Optical Phase Conjugation

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Download this document: http://moodle.univ-metz.fr/
Useful reading. . .
[YY84, Yar97, San99]

F. Sanchez.
*Optique non-linéaire - Cours et problèmes résolus.*
Ellipses, 1999.

A. Yariv.
*Optical Electronics in Modern Communications.*

A. Yariv and P. Yeh.
*Optical waves in crystals. Propagation and control of laser radiation.*

. . . and many others.
Contents

1 Principle and application of phase conjugation
   - Experiment
   - Distortion correction theorem

2 Generation of phase conjugate waves
   - Non Linear Polarization Development
   - Four wave mixing coupled mode formulation
   - Conjugate wave amplitude

3 Self Pumped Phase Conjugation and Holography
   - Four Wave Mixing from a holographic point of view
   - Phase Conjugation without pumping
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A peculiar phenomenon
Discovered in the early 70\textsuperscript{ies}
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Use Non Linear Material

- Third order: non zero $\chi_3$
  - Photorefractivity
  - Stimulated Brillouin Scattering
  - Stimulated Raman Scattering
- Four wave mixing

Wavefront correction

- Distorted incident beam
- Reflected back as is
- Distortion corrected

**Figure:** Phase conjugation principle.

**Source:** Wikipedia
A peculiar phenomenon
Discovered in the early 70\textsuperscript{ies}

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- Wavefront correction
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\textbf{Figure:} Phase conjugation principle.
Beams are reflected back as if time was reversed

Images source:
http://sharp.bu.edu/~slehar/PhaseConjugate/PhaseConjugate.html

Incident wavefronts...
- Are reflected back exactly
- Back and forth wavefronts are identical
Principle and application of phase conjugation

Generation of phase conjugate waves

Self Pumping and Holography

Experiment

Beams are reflected back *as if time was reversed*

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Incident wavefronts...

- Are reflected back **exactly**
- Back and forth wavefronts are **identical**
Attractive applications
All based on wavefront distortion correction

Phase conjugation applications

- All optical image transmission through fibers
- Distortion correction in high power lasers
- Dynamic wave front correction for optical sensors
- Dynamic Holography
- ...
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A phase conjugate wave travels time the wrong way
Phase conjugation is also known as *time reversal*

Take some input monochromatic wave

\[ E_1 = \Re \left[ \psi (r) \exp (i (\omega t - kz)) \right] = \Re \left[ \psi (r) \right] \cos (\omega t - kz) \]

Take the phase conjugate over space only

- \[ E_2 = \Re \left[ \psi (r) \exp (i (-kz)) e^{i\omega t} \right] \]
- \[ E_2 = \Re \left[ \psi (r) \exp (i (\omega t + kz)) \right] \]
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\[
E_1 = \mathcal{R}e \left[ \psi (r) \exp (i (\omega t - kz)) \right] = \mathcal{R}e \left[ \psi (r) \right] \cos (\omega t - kz)
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The distortion correction theorem

If a backward traveling wave is phase conjugate \textit{somewhere} then it is \textit{everywhere}

Take a paraxial forward propagating wave

- Expressed as: \( E_1 (r, t) = \psi_1 (r) e^{i(\omega t - k z)} \)
- Obeys the wave equation: \( \Delta E_1 + \omega^2 \mu_0 \varepsilon (r) E_1 = 0 \)
  - In the paraxial limit: \( \Delta \psi_1 + \left[ \omega^2 \mu_0 \varepsilon (r) - k^2 \right] \psi_1 - 2ik \frac{\partial \psi_1}{\partial z} = 0 \)
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Same second order linear differential equations for loss-less media

\( \varepsilon (r) \in \mathbb{R} \Rightarrow \left[ \psi_2 (z = 0) = a \psi_1 (z = 0) \right] \iff \forall z, \psi_2 (z) = a \psi_1 (z) \)
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Four wave mixing
Third Order Non Linear Optics

Flashback : Second Order
- Relies on $\chi_2 : P_{NL} \propto E^2$
- Two waves mix to generate a third one
  $\omega_1 \pm \omega_2 \rightarrow \omega_3$
- Second Harmonic Generation, Optical Parametric Amplification, Optical Parametric Oscillation...

