A number of different exercises is proposed. You do not need to try to solve them all.

**Exercise 1** Let $R$ be a relation on a set $X$, that is, a subset of $X^2$. Let $R^0$ be the identity relation, that is, $x$ is in relation with $y$, written $xR^0y$, if $x = y$. And let $R^{n+1}$ be the composition of $R$ and $R^n$. The composition of two relations $R$ and $S$, written $R \circ S$, is the relation such that $x(R \circ S)y$ if there is $z$ such that $xRz$ and $zSy$. Hence, $xR^ny$ $(n > 0)$ if there are $n$ − 1 intermediate elements $x_1, \ldots, x_{n−1}$ such that $xRx_1, \ldots, x_{n−1}Ry$. A relation $S$ is reflexive if, for all $x$, $xSx$. A relation $S$ is transitive if, for all $x, y, z$, $xSyz$ whenever $xSy$ and $ySz$.

Prove that:
1. $R^+ = \bigcup_{i>0} R^i$ is transitive.
2. $R^+$ is the smallest transitive relation containing $R$.
3. $R^* = \bigcup_{i\geq0} R^i$ is transitive and reflexive.
4. $R^*$ is the smallest transitive and reflexive relation containing $R$.
5. The function $f(X) = X \circ X$ is monotone, that is, $f(X) \subseteq f(Y)$ if $X \subseteq Y$.
6. $R^*$ is a fixpoint of $f$, that is, $f(R^*) = R^*$.

**Exercise 2** We consider the set of $\lambda$-terms $t = x \mid \lambda xt \mid tt$, where $x$ is a variable taken in an infinite set $X$. Let $\rightarrow_\beta$ be the smallest relation such that:
- $(\lambda xt)u \rightarrow_\beta t{x\mapsto u}$, where $t{x\mapsto u}$ is the term obtained by replacing every free occurrence of $x$ in $t$ by $u$ (and renaming bound variables if necessary)
- $\lambda xt \rightarrow_\beta \lambda xt'$ if $t \rightarrow_\beta t'$
- $tu \rightarrow_\beta t'u$ if $t \rightarrow_\beta t'$
- $tu \rightarrow_\beta tu'$ if $u \rightarrow_\beta u'$

Prove that:
1. $\beta$-reduction introduces no new variables: if $t \rightarrow_\beta t'$, then $\text{FV}(t') \subseteq \text{FV}(t)$.
2. If $v \rightarrow_\beta^* v'$, then $t{y\mapsto v} \rightarrow_\beta^* t{y\mapsto v'}$. 

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3. \( \rightarrow_\beta \) is stable by substitution: if \( t \rightarrow_\beta t' \), then \( t\{y \mapsto v\} \rightarrow_\beta t'\{y \mapsto v\} \).

4. \( \rightarrow_\beta \) is locally confluent, that is, if \( t \rightarrow_\beta u \) and \( t \rightarrow_\beta v \), then there is \( w \) such that \( u \rightarrow^*_\beta w \) and \( v \rightarrow^*_\beta w \).

This last property means that, locally, the order in which we make \( \beta \)-reductions is not important, since we can always find a common reduct. An important property of \( \beta \)-reduction is that it also holds globally: if \( t \rightarrow^*_\beta u \) and \( t \rightarrow^*_\beta v \), then there is \( w \) such that \( u \rightarrow^*_\beta w \) and \( v \rightarrow^*_\beta w \). Thus, \( \beta \)-reduction is in some sense deterministic: it can lead to only one result.

**Exercise 3** Define a 2-tapes Turing machine \((\Sigma, Q, q_i, q_f, \delta)\) computing the function \( n \mapsto 2n \), where \( \Sigma \) is the finite alphabet used by the machine with at least the symbols \( \{\square, 0, 1\} \), \( Q \) is the finite set of states of the machine, \( q_i \) the initial state, \( q_f \) the final state, and \( \delta \) is a partial function from \( Q \times \Sigma^2 \) to \( Q \times \Sigma^2 \times \{-1, 0, +1\} \). To do this, you are free to introduce in \( \Sigma \) symbols different from \( \{\square, 0, 1\} \), and in \( Q \) states different from \( \{q_i, q_f\} \). The machine starts in position 0 in state \( q_i \) with, on tape 1, \( \square \) followed by \( n \) 1's and then 0's, and on tape 2, \( \square \) followed by 0's. It must end in position 0 in state \( q_f \) with, on tape 2, \( \square \) followed by \( 2n \) 1's and then 0's.

**Exercise 4** We consider a set \( \mathcal{X} \) of variables, a set \( \mathcal{F} \) of function symbols and a set \( \mathcal{P} \) of predicate symbols. We assume that every (function or predicate) symbol \( f \) is equipped with an arity \( \alpha_f \in \mathbb{N} \), that is, in a term of the form \( f(t_1, \ldots, t_n) \), we always have \( n = \alpha_f \). A model is given by a set \( A \) and, for each function symbol \( f \) of arity \( n \), a function \( f_A : A^n \to A \) and, for each predicate symbol \( P \) of arity \( n \), a function \( P_A : A^n \to \text{Bool} \) where \( \text{Bool} = \{\text{true}, \text{false}\} \).

