# Gaussian Beams

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# Further reading

[KL66, GB94]

A. Gerrard and J.M. Burch.
*Introduction to matrix methods in optics.*
Dover, 1994.

H. KOGELNIK and T. LI.
Laser beams and resonators.
*Appl. Opt.*, 5(10):1550–1567, Oct 1966.

# Course Outline

1. Fundamentals of Gaussian beam propagation
   - Gaussian beams vs. plane waves
   - The fundamental mode
   - Higher order modes
2. Matrix methods for geometrical and Gaussian optics
   - Linear algebra for geometrical optics
   - A few simple matrices
   - Matrix method for Gaussian beams

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# Plane waves do not exist

## Waves carrying an infinite amount of energy cannot come into existence

### Planes waves

- Plane wave have a homogeneous transversal electric field
- Ponting's vector norm, and power density, are also homogeneous
- Total carried power is infinite

### Practical use of plane wave theory: usual unsaid approximation

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# Plane waves, Gaussian beams. . . what else ?

## Solutions of the wave equations: one finds only those he was searching for

### Solving the wave equation $\overrightarrow{\triangle E} = \frac{1}{c^2}\frac{\partial^2 \overrightarrow{E}}{\partial t^2}$

- Vectorial Partial Derivatives Equations
- Solutions are numerous
- An ansatz[1] is needed to seek solutions

### Gaussian beams as an *ansatz*

- We will find another family of solutions
- We never pretend to get them all

[1] An ansatz is an *a priori* hypothesis on the form of the sought solution.

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# Gaussian *ansatz*

## Plugging the ansatz into the wave equation builds the envelope equation

### Introducing a space dependent envelope

- Plane wave: $\overrightarrow{E_0} \times e^{-i\overrightarrow{k}\cdot\overrightarrow{r}}$
- Gaussian ansatz : $u(x, y, z)\,\overrightarrow{e_x} \times e^{-ikz}$
    - $u(x, y, z)$ : complex beam envelope
    - $\overrightarrow{e_x}$ unit vector
- The envelope $u(x, y, z)$ is our new unknown

### Envelope equation

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### Envelope equation

- Wave equation: $\overrightarrow{\Delta E} = \frac{1}{c^2} \frac{\partial^2 \overrightarrow{E}}{\partial t^2}$
- Scalar harmonic wave equation: $\Delta E + E = 0$
- Envelope equation:
  $\Delta\left[u(x, y, z)\, e^{-ikz}\right] + k^2 u(x, y, z)\, e^{-ikz} = 0$

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- Scalar harmonic wave equation: $\triangle E + \frac{\omega^2}{c^2}E = 0$
- Envelope equation:
$\triangle\left[u\left(x,y,z\right)e^{-ikz}\right] + k^2 u\left(x,y,z\right)e^{-ikz} = 0$

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## Envelope equation

- Scalar harmonic wave equation: $\triangle E + k^2 E = 0$
- Envelope equation: $\frac{\partial^2\left(ue^{-ikz}\right)}{\partial z^2} + \triangle_\perp ue^{-ikz} + k^2 ue^{-ikz} = 0$

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### Envelope equation

- Scalar harmonic wave equation: $\triangle E + k^2 E = 0$
- Envelope equation: $e^{-ikz}\left(\triangle u - 2ik\frac{\partial u}{\partial z}\right) = 0$

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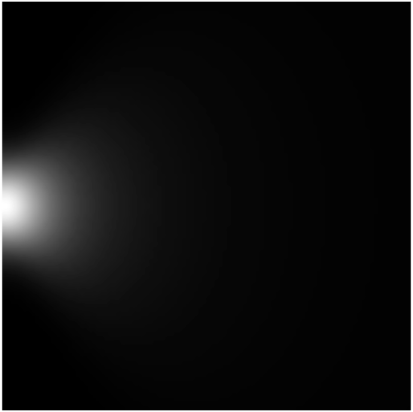
# The paraxial approximation

## Also known as Gauss conditions, Slow Varying Envelope...

### Non Paraxial Beam



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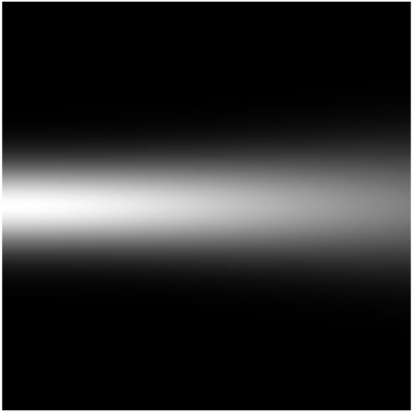

