UE SPM-PHO-S09-112
Optical Phase Conjugation

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Unité de Recherche commune à l'Université Paul Verlaine Metz et à Supélec

Dowload this document : `http://moodle.univ-metz.fr/`

Useful reading...

[YY84, Yar97, San99]

F. Sanchez.
Optique non-linéaire - Cours et problèmes résolus.
Ellipses, 1999.

A. Yariv.
Optical Electronics in Modern Communications.
Oxford Series in Electrical and Computer Engineering. Oxford University Press, 1997.

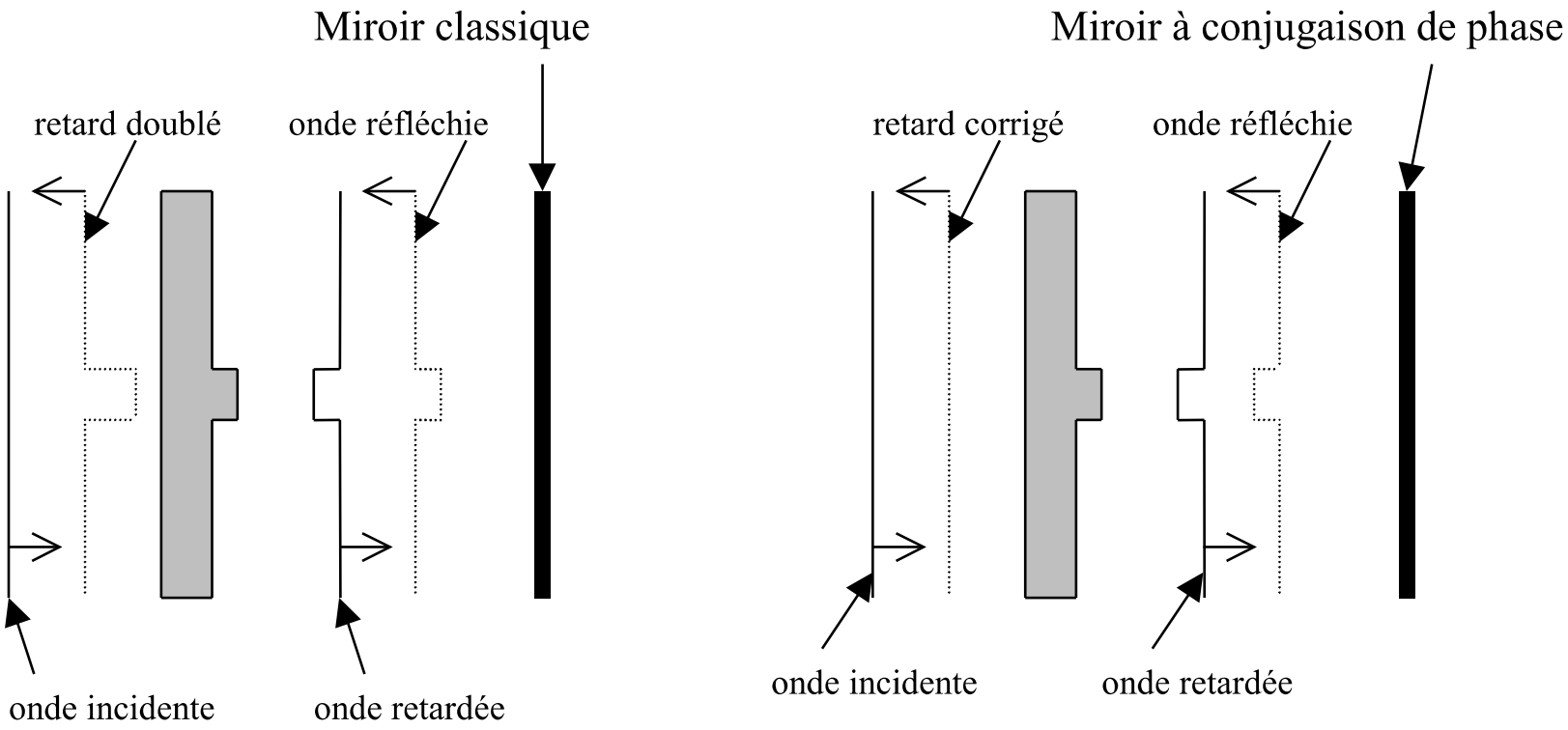
A. Yariv and P. Yeh.
Optical waves in crystals. Propagation and control of laser radiation.
Wiley series in pure and applied optics. Wiley-Interscience, Stanford University, 1984.