Third order
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- Phase conjugation for $\omega_4 = \omega_1 + \omega_2 - \omega_3$? Let’s see...
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Non Linear Polarization Development

Non Linear Polarization $P_{NL}$

General polarization development

$$[P]_i = \varepsilon_0 [\chi]_{ij} [E]_j + 2[d]_{ijk} [E]_j [E]_k + 4[\chi]_{ijkl} [E]_j [E]_k [E]_l$$

Third order non linear development\(^1\) around $\omega_4 = \omega_1 + \omega_2 - \omega_3$

$$[P_{NL}]_i (\omega_4) = 6[\chi]_{ijkl} [E]_j (\omega_1) [E]_k (\omega_2) [E]_l (\omega_3) e^{i(\omega t + k_4 r)}$$

Degenerate Four wave mixing: $\omega = \omega + \omega - \omega$

$$[P_{NL}]_i (\omega) = 6[\chi]_{ijkl} [E]_j (\omega) [E]_k (\omega) [E]_l (\omega) e^{i(\omega t + kr)}$$

\(^1\)As an exercise, you can multiply, sum-up and keep only $\omega_4$ related terms... and find $k_4$
### Non Linear Polarization Development

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#### Degenerate Four wave mixing: $\omega = \omega + \omega - \omega$

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Degenerate Four Wave mixing configuration

- $A_1 = A_2$ intense plane pumps
- $A_3$ is the signal
- We seek $A_4$

In a Non Linear Medium, for $z = 0$ to $z = L$:

$A_1 A_2 A_3 A_4$
Signal wave equation

Let us start with the standard non linear wave equation

$$\Delta E - \mu_0 \varepsilon \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$$

Signal $A_3$ propagation

- Each wave has its own direction and polarization
- They can be treated separately

$$\text{Signal Equation:} \quad \mu_0 \varepsilon \frac{\partial^2 A_3}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$$
Signal wave equation

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- With $\Delta E_3 = \Re e \left[ \left( -k^2 A_3 - 2i k \frac{\partial A_3}{\partial z} + \frac{\partial^2 A_3}{\partial z^2} \right) e^{i(\omega t - kz)} \right]$
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- $\mathcal{R}e \left[ \left( \left( \omega^2 \mu_0 \varepsilon - k^2 \right) A_3 - 2ik \frac{\partial A_3}{\partial z} + \frac{\partial^2 A_3}{\partial z^2} \right) e^{i(\omega t - kz)} \right] = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$
Let us start with the standard non linear wave equation

\[ \Delta E - \mu_0 \varepsilon \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} \]

**Signal \( A_3 \) propagation**

- Each wave has its own direction and polarization
- They can be treated separately
- With \( \Delta E_3 = \Re \left[ \left( -k^2 A_3 - 2ik \frac{\partial A_3}{\partial z} + \frac{\partial^2 A_3}{\partial z^2} \right) e^{i(\omega t - kz)} \right] \)
- \( \Re \left[ \left( \omega^2 \mu_0 \varepsilon - k^2 \right) A_3 - 2ik \frac{\partial A_3}{\partial z} + \frac{\partial^2 A_3}{\partial z^2} \right) e^{i(\omega t - kz)} \] = \( \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} \)

**Dispersion equation**

\[ \omega^2 \mu_0 \varepsilon = k^2 \]
Let us start with the standard non linear wave equation

\[ \Delta E - \mu_0 \varepsilon \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} \]

**Signal \ A_3 \ propagation**

- Each wave has its own direction and polarization
- They can be treated separately
- With \( \Delta E_3 = R\text{e} \left[ \left( -k^2 A_3 - 2ik \frac{\partial A_3}{\partial z} + \frac{\partial^2 A_3}{\partial z^2} \right) e^{i(\omega t - kz)} \right] \)

\[ R\text{e} \left[ \left( -2ik \frac{\partial A_3}{\partial z} + \frac{\partial^2 A_3}{\partial z^2} \right) e^{i(\omega t - kz)} \right] = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} \]

**Assumption**

Slow varying assumed: \( \| k \frac{\partial A_3}{\partial z} \| \gg \| \frac{\partial^2 A_3}{\partial z^2} \| \)
Signal wave equation