# Paraxial approximation and partial derivatives

## Assuming small angles is equivalent to neglecting $z$ derivatives

**Transversal variation** *vs.* **longitudinal variation**

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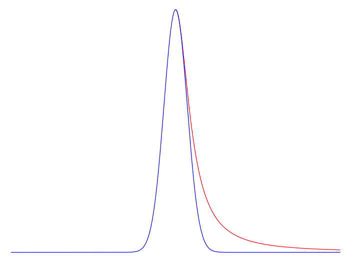

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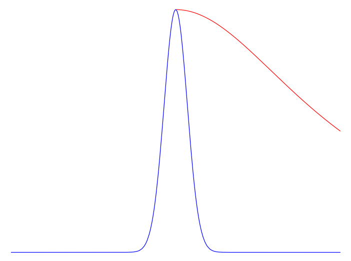

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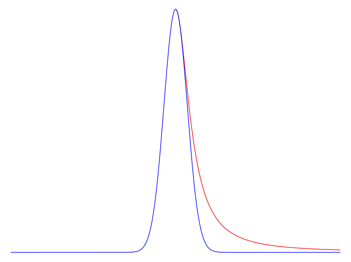

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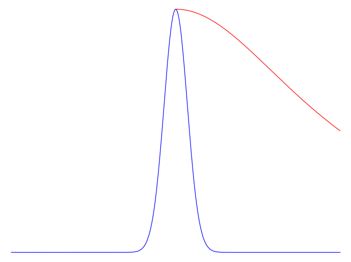

### Transversal Laplacian

$$\frac{\partial^2}{\partial z^2} \ll \frac{\partial^2}{\partial x^2} \qquad \triangle \approx \triangle_\perp$$

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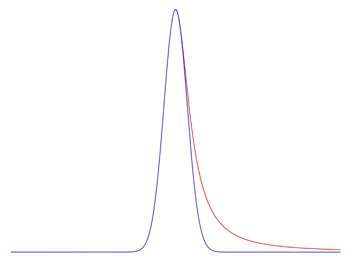

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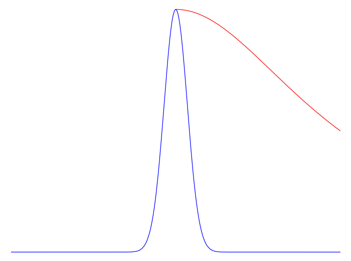

### Paraxial wave equation

$$\triangle_{\perp} u - 2ik\frac{\partial u}{\partial z} = 0$$

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# A first solution to the paraxial wave equation

## The simplest one, though probably the more important

### Wave Propagation Equation

$$\triangle_{\perp} u - 2ik\frac{\partial u}{\partial z} = 0$$

### A simple *ansatz*

$$u = e^{-i\left(P(z)+\frac{k}{2q(z)}r^2\right)}$$

### Complex beam radius $q(z)$

- Real part: phase variations
- Imaginary part: intensity variations
- Plugging ansatz: $q' = 1$
- Integration: $q(z) = q(0) + z$

### Phase shift $P(z)$

- Phase shift with respect to the plane wave
- $qP' + i = 0$

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# The complex beam radius $q(z)$

A closer look to the signification on a complex parameter $u = e^{-i\left(P(z)+\frac{k}{2q(z)}r^2\right)}$

**A complex parameter is linked to two real ones**

$$\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi W^2}$$

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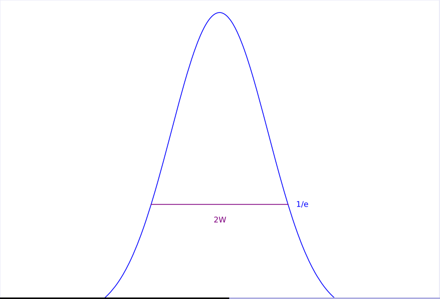

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What is $R$ ?

# Spherical wavefront of radius $R$ at abscissa $z$

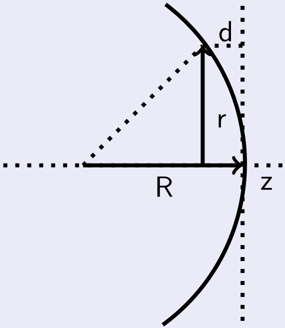
## Phase at abscissa $z$

- Constant phase on sphere
- Phase $\propto d, \quad r \ll R$

## Gaussian ansatz

- $u = e^{-i\left(P(z)+k\frac{r^2}{2R(z)}\right)} e^{-\frac{r^2}{W(z)^2}}$

## $R$ radius spherical wavefront



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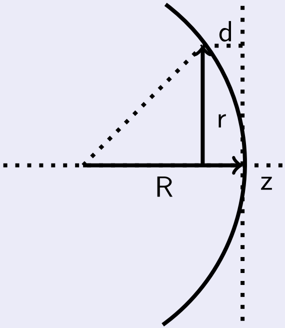

