$

and

$$\begin{aligned}
h(t,a) &= f(t+s+a)\mathrm{e}^{-\int_0^{t+s} \beta(a+u)du} + \int_0^t \mathrm{e}^{-\int_0^\tau \beta(a+u)\mathrm{d}u} \beta(a+\tau) h(t-\tau,0)\,\mathrm{d}\tau\\
&\quad + \int_t^{t+s} \mathrm{e}^{-\int_0^\tau \beta(a+u)\mathrm{d}u} \beta(a+\tau) M_{t+s-\tau} f(0)\,\mathrm{d}\tau\\
&= f(t+s+a)\mathrm{e}^{-\int_0^{t+s} \beta(a+u)\mathrm{d}u} + \int_0^t \mathrm{e}^{-\int_0^\tau \beta(a+u)\mathrm{d}u} \beta(a+\tau) h(t-\tau,0)\,\mathrm{d}\tau\\
&\quad + \int_0^s \mathrm{e}^{-\int_0^{\tau+t} \beta(a+u)\mathrm{d}u} \beta(a+\tau+t) M_{s-\tau} f(0)\,\mathrm{d}\tau\\
&= \left[f(t+a+s)\mathrm{e}^{-\int_0^s \beta(t+a+u)\mathrm{d}u} + \int_0^s \mathrm{e}^{-\int_0^\tau \beta(t+a+u)\mathrm{d}u} \beta(t+a+\tau) M_{s-\tau} f(0)\,d\tau\right] \mathrm{e}^{-\int_0^t \beta(a+u)\mathrm{d}u}\\
&\quad + \int_0^t \mathrm{e}^{-\int_0^\tau \beta(a+u)\mathrm{d}u} \beta(a+\tau) h(t-\tau,0)\,\mathrm{d}\tau\\
&= M_s f(t+a)\mathrm{e}^{-\int_0^t \beta(a+u)\mathrm{d}u} + \int_0^t \mathrm{e}^{-\int_0^\tau \beta(a+u)\mathrm{d}u} \beta(a+\tau) h(t-\tau,0)\,\mathrm{d}\tau.
\end{aligned}$$

By uniqueness of the fixed point we deduce that $g(t,a) = h(t,a)$ for all $t \geqslant 0$.

The conservativeness $M_t 1 = 1$ is straightforward computations and the positivity and the contraction property follow immediately from Theorem 5.

It remains to prove the commutativity for $f \in C_b^1(\mathbb{R}_+)$. From (10) we deduce that $\partial_t M_t f$ is the fixed point of Γ with initial data $\mathcal{A}f$. By uniqueness of the fixed point we deduce that $\partial_t M_t f = M_t \mathcal{A} f$. But since $f \in C_b^1(\mathbb{R}_+)$, $M_t f$ satisfies (7) in the classical sense, *i.e.* $\partial_t M_t f(a) = \mathcal{A} M_t f(a)$, and the proof is complete. □

Remark. *The semigroup $(M_t)_{t\geqslant 0}$ is not strongly continuous, i.e. we do not have $\lim_{t\to 0} \|M_t f - f\|_\infty = 0$ for all $f \in C_b(\mathbb{R}_+)$. However the restriction of the semigroup to the invariant subspace of bounded and uniformly continuous functions is strongly continuous.*

Proposition 8. *Assume that β satisfies (3) and is additionally uniformly continuous on $\mathbb{R}_+$. Then the operator $\mathcal{A}$ is the generator of the semigroup $(M_t)_{t\geqslant 0}$ in the sense that*

$$\forall f \in C_c^1(\mathbb{R}_+), \qquad \Big\| \frac{1}{t}(M_t f - f) - \mathcal{A} f \Big\|_\infty \to 0 \quad \text{when } t \to 0.$$

Proof. Let $f \in C_c^1(\mathbb{R}_+)$. Using (11) we can write

$$\Big\| \frac{1}{t}(M_t f - f) - \mathcal{A} f \Big\|_\infty \leqslant \frac{1}{t} \int_0^t \|\mathcal{A} M_s f - \mathcal{A} f\|_\infty \, \mathrm{d}s.$$