...and many others.

Contents

A peculiar phenomenon

Discovered in the early 70^{ies}



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Use Non Linear Material

- Third order : non zero χ_3
  - Photorefractivity
  - Stimulated Brillouin Scattering
  - Stimulated Raman Scattering
- Four wave mixing

Wavefront correction

- Distorted incident beam
- Reflected back as is
- Distortion corrected

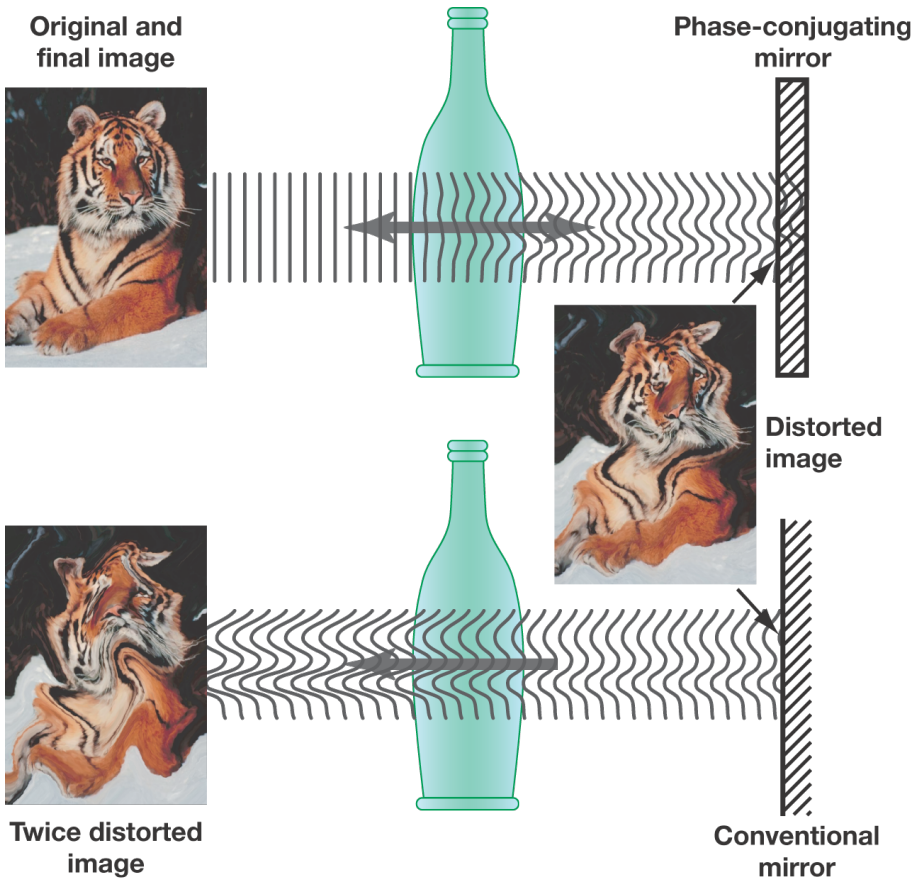

Figure: Phase conjugation principle.
Source : Wikipedia

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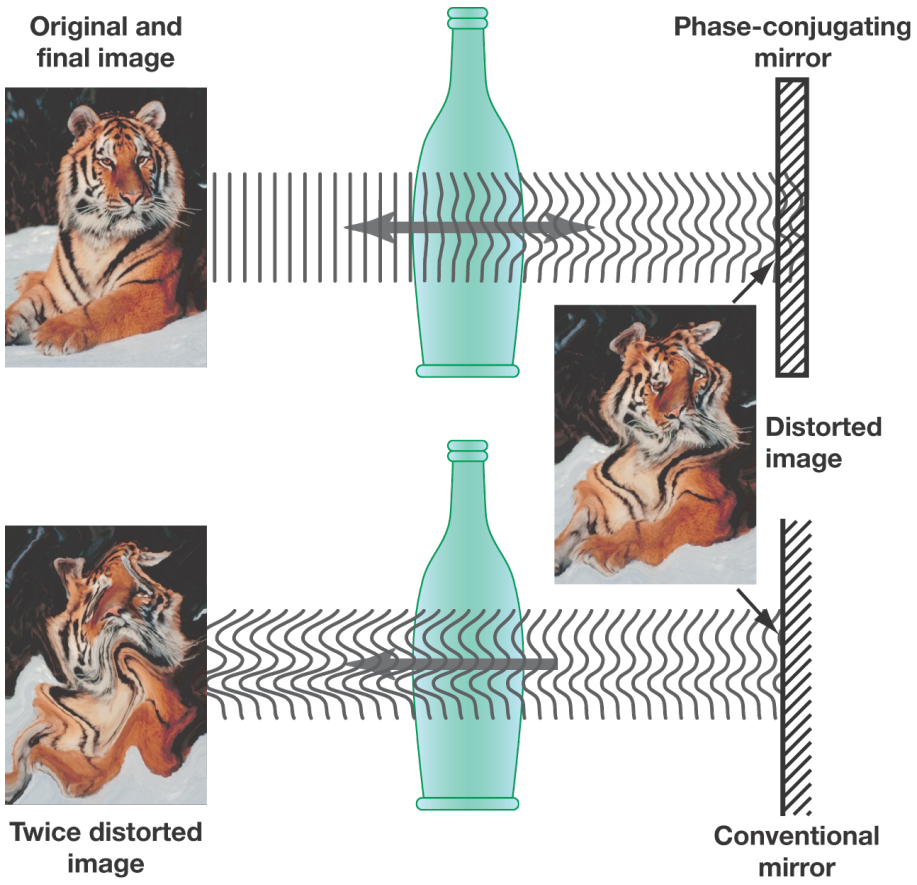

Figure: Phase conjugation principle.
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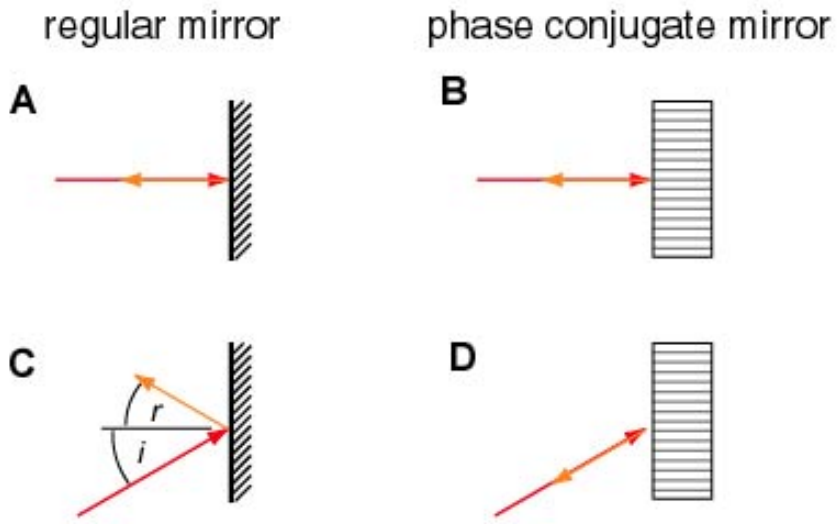
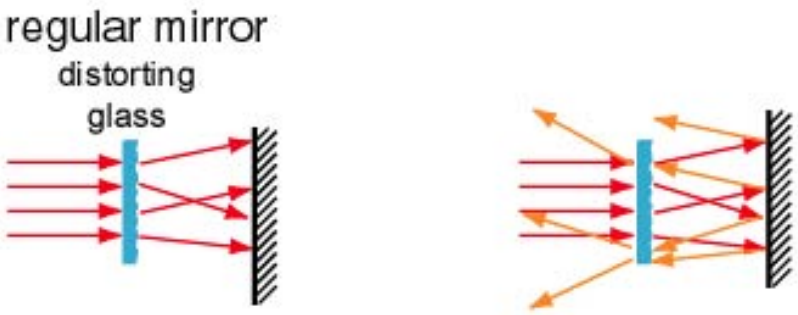
Beams are reflected back *as if time was reversed*