# Spherical wavefront of radius $R$ at abscissa $z$

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- $(R - d)^2 + r^2 = R^2$

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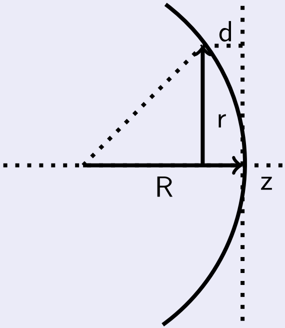

# Spherical wavefront of radius $R$ at abscissa $z$

## Phase at abscissa $z$

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- $d = R - \sqrt{R^2 - r^2}$

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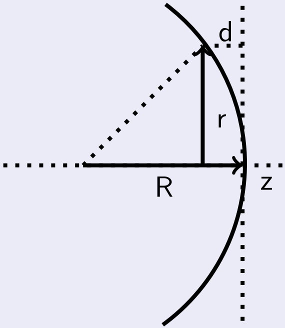

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- $d = R\left(1 - \sqrt{1 - \frac{r^2}{R^2}}\right)$

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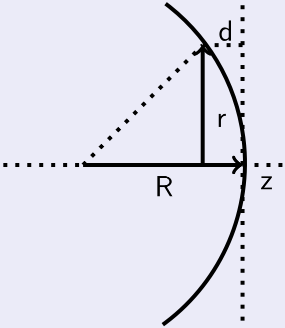

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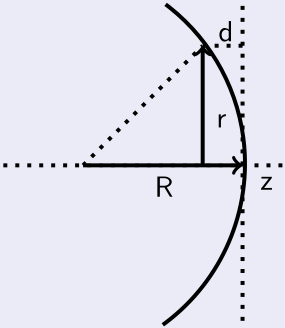

# Gaussian Beam Complex Amplitude

## Where the Gaussian beam amplitude is derived from the ansatz and $q' = 1$

### A quick summary

- Ansatz : $u = e^{-i\left(P(z) + \frac{k}{2q(z)} r^2\right)}$
- Complex beam radius : $\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi W^2}$
- Beam radius equation : $q' = 1$

### Assuming a plane wavefront for $z = 0$

- $q(0) = i\frac{\pi W_0^2}{\lambda}$

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## Assuming a plane wavefront for $z = 0$

- $q(0) = i\frac{\pi W_0^2}{\lambda}$
- $W^2(z) = W_0^2\left[1 + \left(\frac{\lambda z}{\pi W_0^2}\right)^2\right]$
- $R(z) = z\left[1 + \left(\frac{\pi W_0^2}{\lambda z}\right)^2\right]$

# Gaussian Beam Complex Amplitude

Where the Gaussian beam amplitude is derived from the ansatz and $q' = 1$

## A quick summary

- Ansatz : $u = e^{-i\left(P(z) + \frac{k}{2q(z)} r^2\right)}$
- Complex beam radius : $\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi W^2}$
- Beam radius equation : $q(z) = q(0) + z$

## Assuming a plane wavefront for $z = 0$

- $q(0) = i\frac{\pi W_0^2}{\lambda}$
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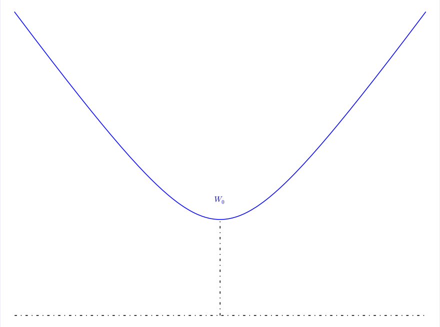


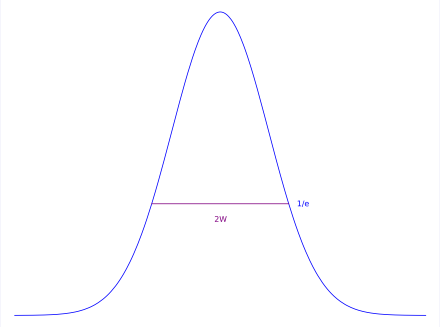

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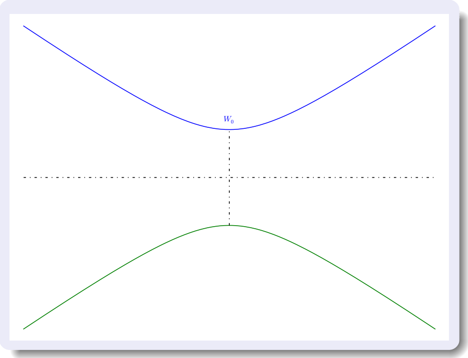