Then, given a finite valuation \( \mu : \mathcal{X} \to A \), the interpretation of a formula \( \phi \), written \( [\phi]_\mu \), is defined as follows:

- \([x]_\mu = \mu(x)\)
- \([f(t_1, \ldots, t_n)]_\mu = f_A([t_1]_\mu, \ldots, [t_n]_\mu)\)
- \([P(t_1, \ldots, t_n)]_\mu = P_A([t_1]_\mu, \ldots, [t_n]_\mu)\)
- \([\bot]_\mu = \text{false}\)
- \([\neg \phi]_\mu = \text{not}([\phi]_\mu)\)
- \([\phi \lor \psi]_\mu = \text{or}([\phi]_\mu, [\psi]_\mu)\)
- \([\phi \land \psi]_\mu = \text{and}([\phi]_\mu, [\psi]_\mu)\)
- \([\phi \Rightarrow \psi]_\mu = \text{impl}([\phi]_\mu, [\psi]_\mu)\)
- \([\forall x \phi]_\mu = \text{forall}([\phi]_{\mu,x} | x \in A)\) if \( x \notin \text{dom}(\mu)\)
- \([\exists x \phi]_\mu = \text{exists}([\phi]_{\mu,x} | x \in A)\) if \( x \notin \text{dom}(\mu)\)
where \( \text{dom}(\mu) \) is the set of variables on which \( \mu \) is defined, the boolean functions \( \text{not}, \text{or}, \ldots \) are defined as usual, and \( \text{forall}, \text{exists} : \mathcal{P}(\text{Bool}) \to \text{Bool} \) are defined as follows:

- \( \text{forall}(S) = \text{true} \iff \text{false} \notin S \)
- \( \text{exists}(S) = \text{true} \iff \text{true} \in S \)

The universal closure of a formula \( \phi \) with free variables \( x_1, \ldots, x_n \) is \( \forall \phi = \forall x_1 \ldots \forall x_n \phi \). A formula \( \phi \) is valid if, in every model \( A, \llbracket \forall \phi \rrbracket = \text{true} \).

Let the provability relation \( \vdash \) be the smallest relation on pairs \((\Gamma, \phi)\) where \( \Gamma \) is a finite set of formulas (the assumptions) and \( \phi \) a formula, such that:

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\text{axiom} \quad \phi \in \Gamma \quad \Gamma \vdash \phi \\
(\bot\text{-elim}) \quad \Gamma \vdash \bot \quad \Gamma \vdash \phi \\
(\Rightarrow\text{-intro}) \quad \Gamma \cup \{\phi\} \vdash \psi \quad \Gamma \vdash \phi \Rightarrow \psi \\
(\Rightarrow\text{-elim}) \quad \Gamma \vdash \phi \Rightarrow \psi \quad \Gamma \vdash \phi \quad \Gamma \vdash \psi \\
(\wedge\text{-intro}) \quad \Gamma \vdash \phi \quad \Gamma \vdash \psi \quad \Gamma \vdash \phi \land \psi \\
(\wedge\text{-elim-left}) \quad \Gamma \vdash \phi \land \psi \quad \Gamma \vdash \phi \\
(\wedge\text{-elim-right}) \quad \Gamma \vdash \phi \land \psi \quad \Gamma \vdash \psi \\
(\vee\text{-intro-left}) \quad \Gamma \vdash \phi \quad \Gamma \vdash \phi \lor \psi \\
(\vee\text{-intro-right}) \quad \Gamma \vdash \psi \quad \Gamma \vdash \phi \lor \psi \\
(\vee\text{-elim}) \quad \Gamma \vdash \phi \lor \psi \quad \Gamma \cup \{\phi\} \vdash \chi \quad \Gamma \cup \{\psi\} \vdash \chi \quad \Gamma \vdash \chi \\
(\forall\text{-intro}) \quad \Gamma \vdash \phi \quad x \notin \text{FV}(\Gamma) \quad \Gamma \vdash \forall x \phi \\
(\forall\text{-elim}) \quad \Gamma \vdash \forall x \phi \quad \Gamma \vdash \phi\{x \mapsto t\}
(∃-intro) \[ \Gamma \vdash \phi \{ x \mapsto t \} \]
\[ \Gamma \vdash \exists x \phi \]

(∃-elim) \[ \Gamma \vdash \exists x \phi \quad \Gamma \cup \{ \phi \} \vdash \chi \quad x \notin \text{FV}(\Gamma) \cup \text{FV}(\chi) \]
\[ \Gamma \vdash \chi \]

(EM) \[ \Gamma \vdash \phi \lor \neg \phi \]

Prove that the provability relation is correct, that is, in every model \( A \), if \( \Gamma \vdash \phi \) and \( \Gamma = \{ \phi_1, \ldots, \phi_n \} \), then \( \phi_1 \land \ldots \land \phi_n \Rightarrow \phi \) is valid.