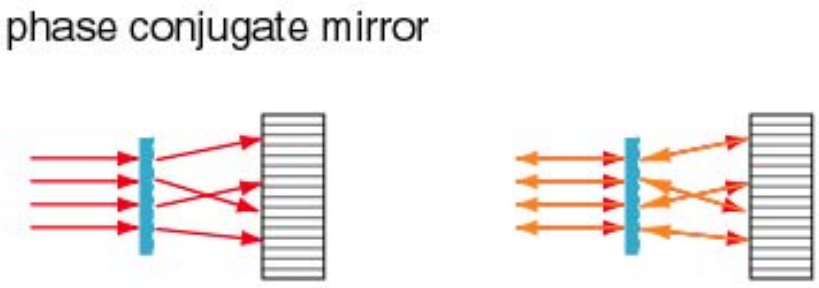
Images source :
http://sharp.bu.edu/~slehar/PhaseConjugate/PhaseConjugate.html

Incident wavefronts. . .

- Are reflected back exactly
- Back and forth wavefronts are identical







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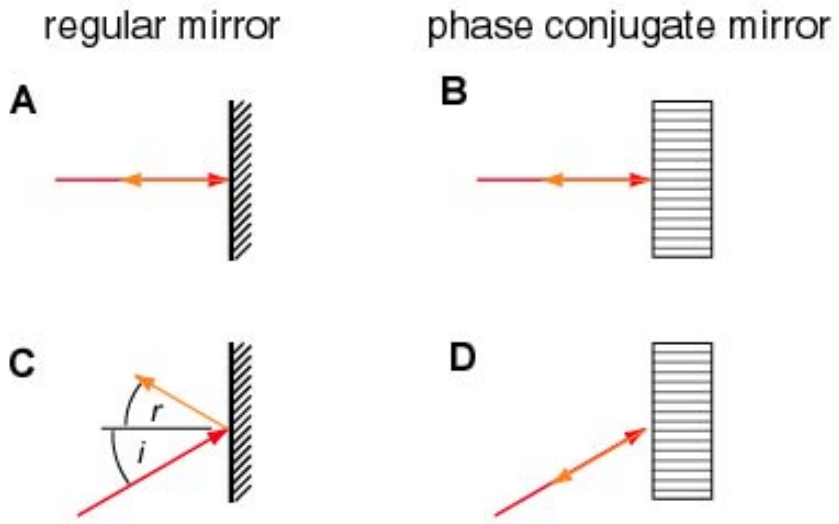

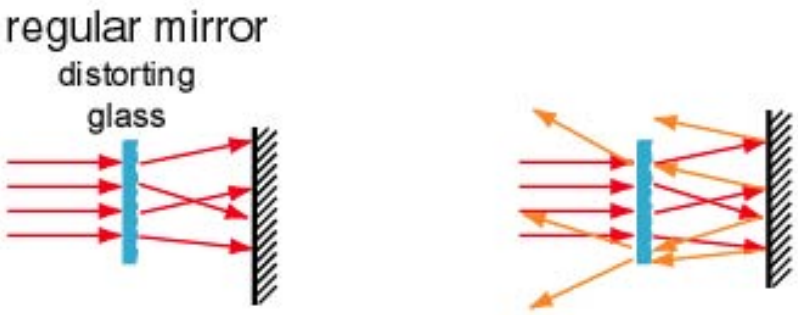

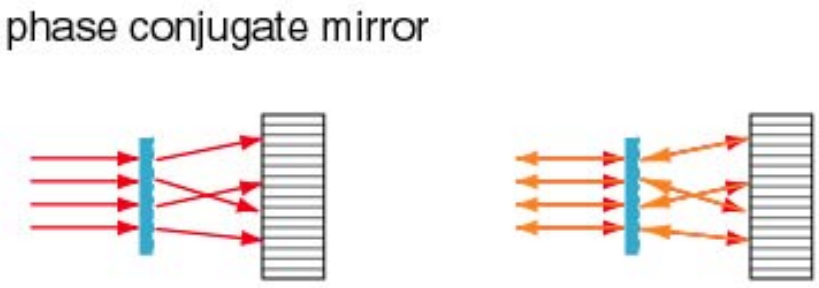

Attractive applications

All based on wavefront distortion correction

Phase conjugation applications

- All optical image transmission through fibers
- Distortion correction in high power lasers
- Dynamic wave front correction for optical sensors
- Dynamic Holography
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A phase conjugate wave travels time the wrong way