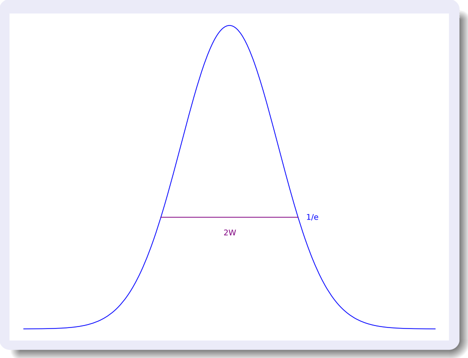

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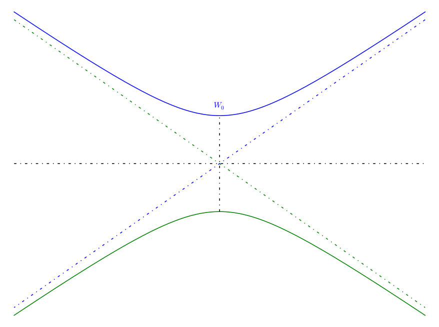


## Asymptotes

- $\frac{\lambda z}{\pi W_0^2} \gg 1 \quad \Rightarrow \quad W(z) \approx \frac{\lambda z}{\pi W_0}$
- $\gamma = \frac{\lambda}{\pi W_0}$

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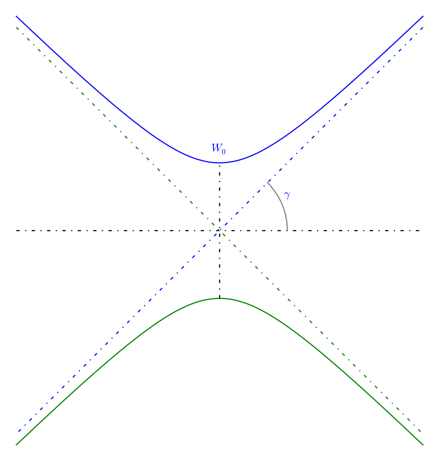


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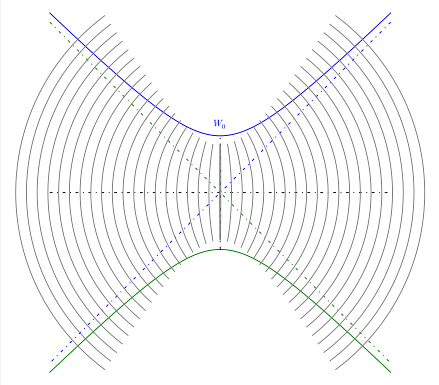
# Gaussian wavefront curvature

$$R(z) = z\left[1 + \left(\frac{\pi W_0^2}{\lambda z}\right)^2\right]$$

## Plane and spherical limits

- For small $z$ : $R = \infty$
  plane wavefront
- For high $z$ : $R \approx z$
  spherical wavefront



# The Rayleigh length

## A quantitative criterion to decide whether a Gaussian beam is plane or spherical

**Plane for small $z$** $\quad \frac{\lambda z}{\pi W_0^2} \ll 1$

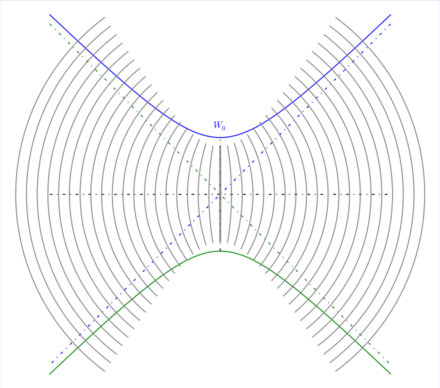
- $W(z) \approx W_0$
- $\lim_{z \to 0} R(z) = \infty$

**Spherical for high $z$** $\quad \frac{\lambda z}{\pi W_0^2} \gg 1$

- $W(z) \approx \dfrac{\lambda z}{\pi W_0}$
- $R(z) \approx z$

**The Rayleigh length is the limit**

$$L_R = \frac{\pi W_0^2}{\lambda}$$



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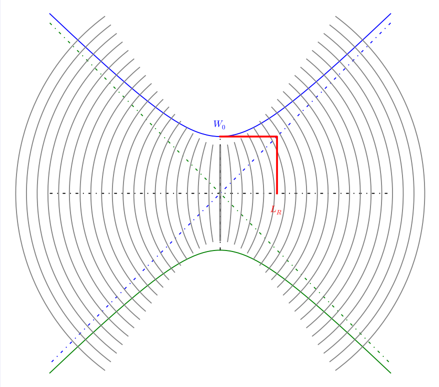