Let us start with the standard non linear wave equation

\[ \Delta E - \mu_0 \varepsilon \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} \]

Signal \( A_3 \) propagation

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Evaluation of the non linear polarization \( P_{NL} \)

Let us take a look at the terms which involve \( e^{i(\omega t \pm kz)} \)
Stripping the non linear polarization to useful terms

Keeping only the relevant terms which contain $e^{i(\omega t \pm kz)}$

Expansion of third order non linear polarization

$$[P_{NL}]_i = \mathcal{R}e \left[ 6 \left( \begin{array}{cccc} [\chi]_{ijkl} & [A_1]_j & [A_2]_k & [A_4]_l \\ + & [\chi]_{ijji} & [A_1]_j & [A_1]_j \\ + & [\chi]_{ikk} & [A_2]_k & [A_2]_k \\ + & [\chi]_{iiii} & [A_3]_i & [A_3]_i \\ + & [\chi]_{illi} & [A_4]_l & [A_4]_l \end{array} \right) e^{i(\omega t - kz)} \right]$$

Simplifying assumptions

- Intense pump beam terms are dominant
- Polarizations are
  - either all the same, only $[\chi]_{iiii}$ involved
  - or $(A_1/A_2) \perp (A_3/A_4)$, only $[\chi]_{ijji}, i \neq j$ involved
Principle and application of phase conjugation

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**Stripping the non linear polarization to useful terms**

Keeping only the relevant terms which contain $e^{i(\omega t \pm kz)}$

**Expansion of third order non linear polarization**

$$ [P_{NL}]_i = \Re e \left[ 6 \left( \begin{array}{cccc} [\chi]_{ijkl} & [A_1]_j & [A_2]_k & [A_4]_l \\ + [\chi]_{jjji} & [A_1]_j & [A_1]_j & [A_3]_i \\ + [\chi]_{kkki} & [A_2]_k & [A_2]_k & [A_3]_i \\ + [\chi]_{iiii} & [A_3]_i & [A_3]_i & [A_3]_i \\ + [\chi]_{iiii} & [A_4]_i & [A_4]_i & [A_3]_i \end{array} \right) \right] e^{i(\omega t - kz)} $$

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  - either all the same, only $[\chi]_{iiii}$ involved
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$$ [P_{NL}]_i = \chi^{(3)} \left[ (\|A_1\|^2 + \|A_2\|^2) A_3 + A_1 A_2 A_3 \right] e^{i(\omega t - kz)} $$

$$ \chi^{(3)} = 6 [\chi]_{iiii} \text{ or } \chi^{(3)} = 6 [\chi]_{jjji} $$
Stripping the non linear polarization to useful terms
Keeping only the relevant terms which contain $e^{i(\omega t \pm kz)}$

### Expansion of third order non linear polarization

$$[P_{NL}]_i = \mathcal{R}e \left[ 6 \left( \begin{array}{c} [\chi]_{ijkl} [A_1]_j [A_2]_k [A_4]_l \\ + [\chi]_{ijji} [A_1]_j [A_1]_j [A_3]_i \\ + [\chi]_{ikki} [A_2]_k [A_2]_k [A_3]_i \end{array} \right) e^{i(\omega t - kz)} \right]$$

### Simplifying assumptions

- Intense pump beam terms are dominant
- Polarizations are
  - either all the same, only $[\chi]_{iii}$ involved
  - or $(A_1/A_2) \perp (A_3/A_4)$, only $[\chi]_{iji}$, $i \neq j$ involved

$$[P_{NL}]_i = \chi^{(3)} \left( \|A_1\|^2 + \|A_2\|^2 \right) A_3 + A_1 A_2 A_4 e^{i(\omega t - kz)}$$

$$\chi^{(3)} = 6[\chi]_{iii} \text{ or } \chi^{(3)} = 6[\chi]_{iji}$$
Stripping the non linear polarization to useful terms
Keeping only the relevant terms which contain $e^{i(\omega t \pm k z)}$