Phase conjugation is also known as *time reversal*

Take some input monochromatic wave

$$E_1 = \mathcal{R}e\left[\psi\left(r\right)\exp\left(i\left(\omega t - kz\right)\right)\right] = \mathcal{R}e\left[\psi\left(r\right)\right]\cos\left(\omega t - kz\right)$$

Take the phase conjugate *over space only*

- $E_2 = \mathcal{R}e\left[\overline{\psi\left(r\right)\exp\left(i\left(-kz\right)\right)}e^{i\omega t}\right]$
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The distortion correction theorem

If a backward traveling wave is phase conjugate *somewhere* then it is *everywhere*

Take a paraxial forward propagating wave

- Expressed as: $E_1(r,t) = \psi_1(r)\, e^{i(\omega t - kz)}$
- Obeys the wave equation: $\Delta E_1 + \omega^2 \mu_0 \varepsilon(r) E_1 = 0$
- In the paraxial limit: $\Delta\psi_1 + \left[\omega^2\mu_0\varepsilon(r) - k^2\right]\psi_1 - 2ik\frac{\partial\psi_1}{\partial z} = 0$
- Conjugate equation: $\Delta\overline{\psi_1} + \left[\omega^2\mu_0\overline{\varepsilon(r)} - k^2\right]\overline{\psi_1} + 2ik\frac{\partial\overline{\psi_1}}{\partial z} = 0$

Had we taken a backward propagating wave

- Expressed as: $E_2(r,t) = \psi_2(r)\, e^{i(\omega t + kz)}$
- Paraxial equation ($z \to -z$):
  $\Delta\psi_2 + \left[\omega^2\mu_0\varepsilon(r) - k^2\right]\psi_2 + 2ik\frac{\partial\psi_2}{\partial z} = 0$

Same second order linear differential equations for loss-less media

$$\varepsilon(r) \in \mathbb{R} \Rightarrow \left[\psi_2(z=0) = a.\overline{\psi_1(z=0)} \Leftrightarrow \forall z, \psi_2(z) = a.\overline{\psi_1(z)}\right]$$

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Four wave mixing

Third Order Non Linear Optics

Flashback : Second Order

- Relies on χ_2 : $P_{NL} \propto E^2$
- Two waves mix to generate a third one
- $\omega_1 \pm \omega_2 \rightarrow \omega_3$
- Second Harmonic Generation, Optical Parametric Amplification, Optical Parametric Oscillation...

Third order

- Relies on χ_3 : $P_{NL} \propto E^3$
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- $\omega_1 \pm \omega_2 \pm \omega_3 \rightarrow \omega_4$
- Phase conjugation for $\omega_4 = \omega_1 + \omega_2 - \omega_3$? Let's see...

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Non Linear Polarization P_{NL}

General polarization development

$$[P]_i = \varepsilon_0 [\chi]_{ij} [E]_j + 2[d]_{ijk} [E]_j [E]_k + 4[\chi]_{ijkl} [E]_j [E]_k [E]_l$$

Third order non linear development[1] around $\omega_4 = \omega_1 + \omega_2 - \omega_3$

$$[P_{NL}]_i (\omega_4) = 6[\chi]_{ijkl} [E]_j (\omega_1) [E]_k (\omega_2) \overline{[E]_l (\omega_3)} e^{i(\omega_4 t + k_4 r)}$$

Degenerate Four wave mixing : $\omega = \omega + \omega - \omega$

$$[P_{NL}]_i (\omega) = 6[\chi]_{ijkl} [E]_j (\omega) [E]_k (\omega) \overline{[E]_l (\omega)} e^{i(\omega t + kr)}$$