# The homogeneous phase shift $P(z)$

$u = e^{-i\left(P(z)+\frac{k}{2q(z)}r^2\right)}$

## Recall the equation

$$qP' + i = 0 \quad \Leftrightarrow \quad P'(z) = -\frac{i}{z + iL_R}$$

## Integrate it

$$iP(z) = \ln\left(1 - i\frac{z}{L_R}\right)$$

## Complex phase meaning

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# The fundamental Gaussian mode

## General expression

$$E(r,z) = \frac{W_0}{W(z)} e^{-i[kz+P(z)]-r^2\left(\frac{1}{W(z)^2}+i\frac{k}{2R(z)}\right)}$$

## With, in (nearly) the order of appearance on the screen

- $W^2(z) = W_0^2\left[1+\left(\frac{z}{L_R}\right)^2\right]$
- $R(z) = z\left[1+\left(\frac{L_R}{z}\right)^2\right]$
- $P(z) = -\tan^{-1}\left(\frac{z}{L_R}\right)$
- Rayleigh length $L_R = \frac{\pi W_0^2}{\lambda}$
- Diffraction half angle: $\gamma \approx \frac{\lambda}{\pi W_0} = \frac{W_0}{L_R}$

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# High order Hermite-Gaussian modes

## A Cartesian family of higher order modes

### Ansatz

$$u(x,y,z) = g\left(\frac{x}{W(z)}\right) h\left(\frac{y}{W(z)}\right) e^{-i\left(P(z)+\frac{k}{2q(z)}\left(x^2+y^2\right)\right)}$$

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- $q' = 1$
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- $\mathcal{H}_0 = 1, \quad \mathcal{H}_1 = x, \quad \mathcal{H}_2 = 4x^2 - 1, \quad \mathcal{H}_3 = 8x^3 - 12x \ldots$
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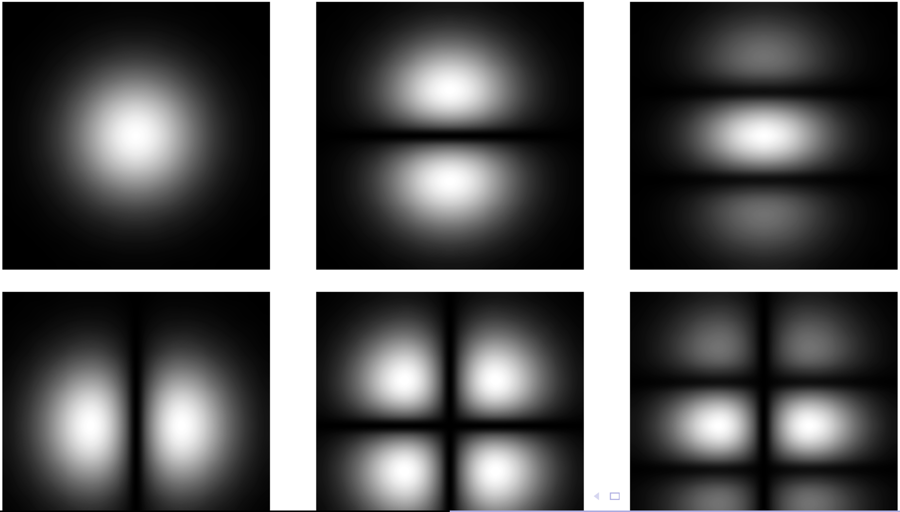
## The intensity if proportional to the squared envelope



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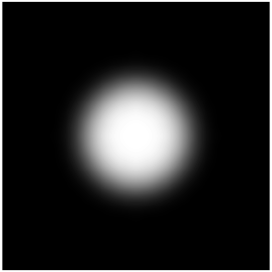
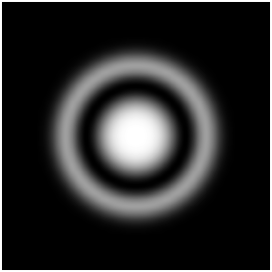