It remains to check that $\|\mathcal{A} M_s f - \mathcal{A} f\|_\infty \to 0$ when $s \to 0$. Since f is uniformly continuous we have $\|M_t f - f\|_\infty \to 0$ when $t \to 0$. Additionally we have

$$\partial_a M_t f(a) - f'(a) = \big[f'(t+a)\mathrm{e}^{-\int_a^{a+t}\beta(u)\mathrm{d}u} - f'(a)\big] + \big[(\beta(a) - \beta(t+a))(f(t+a) - f(0))\big]\mathrm{e}^{-\int_a^{t+a}\beta(u)\mathrm{d}u}$$
$$+ \int_a^{t+a} \mathrm{e}^{-\int_a^\tau \beta(u)\mathrm{d}u}(\beta(\tau) - \beta(a))\partial_t M_{t-\tau+a} f(0)\,\mathrm{d}\tau.$$

Since $f \in C_c^1(\mathbb{R}_+)$ we have that $f' \in C_c(\mathbb{R}_+)$ is uniformly continuous so, using also that β is bounded,

$$\big\| f'(t+\cdot)e^{-\int_\cdot^{\cdot+t}\beta(u)\mathrm{d}u} - f'(\cdot)\big\|_\infty \to 0 \quad \text{when} \quad t \to 0.$$

Similarly since β is uniformly continuous and f is bounded we have

$$\big\| \big[(\beta(\cdot) - \beta(t+\cdot))(f(t+\cdot) - f(0))\big]\mathrm{e}^{-\int_\cdot^{\cdot+t}\beta(u)\mathrm{d}u}\big\|_\infty \to 0 \quad \text{when} \quad t \to 0.$$

The last term also converges uniformly to 0 when $t \to 0$ since the integrand is bounded. Finally we can conclude that $\|\mathcal{A} M_s f - \mathcal{A} f\|_\infty \to 0$ when $s \to 0$ and then $\big\| \frac{1}{t}(M_t f - f) - \mathcal{A} f \big\|_\infty \to 0$ when $t \to 0$. □

4 Existence and uniqueness of a measure solution

We are now ready to prove the existence and uniqueness of a measure solution to Equation (1). We define the dual semigroup $(M_t)_{t\geqslant 0}$ on $\mathcal{M}(\mathbb{R}_+) = C_0(\mathbb{R}_+)'$ by

$$\mu M_t : f \in C_0(\mathbb{R}_+) \mapsto \mu(M_t f) = \int_{\mathbb{R}_+} M_t f \, \mathrm{d}\mu.$$

In other words we have by definition

$$\forall f \in C_0(\mathbb{R}_+), \qquad (\mu M_t) f = \mu(M_t f), \qquad i.e. \quad \int_{\mathbb{R}_+} f \, \mathrm{d}(\mu M_t) = \int_{\mathbb{R}_+} M_t f \, \mathrm{d}\mu.$$

The following lemma ensures that this identity is also satsified for $f \in C_b(\mathbb{R}_+)$. From now on we will denote without ambiguity the quantity $(\mu M_t) f = \mu(M_t f)$ by $\mu M_t f$.

Lemma 9. *For all $f \in C_b(\mathbb{R}_+)$ we have $(\mu M_t)f = \mu(M_t f)$.*

Proof. The identity is true by definition for any $f \in C_0(\mathbb{R}_+)$. It is easy to check that if $f \in C_0(\mathbb{R}_+)$ then $\Gamma(f) \in C_0(\mathbb{R}_+)$, so that $f \in C_0(\mathbb{R}_+) \implies M_t f \in C_0(\mathbb{R}_+)$. Let $f \in C_b(\mathbb{R}_+)$ and f_n as defined in (2). The sequence $(f_n)_{n\in\mathbb{N}}$ lies in $C_0(\mathbb{R}_+)$ so $(\mu M_t)f_n = \mu(M_t f_n)$ for all $n \in \mathbb{N}$. By monotone convergence we clearly have $(\mu M_t)f_n \to (\mu M_t)f$. We will prove that $\mu(M_t f_n) \to \mu(M_t f)$ by dominated convergence. First by positivity of M_t we have $|M_t f_n(a)| \leqslant M_t|f_n|(a) \leqslant M_t|f|(a) \leqslant \|f\|_\infty$. Additionally since $f_n(a) = f(a)$ for all $a \in [0, n]$, we have already seen that for all $a \in [0, n-t]$, $M_t f_n(a) = M_t f(a)$, so for all $a \geqslant 0$ we have $M_t f_n(a) \to M_t f(a)$. □