[1] As an exercise, you can multiply, sum-up and keep only ω_4 related terms. . . and find k_4

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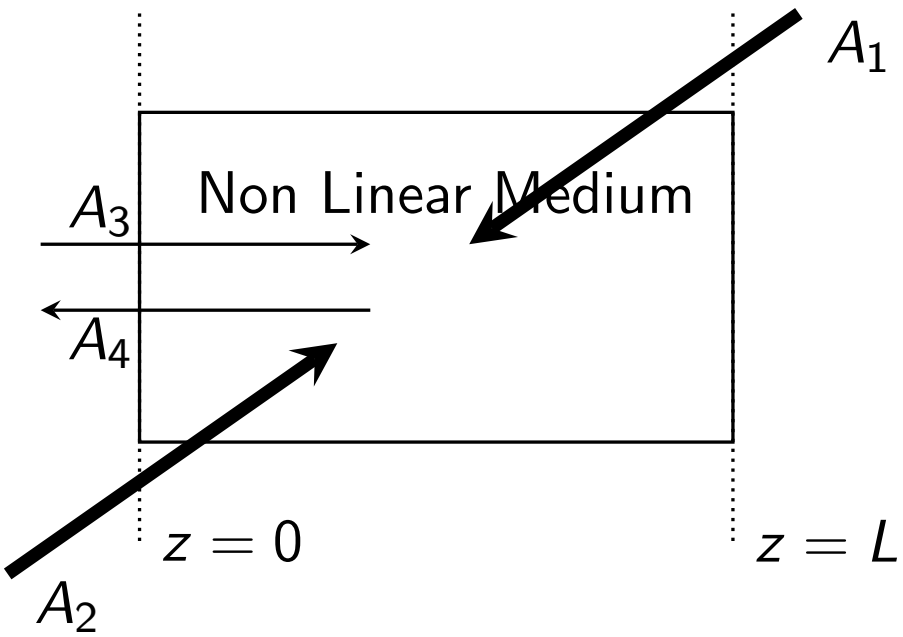
Degenerate Four Wave mixing configuration

Degenerate configuration

- $A_1 = \overline{A_2}$ intense plane pumps
- A_3 is the signal
- We seek A_4



Signal wave equation

Let us start with the standard non linear wave equation

$$\Delta E - \mu_0 \varepsilon \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$$

Signal A_3 propagation

- Each wave has its own direction and polarization
- They can be treated separately
- With $\Delta E_3 = Re\left[\left(-k^2 A_3 - 2ik\frac{\partial A_3}{\partial z} + \frac{\partial^2 A_3}{\partial z^2}\right) e^{i(\omega t - kz)}\right]$
- $Re\left[\left((\omega^2 \mu_0 \varepsilon - k^2) A_3 - 2ik\frac{\partial A_3}{\partial z} + \frac{\partial^2 A_3}{\partial z^2}\right) e^{i(\omega t - kz)}\right] = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$

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$$\Delta E - \mu_0 \varepsilon \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$$

Signal A_3 propagation

- Each wave has its own direction and polarization
- They can be treated separately
- With $\Delta E_3 = \mathcal{R}e\left[\left(-k^2 A_3 - 2ik\frac{\partial A_3}{\partial z} + \frac{\partial^2 A_3}{\partial z^2}\right) e^{i(\omega t - kz)}\right]$
- $\mathcal{R}e\left[\left(\left(\omega^2 \mu_0 \varepsilon - k^2\right) A_3 - 2ik\frac{\partial A_3}{\partial z} + \frac{\partial^2 A_3}{\partial z^2}\right) e^{i(\omega t - kz)}\right] = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$

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Dispersion equation

$$\omega^2 \mu_0 \varepsilon = k^2$$

Signal wave equation

Let us start with the standard non linear wave equation

$$\Delta E - \mu_0 \varepsilon \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$$

Signal A_3 propagation

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- $\mathcal{R}e\left[\left(-2ik\frac{\partial A_3}{\partial z} + \frac{\partial^2 A_3}{\partial z^2}\right) e^{i(\omega t - kz)}\right] = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$

Assumption

Slow varying assumed : $\|k\frac{\partial A_3}{\partial z}\| \gg \|\frac{\partial^2 A_3}{\partial z^2}\|$