$LG(0,0)$

$LG(0,1)$ : vortex



$LG(0,2)$

# Intensity profiles of other Laguerre Gaussian (LG) modes

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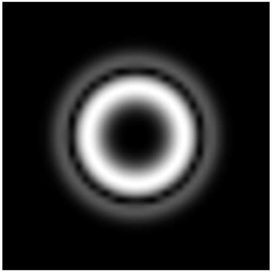
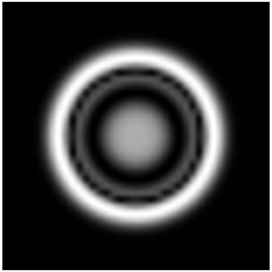
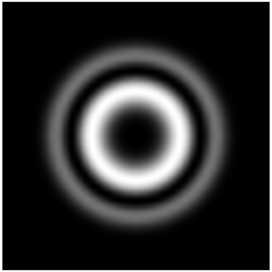


$LG(1,2)$



$LG(1,3)$



$LG(2,3)$

# Homogeneous phase shift is different for high order modes

$qP' + (1 + m + n)j = 0$ $\qquad$ $qP' + (1 + 2p + l)j = 0$

**A small phase difference between modes around the beam waist**

- Slightly different optical paths for different orders
- Slightly different oscillating frequencies in lasers
- Usually forgotten

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# Geometrical optics framework

## Where is is shown that rays are not so thin as you may think

### Geometrical optics *do not* deal with thin rays

- A thin ray has a thin waist: it should diffract $\gamma = \frac{\lambda}{\pi W_0}$
- Thin rays are seldom alone: their meaning is collective
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### Do geometrical optics deal with plane and spherical waves ?

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- Parallel rays imply a plane wavefront
- Converging or diverging rays imply a spherical wavefront
- But neither of them has an infinite extension !

# Geometrical optics framework

Where is is shown that rays are not so thin as you may think

## Geometrical optics *do not* deal with thin rays

- A thin ray has a thin waist: it should diffract $\gamma = \frac{\lambda}{\pi W_0}$
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**Transversely limited plane waves** — **Parallel rays**

Gaussian Beams within their Rayleigh zone

**Transversely limited spherical waves** — **Con(Di)verging rays**

Gaussian Beams far from their Rayleigh zone

**Orders of magnitude**

- He-Ne laser: $W_0 \approx 1\text{mm}, \lambda = 633\text{nm}, L_R \approx 5\text{m}$
- GSM Antenna: $W_0 \approx 1\text{m}, \lambda \approx 33\text{cm}, L_R \approx 10\text{m}$

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Geometrical optics stems entirely from Descartes law $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

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Paraxial approximation : $\theta \ll 1$ $n_1\theta_1 \approx n_2\theta_2$

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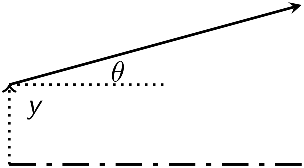
# Matrix geometrical optics

## A 2 dimensional linear algebra framework

**The ray vector** $v = \begin{pmatrix} y \\ \theta \end{pmatrix}$

- $y$ : distance from the axis
- $\theta$ : angle to the axis




**An optical system** $v' = Mv$

- $M$ is a $2 \times 2$ real matrix
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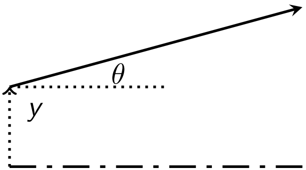


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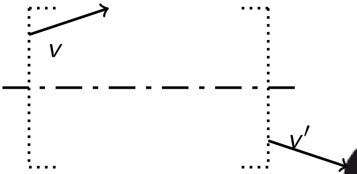
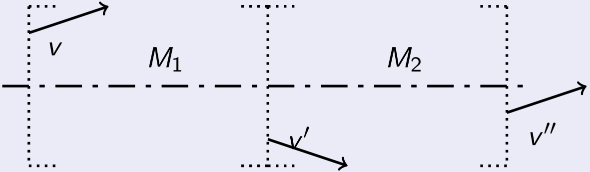


# Optical system composition

## Optical system composition reduced to matrix product

### Optical System Composition




### Matrix Composition

- $v' = M_1 \cdot v$
- $v'' = M_2 \cdot v'$
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### Complex systems

Compose simple systems

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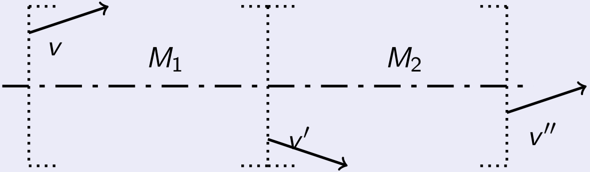


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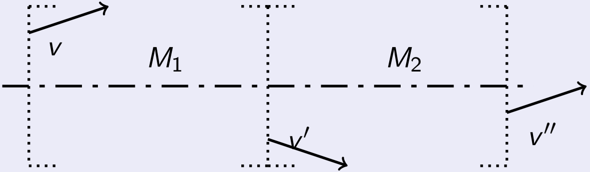