Proposition 10. *The left semigroup $(M_t)_{t\geqslant 0}$ is a positive and conservative contraction, i.e for all $t \geqslant 0$ we have*

$$\mu \geqslant 0 \quad \implies \quad \mu M_t \geqslant 0 \quad \textit{and} \quad \|\mu M_t\|_{TV} = \|\mu\|_{TV},$$

$$\forall \mu \in \mathcal{M}(\mathbb{R}_+), \qquad \|\mu M_t\|_{TV} \leqslant \|\mu\|_{TV}.$$

Proof. It is a consequence of Proposition 7. If $\mu \geqslant 0$, then for any $f \geqslant 0$ we have $(\mu M_t)f = \mu(M_t f) \geqslant 0$. Additionnally $\|\mu M_t\|_{TV} = (\mu M_t)1 = \mu(M_t 1) = \mu 1 = \|\mu\|_{TV}$. For $\mu \in \mathcal{M}(\mathbb{R}_+)$ not necessarily positive, we have

$$\|\mu M_t\|_{TV} = \sup_{\|f\|_\infty \leqslant 1} |\mu M_t f| \leqslant \sup_{\|g\|_\infty \leqslant 1} |\mu g| = \|\mu\|_{TV}.$$

□

Remark. *The left semigroup is not strongly continuous, i.e. we do not have $\lim_{t\to 0} \|\mu M_t - \mu\|_{TV} = 0$ for all $\mu \in \mathcal{M}(\mathbb{R}_+)$. This is due to the non continuity of the transport semigroup for the total variation distance: for instance for $a \in \mathbb{R}$ we have $\|\delta_{a+t} - \delta_a\|_{TV} = 2$ for any $t > 0$. But the left semigroup is weak* continuous. This is an immediate consequence of the strong continuity of the right semigroup on the space of bounded and uniformly continuous functions (which contains $C_0(\mathbb{R}_+)$). The left semigroup is even narrowly continuous as we will see in the proof of the next theorem.*

Theorem 11. *For any $\mu^{\text{in}} \in \mathcal{M}(\mathbb{R}_+)$, the orbit map $t \mapsto \mu^{\text{in}} M_t$ is the unique measure solution to Equation (1).*

Proof. Existence. In this part we use Proposition 7 at different places. Let $\mu \in \mathcal{M}(\mathbb{R}_+)$. We start by checking that $t \mapsto \mu M_t$ is narrowly continuous. Let $f \in C_b(\mathbb{R}_)$. Due to the semigroup property, it is sufficient to check that $\lim_{t\to 0} \mu M_t f = \mu f$. But from (8) we have

$$|\mu M_t f - \mu f| \leqslant \underbrace{\left| \mu f - \int_{\mathbb{R}_+} f(a+t) \mathrm{e}^{-\int_0^t \beta(a+u)\mathrm{d}u} \mathrm{d}\mu(a) \right|}_{\xrightarrow[t\to 0]{} 0 \text{ by dominated convergence}} + \underbrace{\left| \mu \int_0^t \mathrm{e}^{-\int_0^\tau \beta(\cdot+u)\mathrm{d}u} \beta(\cdot+\tau) M_{t-\tau} f(0)\,\mathrm{d}\tau \right|}_{\leqslant \|\mu\|_{TV} \|\beta\|_\infty \|f\|_\infty t \xrightarrow[t\to 0]{} 0}.$$