Signal wave equation

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Evaluation of the non linear polarization P_{NL}

Let us take a look at the terms which involve $e^{i(\omega t \pm kz)}$

Stripping the non linear polarization to useful terms

Keeping only the relevant terms which contain $e^{i(\omega t \pm kz)}$

Expansion of third order non linear polarization

$$[P_{NL}]_i = \mathcal{R}e\left[6\begin{pmatrix} & [\chi]_{ijkl} & [A_1]_j & [A_2]_k & \overline{[A_4]_l} \\ + & [\chi]_{ijji} & [A_1]_j & \overline{[A_1]_j} & \overline{[A_3]_i} \\ + & [\chi]_{ikki} & [A_2]_k & \overline{[A_2]_k} & [A_3]_i \\ + & [\chi]_{iiii} & [A_3]_i & \overline{[A_3]_i} & [A_3]_i \\ + & [\chi]_{illi} & [A_4]_l & \overline{[A_4]_l} & [A_3]_i \end{pmatrix} e^{i(\omega t - kz)}\right]$$

Simplifying assumptions

- Intense pump beam terms are dominant
- Polarizations are
  - either all the same, only $[\chi]_{iiii}$ involved
  - or $(A_1/A_2) \perp (A_3/A_4)$, only $[\chi]_{ijji}, i \neq j$ involved

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  - $[P_{NL}]_i = \chi^{(3)}\left[(\|A1\|^2 + \|A2\|^2)A_3 + A_1 A_2 \overline{A_4}\right] e^{i(\omega t - kz)}$

    $\chi^{(3)} = 6[\chi]_{iii}$ or $\chi^{(3)} = 6[\chi]_{iii}$

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Expansion of third order non linear polarization

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$$\chi^{(3)} = 6[\chi]_{iiii} \text{ or } \chi^{(3)} = 6[\chi]_{ijji}$$

Resulting coupled wave propagation equation

Coupled wave equation resulting of P_{NL}

$$-2ik\frac{\partial A_3}{\partial z}e^{i(\omega t-kz)} = \mu_0\frac{\partial^2 P_{NL}}{\partial t^2}$$

Further simplification

- Homogeneous refraction index modulation : Kerr effect
  - Simple phase factor change
  - Remove it from equation $A_i' = A_i e^{-i\frac{\omega}{2}\sqrt{\frac{\mu_0}{\varepsilon}}\chi^{(3)}\left(\|A_1\|^2+\|A_2\|^2\right)z}$
- Set $\kappa = \frac{\omega}{2}\sqrt{\frac{\mu_0}{\varepsilon}}\chi^{(3)}A_1A_2$

Simplified coupling equations

- $\frac{\partial A_3}{\partial z} = j\kappa A_4^*$
- $\frac{\partial A_4^*}{\partial z} = j\kappa A_3^*$ obtained through the same Fourier transformation

Resulting coupled wave propagation equation

Coupled wave equation resulting of P_{NL}

$$\frac{\partial A_3}{\partial z} = -i\frac{\omega}{2}\sqrt{\frac{\mu_0}{\varepsilon}}\chi^{(3)}\left[\left(\|A1\|^2 + \|A2\|^2\right)A_3 + A_1 A_2 \overline{A_4}\right]$$

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- Set $\overline{\kappa} = \frac{\omega}{2}\sqrt{\frac{\mu_0}{\varepsilon}}\chi^{(3)}A_1 A_2$

Simplified coupling equations

- $\frac{\partial \overline{A_4}}{\partial z} = \overline{\kappa} A_3$
- $\frac{\partial A_3}{\partial z} = \overline{\kappa}\overline{A_4}$ obtained through the same kind of reasoning

Resulting coupled wave propagation equation

Coupled wave equation resulting of P_{NL}

$$\frac{\partial A_3}{\partial z} = -i\frac{\omega}{2}\sqrt{\frac{\mu_0}{\varepsilon}}\chi^{(3)}\left[\left(\|A1\|^2 + \|A2\|^2\right)A_3 + A_1 A_2 \overline{A_4}\right]$$