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# Propagation in a homogeneous medium

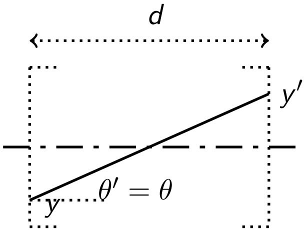
$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_d \begin{pmatrix} y \\ \theta \end{pmatrix}$$

## Light propagates in straight line

- No direction change: $\theta' = \theta$
- $y' = y + d \sin(\theta)$

## $M_d$

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



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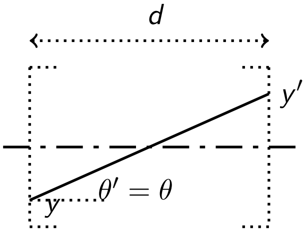

# Passing through a plane interface

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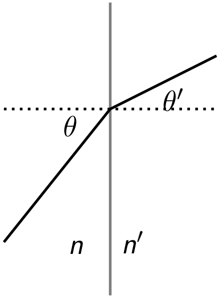

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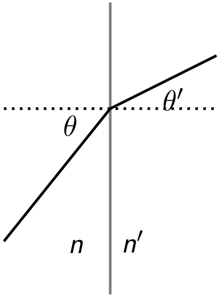

# Passing through a thin lens

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_l \begin{pmatrix} y \\ \theta \end{pmatrix}$$

## Two characteristic rays

- No propagation: $y' = y$
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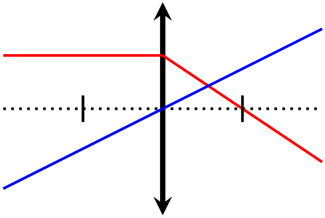

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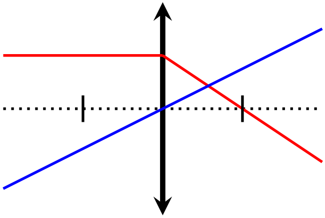

# Passing through a (thin) spherical interface

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_s \begin{pmatrix} y \\ \theta \end{pmatrix}$$

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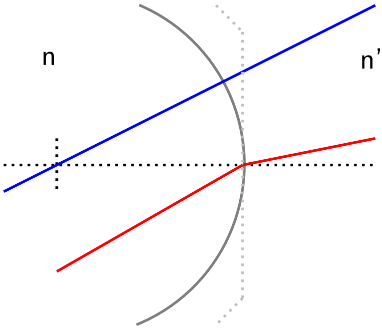

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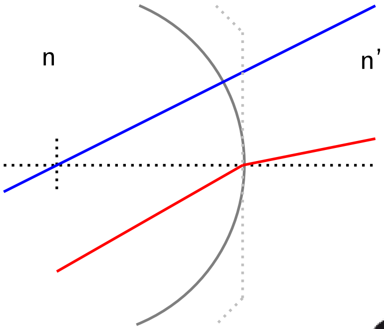
## $\theta' = ay + b\theta$

Solve for $(a, b)$ using

- $(y, \theta') = (R\theta, \theta)$
- $(y, \theta') = (0, \frac{n}{n'}\theta)$



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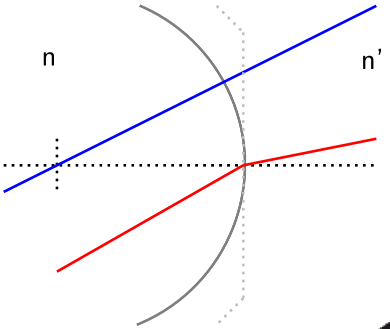

# Mirrors

## Unfolding the light

### Plane mirrors as if they did not exist

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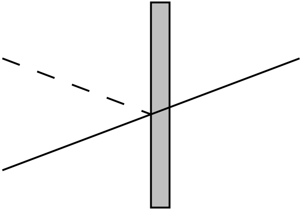

### Spherical Mirrors are thin lenses

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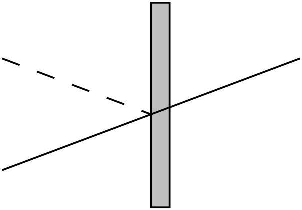

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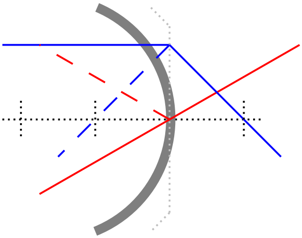