Now we check that $(\mu M_t)_{t\geqslant 0}$ satisfies (4). Let $f \in C_c^1(\mathbb{R}_+)$. Starting from (11) and using the Fubini's theorem we have

$$(\mu M_t)f = \mu(M_t f) = \mu f + \mu \int_0^t \mathcal{A} M_s f\,\mathrm{d}s = \mu f + \mu \int_0^t M_s \mathcal{A} f\,\mathrm{d}s = \mu f + \int_0^t (\mu M_s)\mathcal{A} f\,\mathrm{d}s.$$

Alternatively we can prove that μM_t satisfies (5). For $f \in C_b^1(\mathbb{R}_+)$ and $t \geqslant 0$ we can write

$$\frac{1}{h}\big(\mu M_{t+h} f - \mu M_t f\big) = \int_{\mathbb{R}_+} \frac{1}{h}\big(M_h M_t f - M_t f\big)\,\mathrm{d}\mu \xrightarrow[h\to 0]{} \int_{\mathbb{R}_+} \mathcal{A} M_t f\,\mathrm{d}\mu = \mu M_t \mathcal{A} f$$

by dominated convergence, since for all $h \in]0, 1]$ and all $a \geqslant 0$ we have

$$\left| \frac{1}{h}\big(M_{t+h} f(a) - M_t f(a)\big) \right| \leqslant \sup_{(t',a')\in[t,t+1]\times\mathbb{R}_+} \left| \partial_t\, M_{t'} f(a') \right| \ \in L^1(\mathbb{R}_+, \mathrm{d}\mu).$$

Uniqueness. We first give a proof which uses Proposition 8 (and so requires the uniform continuity of β). We want to check that if $t \mapsto \mu_t$ is a measure solution to Equation (1), then $\mu_t = \mu^{\rm in} M_t$ for all $t \geqslant 0$. We consider for $t > 0$ fixed and $f \in C_c^1(\mathbb{R}_+)$ the mapping $s \mapsto \mu_s M_{t-s} f$ and we compute its derivative. For $0 < s < s+h < t$ we have

$$\frac{1}{h}\big(\mu_{s+h}M_{t-s-h}f - \mu_s M_{t-s}f\big) = \frac{1}{h}\big(\mu_{s+h}M_{t-s}f - \mu_s M_{t-s}f\big) - \frac{1}{h}\big(\mu_{s+h}M_{t-s}f - \mu_{s+h}M_{t-s-h}f\big).$$

Since $M_{t-s}f \in C_b^1(\mathbb{R}_+)$ we get from Proposition 2 that

$$\frac{1}{h}\big(\mu_{s+h}M_{t-s}f - \mu_s M_{t-s}f\big) \xrightarrow[h\to 0]{} \mu_s \mathcal{A} M_{t-s}f.$$

For the second term we use Proposition 8 and the weak continuity of $t \mapsto \mu_t$ to write

$$\begin{aligned}
&\Big|\frac{1}{h}\big(\mu_{s+h}M_{t-s}f - \mu_{s+h}M_{t-s-h}f\big) - \mu_s\mathcal{A}M_{t-s}f\Big| \\
&\qquad\leqslant \Big|\frac{1}{h}\big(\mu_{s+h}M_{t-s}f - \mu_{s+h}M_{t-s-h}f\big) - \mu_{s+h}\mathcal{A}M_{t-s}f\Big| + \big|\mu_{s+h}\mathcal{A}M_{t-s}f - \mu_s\mathcal{A}M_{t-s}f\big| \\
&\qquad\leqslant \sup_{0\leqslant u\leqslant t}\operatorname{ess}\|\mu_u\|_{TV}\ \underbrace{\Big\|\frac{1}{h}\big(M_{t-s}f - M_{t-s-h}f\big) - \mathcal{A}M_{t-s}f\Big\|_\infty}_{\xrightarrow[h\to 0]{}0} + \underbrace{\big|\mu_{s+h}\mathcal{A}M_{t-s}f - \mu_s\mathcal{A}M_{t-s}f\big|}_{\xrightarrow[h\to 0]{}0}.
\end{aligned}$$

We conclude that $\frac{\mathrm{d}}{\mathrm{d}s}\big(\mu_s M_{t-s}f\big) = 0$, so $s \mapsto \mu_s M_{t-s}f$ is constant and $\mu_t f = \mu_0 M_t f = \mu^{\rm in} M_t f$.