### Expansion of third order non linear polarization

\[
[P_{NL}]_i = \Re \left[6 \left( [\chi]_{ijkl} [A_1]_j [A_2]_k [A_4]_l + [\chi]_{ijji} [A_1]_j [A_1]_j [A_3]_i + [\chi]_{ikki} [A_2]_k [A_2]_k [A_3]_i \right) e^{i(\omega t - k z)} \right]
\]

### Simplifying assumptions

- Intense pump beam terms are dominant
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[P_{NL}]_i = \chi^{(3)} \left[ (\|A_1\|^2 + \|A_2\|^2) A_3 + A_1 A_2 A_4 \right] e^{i(\omega t - k z)}
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Resulting coupled wave propagation equation

Coupled wave equation resulting of $P_{NL}$

$$-2ik \frac{\partial A_3}{\partial z} e^{i(\omega t - kz)} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$$

Further simplification

- Homogeneous refraction index modulation: Kerr effect
- Simple phase factor change
- Remove it from equation $A'_1 = A_1 e^{-ikz} \sqrt{\|A_1\|^2 + \|A_3\|^2}$

Simplified coupling equations

$$\frac{\partial A'_3}{\partial z} = \kappa A'_4$$
$$\frac{\partial A'_4}{\partial z} = \kappa A'_3$$

obtained through the same kind of derivation
### Resulting coupled wave propagation equation

**Coupled wave equation resulting of** $P_{NL}$

\[
\frac{\partial A_3}{\partial z} = -i \frac{\omega}{2} \sqrt{\frac{\mu_0}{\varepsilon}} \chi^{(3)} \left[ (\|A_1\|^2 + \|A_2\|^2) A_3 + A_1 A_2 A_4 \right]
\]

**Further simplification**

- Homogeneous refraction index modulation : Kerr effect
- Simple phase factor change
- Remove it from equation $A_i' = A_i e^{-i/2 \sqrt{\mu_0/\varepsilon} \chi^{(3)}(\|A_1\|^2 + \|A_2\|^2) z}$

Set $\kappa = \frac{\omega}{2} \sqrt{\frac{\mu_0}{\varepsilon}} \chi^{(3)} A_1 A_2$

**Simplified coupling equations**

\[
\begin{align*}
\frac{\partial A_3'}{\partial z} &= i \kappa A_4' \\
\frac{\partial A_4'}{\partial z} &= i \kappa A_3'
\end{align*}
\]

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Resulting coupled wave propagation equation

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\frac{\partial A_3}{\partial z} = -i \frac{\omega}{2} \sqrt{\frac{\mu_0}{\varepsilon}} \chi^{(3)} \left[ (\|A_1\|^2 + \|A_2\|^2) A_3 + A_1 A_2 \overline{A_4} \right]
\]

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\[
\frac{\partial A_3}{\partial z} = -i \frac{\omega}{2} \sqrt{\frac{\mu_0}{\varepsilon}} \chi^{(3)} \left[ (|A_1|^2 + |A_2|^2) A_3 + A_1 A_2 \overline{A_4} \right]
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- Homogeneous refraction index modulation: Kerr effect
  - Simple phase factor change
  - Remove it from equation \( A'_i = A_i e^{-i \frac{\omega}{2} \sqrt{\frac{\mu_0}{\varepsilon}} \chi^{(3)} (|A_1|^2+|A_2|^2)} z \)
- Set \( \kappa = \frac{\omega}{2} \sqrt{\frac{\mu_0}{\varepsilon}} \chi^{(3)} A_1 A_2 \)

**Simplified coupling equations**

- \( \frac{\partial A_4}{\partial z} = i \kappa A_4 \)
- \( \frac{\partial A_3}{\partial z} = i \kappa A_3 \)

obtained through the same kind of derivation.
Resulting coupled wave propagation equation

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$$\frac{\partial A_3}{\partial z} = -i \frac{\omega}{2} \sqrt{\frac{\mu_0}{\varepsilon}} \chi^{(3)} \left[ (\|A_1\|^2 + \|A_2\|^2) A_3 + A_1 A_2 \overline{A_4} \right]$$

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1. Principle and application of phase conjugation
   - Experiment
   - Distortion correction theorem

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   - Conjugate wave amplitude

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   - Four Wave Mixing from a holographic point of view
   - Phase Conjugation without pumping
**Conjugate wave amplitude**