# Matrix property

## A determinant property stemming from all the simple matrices determinants

$n$: start index $\quad$ $n'$: stop index

$$\forall M,\ \det(M) = \frac{n}{n'}$$

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# Gaussian modes, propagation and lenses

A Gaussian mode does not change upon propagation of by passing through thin interfaces or lenses

$$g\left(\frac{x}{W(z)}\right) h\left(\frac{y}{W(z)}\right) e^{-i\left(P(z)+\frac{k}{2q(z)}\left(x^2+y^2\right)\right)}$$

- $z$ independent modulation of the fundamental mode
- Free space $q' = 1$ common property
- Thin lens does not change mode profile

**Common $R(z)$ and $W(z)$ behavior**

All the modes share the same laws on $q(z)$, $R(z)$ and $W(z)$

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# Gaussian beam propagation: the ABCD law

## The transformation of the complex radius $q$ for simple optical systems

### Free space $\qquad q' = 1$

- $q_1 = q_0 + d$
- $M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$

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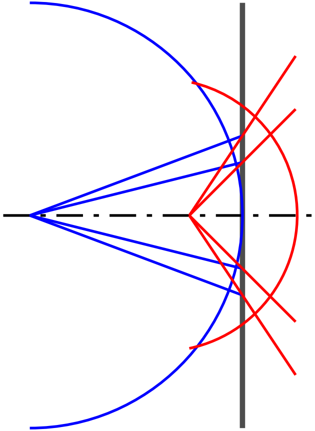

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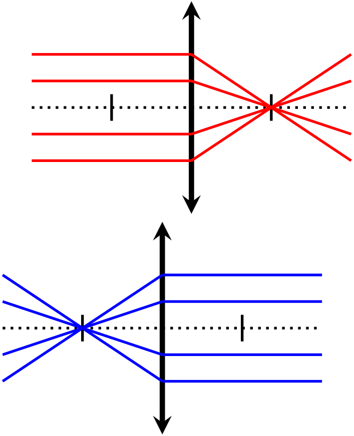

**Thin lens** $\frac{1}{R_1} = \frac{1}{R_0} - \frac{1}{f}$

- $\frac{1}{q_1} = \frac{1}{q_0} - \frac{1}{f} \Rightarrow q_1 = \frac{1}{-\frac{1}{f} q_0 + 1}$
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**Geometrical and Gaussian optics are linked through paraxial approx.**

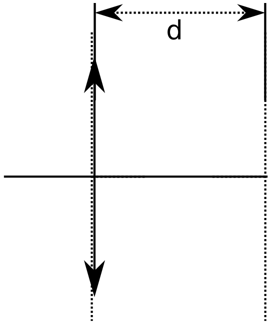
Gaussian beam propagation can be evaluated, for any mode, using simple matrix geometrical optics

# Focal lens example

## Using the ABCD law to verify that parallel rays do converge on the focal plane

### Parallel input beam

- Input plane : just before lens
- Output plane : after length $d$
- Input beam at waist:
  $q_0 = i\frac{\pi W_0^2}{\lambda}$



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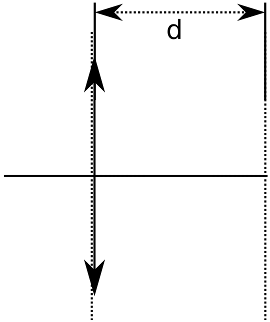
### Propagation matrix

- lens: $M_f = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$
- distance $d$ : $M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$



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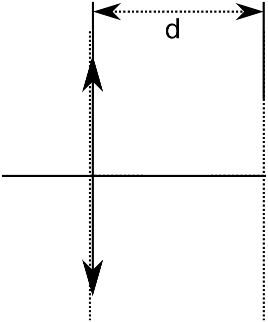

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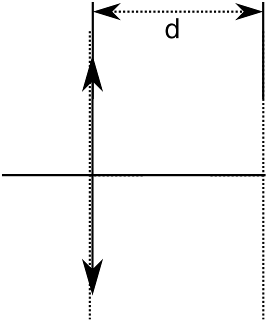

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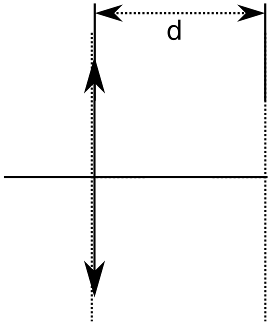
## ABCD law

$$q_1 = \frac{df + i(f-d)L_{R_0}}{f - iL_{R_0}}$$




## $d$ for plane wavefront: imaginary $q_1$

$$d = \frac{f}{1 + \left(\frac{f}{L_{R_0}}\right)^2}$$