We can also prove the uniqueness by using Proposition (3) (and so without needing β to be uniformly continuous). By linearity we can assume that $\mu^{\rm in} = 0$ and we want to prove that the unique family $(\mu_t)_{t\geqslant 0}$ which satisfies

$$\int_0^\infty \int_{\mathbb{R}_+} \big[\partial_t\varphi(t,a) + \partial_a\varphi(t,a) - \beta(a)\varphi(t,a) + \beta(a)\varphi(t,0)\big]\,\mathrm{d}\mu_t(a)\,\mathrm{d}t = 0$$

for all $\varphi \in C_b^1(\mathbb{R}_+\times\mathbb{R}_+)$ with compact support in time, is the trivial family. If we can prove that for all $\psi \in C_c^1(\mathbb{R}_+\times\mathbb{R}_+)$ there exists $\varphi \in C_b^1(\mathbb{R}_+\times\mathbb{R}_+)$ compactly supported in time such that for all $t, a \geqslant 0$

$$\partial_t\varphi(t,a) + \partial_a\varphi(t,a) - \beta(a)\varphi(t,a) + \beta(a)\varphi(t,0) = \psi(t,a), \tag{12}$$

then we get the conclusion. Let $\psi \in C_c^1(\mathbb{R}_+\times\mathbb{R}_+)$ and let $T > 0$ such that $\operatorname{supp}\psi \subset [0,T)\times\mathbb{R}_+$. Using the same method as for (7), we can prove the existence of a solution $\varphi \in C_b^1([0,T]\times\mathbb{R}_+)$ to (12) with terminal condition $\varphi(T,a) = 0$. Since $\psi \in C_c^1([0,T)\times\mathbb{R}_+)$ we easily check that the extension of $\varphi(t,a)$ by 0 for $t > T$ belongs to $\varphi \in C_b^1(\mathbb{R}_+\times\mathbb{R}_+)$, is compactly supported in time, and satisfies (12).

□

5 Exponential convergence to the invariant measure

In this last section we prove the exponential convergence of the measure solutions to a stationnary distribution. The proof relies on a coupling argument expressed by a so-called Doeblin's condition. This condition guarantees that the total variation distance between two solutions with the same mass decreases exponentially fast. Choosing the invariant measure as a reference then allows to get the asymptotic stability result.

Theorem 12. *Assume that the semigroup $(M_t)_{t\geqslant 0}$ satisfies the Doeblin's condition*

$$\exists\, t_0 > 0,\ 0 < c < 1,\ \nu \text{ probability measure},\ \forall f \geqslant 0,\ \forall a \geqslant 0, \quad M_{t_0}f(a) \geqslant c\,(\nu f).$$

Then for $\alpha := \frac{-\log(1-c)}{t_0} > 0$ we have for all $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}_+)$ such that $\mu_1(\mathbb{R}_+) = \mu_2(\mathbb{R}_+)$

$$\forall t \geqslant 0, \qquad \|\mu_1 M_t - \mu_2 M_t\|_{TV} \leqslant \mathrm{e}^{-\alpha(t-t_0)}\|\mu_1 - \mu_2\|_{TV}.$$

Proof. Step 1: We prove the result for $\mu_1 = \delta_{a_1}$ et $\mu_2 = \delta_{a_2}$ for any $a_1, a_2 \geqslant 0$. Let $f \in C_0(\mathbb{R}_+)$ and write for $t \geqslant t_0$

$$\begin{aligned} M_t f(a_1) - M_t f(a_2) &= (1-c)\Big(\frac{M_t f(a_1) - c(\nu f)}{1-c} - \frac{M_t f(a_2) - c(\nu f)}{1-c}\Big) \\ &= (1-c)\big[M_{t-t_0} U f(a_1) - M_{t-t_0} U f(a_2)\big] \end{aligned}$$