**General solution**

Boundary conditions at $z = 0$ and $z = L$

\[ A_3 \text{ is forward propagating} \]
\[ A'_3 (z) = -i \frac{|\kappa| \sin (|k| z)}{\kappa \cos (|k| L)} A'_4 (L) + \frac{\cos (|k| (z - L))}{\cos (|k| L)} A'_3 (0) \]

\[ A'_4 (z) = \frac{\cos (|k| z)}{\cos (|k| L)} A'_4 (L) + i \frac{|\kappa| \sin (|k| (z - L))}{|k| \cos (|k| L)} A'_3 (0) \]

**One beam experiment**

\[ A'_4 (L) = 0 \]

\[ A'_3 (L) = \frac{A'_3 (0)}{\cos (|k| L)} \]

Coherent amplifier
Conjugate wave amplitude

**General solution**

Boundary conditions at \( z = 0 \) and \( z = L \)

**\( A_3 \) is forward propagating**

\[
A_3' (z) = -i \frac{\kappa |k| z}{\kappa \cos(|k| L)} A_4'(L) + \frac{\cos(|k|(z - L))}{\cos(|k| L)} A_3'(0)
\]

\[
A_4' (z) = \frac{\cos(|k| z)}{\cos(|k| L)} A_4'(L) + i \frac{\kappa}{|k| \cos(|k| L)} \sin(|k|(z - L)) A_3'(0)
\]

**One beam experiment**

\( A_4'(L) = 0 \)

\[
A_3'(L) = \frac{A_3'(0)}{\cos(|k| L)}
\]

\[
A_4'(0) = -i \frac{\kappa}{|k| \tan(|k| L)} A_3'(0)
\]

Coherent amplifier

Reflectivity can exceed 1
### Conjugate wave amplitude

#### General solution

**Boundary conditions at** $z = 0$ **and** $z = L$

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>$A'_3(z) = -i \frac{\kappa</td>
<td>k</td>
</tr>
</tbody>
</table>

#### One beam experiment

**$A'_4(L) = 0$**

- $A'_3(L) = \frac{A'_3(0)}{\cos(|k|L)}$
- $A'_4(0) = -i \frac{\kappa}{|\kappa|} \tan(|\kappa|L) \overline{A'_3(0)}$

**Coherent amplifier**

**Reflectivity can exceed 1**
## General solution

### Boundary conditions at $z = 0$ and $z = L$

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### One beam experiment

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What if \( \cos(|k|L) = 0 \)?

- Infinite gain
- \( A_3 \) and \( A_4 \) start from noise
- Spontaneous oscillations
One Beam experiment and phase factor

One beam experiment

- $A'_3(L) = \frac{A'_3(0)}{\cos(|k|L)}$
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What if $\cos(|k|L) = 0$?

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One beam experiment

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Coherent amplifier
Reflectivity can exceed 1

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Four Wave Mixing from a holographic point of view

Four Wave Mixing is Real Time Holography

Write Hologram

\[ T \propto \|A_1 + A_3\|^2 = \|A_1\|^2 + \|A_3\|^2 + A_1A_3 + A_3A_1 \]

Read Hologram with \( A_2 = \overline{A_1} \)

\[ A_4 \propto TA_2 = (\|A_1\|^2 + \|A_3\|^2)A_2 + A_2A_1\overline{A_3} + A_2A_3\overline{A_1} \]
Four Wave Mixing from a holographic point of view

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---

**Write Hologram**

\[
T \propto \|A_1 + A_3\|^2 = \|A_1\|^2 + \|A_3\|^2 + A_1 A_3 + A_3 A_1
\]

**Read Hologram with** \(A_2 = \overline{A_1}\)

\[
A_4 \propto T \overline{A_2} = (\|A_1\|^2 + \|A_3\|^2) \overline{A_2} + A_2 A_1 \overline{A_3} + A_2 A_3 \overline{A_1}
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CAT Conjugator using Beam Fanning

Use Mirrors
CAT Conjugator using Beam Fanning

Figure: Beam fanning in photorefractive Baryum Titanate
CAT Conjugator using Beam Fanning

Figure: Beam fanning in photorefractive Baryum Titanate