where we have set $U := \frac{1}{1-c}(M_{t_0} - c\,\nu) \in \mathcal{L}(C_b(\mathbb{R}_+))$. Since $U \geqslant 0$ and $U1 = 1$ we have $\|Uf\|_\infty \leqslant \|f\|_\infty$. We deduce that

$$\begin{aligned} \sup_{f,\,\|f\|_\infty \leqslant 1} \big|M_t f(a_1) - M_t f(a_2)\big| &\leqslant (1-c) \sup_{f,\,\|f\|_\infty \leqslant 1} \big|M_{t-t_0} U f(a_1) - M_{t-t_0} U f(a_2)\big| \\ &\leqslant (1-c) \sup_{f,\,\|f\|_\infty \leqslant 1} \big|M_{t-t_0} f(a_1) - M_{t-t_0} f(a_2)\big| \end{aligned}$$

which is exactly

$$\|\delta_{a_1} M_t - \delta_{a_2} M_t\|_{TV} \leqslant (1-c)\|\delta_{a_1} M_{t-t_0} - \delta_{a_2} M_{t-t_0}\|_{TV}.$$

By induction we get for $n = \lfloor \frac{t}{t_0} \rfloor$

$$\|\delta_{a_1} M_t - \delta_{a_2} M_t\|_{TV} \leqslant (1-c)^n \|\delta_{a_1} M_{t-nt_0} - \delta_{a_2} M_{t-nt_0}\|_{TV} \leqslant \mathrm{e}^{n\log(1-c)}\|\delta_{a_1} - \delta_{a_2}\|_{TV}$$

and this gives the conclusion since

$$n\log(1-c) \leqslant \Big(\frac{t}{t_0} - 1\Big)\log(1-c) = -\alpha(t-t_0).$$

Step 2: We extend the result to general measures. Let $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}_+)$ such that $\mu_1(\mathbb{R}_+) = \mu_2(\mathbb{R}_+)$, so that $(\mu_1 - \mu_2)_+(\mathbb{R}_+) = (\mu_2 - \mu_1)_+(\mathbb{R}_+) = \frac{1}{2}\|\mu_1 - \mu_2\|_{TV}$. For any $t \geqslant 0$ we have

$$\begin{aligned} \|\mu_1 M_t - \mu_2 M_t\|_{TV} &= \|(\mu_1 - \mu_2)_+ M_t + (\mu_1 - \mu_2)_- M_t\|_{TV} \\ &= \|(\mu_1 - \mu_2)_+ M_t - (\mu_2 - \mu_1)_+ M_t\|_{TV} \\ &= \sup_f \Big| \int_a M_t f(a) \mathrm{d}(\mu_1 - \mu_2)_+(a) - \int_{a'} M_t f(a') \mathrm{d}(\mu_2 - \mu_1)_+(a')\Big| \\ &= \frac{1}{(\mu_1 - \mu_2)_+(\mathbb{R}_+)} \sup_f \Big| \iint_{aa'} \big[M_t f(a) - M_t f(a')\big] \mathrm{d}(\mu_1 - \mu_2)_+(a) \mathrm{d}(\mu_2 - \mu_1)_+(a')\Big| \\ &\leqslant \sup_{a,a'} \|\delta_a M_t - \delta_{a'} M_t\|_{TV}\,(\mu_1 - \mu_2)_+(\mathbb{R}_+) \\ &\leqslant \mathrm{e}^{-\alpha(t-t_0)} \sup_{a,a'} \|\delta_a - \delta_{a'}\|_{TV}\,\frac{1}{2}\|\mu_1 - \mu_2\|_{TV} = \mathrm{e}^{-\alpha(t-t_0)}\|\mu_1 - \mu_2\|_{TV}. \end{aligned}$$

□

Proposition 13. *The renewal semigroup $(M_t)_{t\geqslant 0}$ satisfies the Doeblin's condition with $t_0 = a_* + \eta$, $c = \eta\,\beta_{\min}\,\mathrm{e}^{-\beta_{\max}(a_*+\eta)}$, and ν the uniform probability measure on $[0,\eta]$, for any choice of $\eta > 0$.*

Proof. We iterate once the Duhamel formula (8) to get for any $f \geqslant 0$

$$\begin{aligned} f(t,a) &= f_0(t+a)\mathrm{e}^{-\int_0^t \beta(a+u)\mathrm{d}u} + \int_0^t \mathrm{e}^{-\int_0^\tau \beta(a+u)\mathrm{d}u} \beta(a+\tau) f(t-\tau, 0)\,\mathrm{d}\tau \\ &= f_0(t+a)\mathrm{e}^{-\int_0^t \beta(a+u)\mathrm{d}u} + \int_0^t \mathrm{e}^{-\int_0^\tau \beta(a+u)\mathrm{d}u} \beta(a+\tau) f_0(t-\tau)\mathrm{e}^{-\int_0^{t-\tau}\beta(u)\mathrm{d}u}\,\mathrm{d}\tau + (\geqslant 0) \\ &\geqslant \int_0^t \mathrm{e}^{-\int_0^\tau \beta(a+u)\mathrm{d}u} \beta(a+\tau) f_0(t-\tau)\mathrm{e}^{-\int_0^{t-\tau}\beta(u)\mathrm{d}u}\,\mathrm{d}\tau \\ &= \int_0^t \mathrm{e}^{-\int_0^{t-\tau} \beta(a+u)\mathrm{d}u} \beta(a+t-\tau) f_0(\tau)\mathrm{e}^{-\int_0^{\tau}\beta(u)\mathrm{d}u}\,\mathrm{d}\tau. \end{aligned}$$

Consider the probability measure ν defined by $\nu f = \frac{1}{\eta}\int_0^\eta f(a)\,\mathrm{d}a$ for some $\eta > 0$. Then for any $t \geqslant a_* + \eta$ and any $a \geqslant 0$ we have

$$\begin{aligned} f(t,a) &\geqslant \int_0^t \mathrm{e}^{-\int_0^{t-\tau}\beta(a+u)\mathrm{d}u}\beta(a+t-\tau)f_0(\tau)e^{-\int_0^\tau \beta(u)\mathrm{d}u}\,\mathrm{d}\tau \\ &\geqslant \int_0^\eta \mathrm{e}^{-\beta_{\max}(t-\tau)}\beta_{\min}f_0(\tau)\mathrm{e}^{-\beta_{\max}\tau}\,\mathrm{d}\tau \\ &= \eta\,\beta_{\min}\,\mathrm{e}^{-\beta_{\max}t}(\nu f_0). \end{aligned}$$

□

As we have already seen in the introduction, the conservative equation (1) admits a unique invariant probability measure, *i.e.* there exists a unique probability measure μ_∞ such that $\mu_\infty M_t = \mu_\infty$ for all $t \geqslant 0$. This probability measure has a density with respect to the Lebesgue measure

$$\mathrm{d}\mu_\infty = N(a)\,\mathrm{d}a$$

where N is explicitly given by

$$N(a) = N(0)\,\mathrm{e}^{-\int_0^a \beta(u)\mathrm{d}u}$$

with $N(0)$ such that $\int_0^\infty N(a)\,\mathrm{d}a = 1$.

Corollary 14. *For all $\mu \in \mathcal{M}(\mathbb{R}_+)$ and all $\eta > 0$ we have*

$$\forall t \geqslant 0, \qquad \|\mu M_t - (\mu 1)\mu_\infty\|_{TV} \leqslant \mathrm{e}^{-\alpha(t-t_0)}\|\mu - (\mu 1)\mu_\infty\|_{TV},$$

where $t_0 = a_* + \eta$, $c = \eta\,\beta_{\min}\,\mathrm{e}^{-\beta_{\max}(a_*+\eta)}$, and $\alpha = \frac{-\log(1-c)}{t_0}$.

Notice that in the case $a_* = 0$ we obtain by passing to the limit $\eta \to 0$

$$\forall t \geqslant 0, \qquad \|\mu M_t - (\mu 1)\mu_\infty\|_{TV} \leqslant \mathrm{e}^{-\beta_{\min}t}\,\|\mu - (\mu 1)\mu_\infty\|_{TV}.$$