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General solution

Boundary conditions at $z = 0$ and $z = L$

A_3 is forward propagating — **A_4 is backward propagating**

- $A_3'(z) = -i\dfrac{|\kappa|\sin(|k|z)}{\kappa\cos(|k|L)}\overline{A_4'(L)} + \dfrac{\cos(|k|(z-L))}{\cos(|k|L)}A_3'(0)$
- $A_4'(z) = \dfrac{\cos(|k|z)}{\cos(|k|L)}A_4'(L) + i\dfrac{\overline{\kappa}\sin(|k|(z-L))}{|k|\cos(|k|L)}\overline{A_3'(0)}$

One beam experiment — $A_4'(L) = 0$

- $A_3'(L) = \dfrac{A_3'(0)}{\cos(|k|L)}$ — Coherent amplifier
- $A_4'(0) = -i\dfrac{\overline{\kappa}}{|k|}\tan(|k|L)\overline{A_3'(0)}$ — Phase conjugate wave

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One Beam experiment and phase factor

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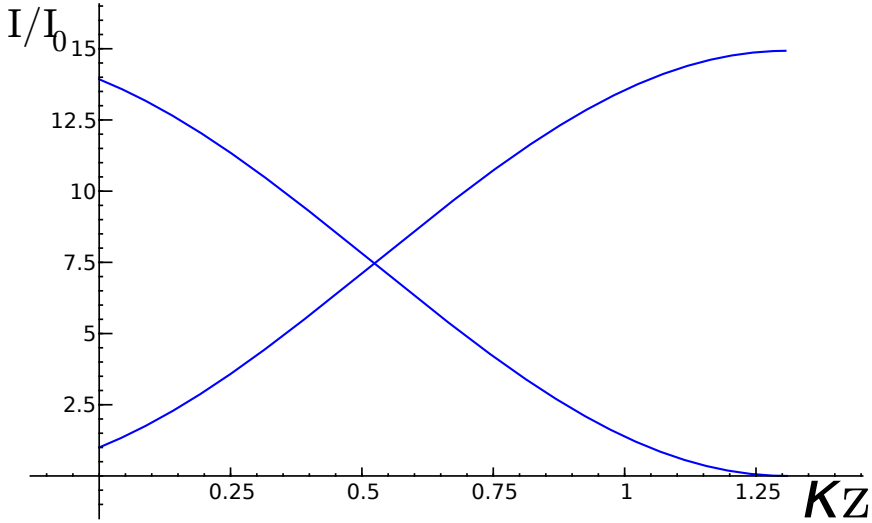

What if $\cos(|k| L) = 0$?

- Infinite gain
- A_3 and A_4 start from noise
- Spontaneous oscillations

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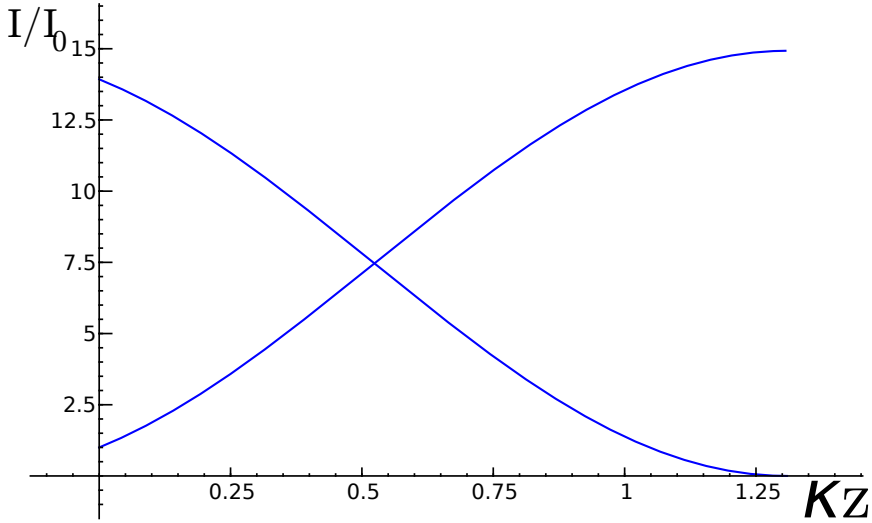

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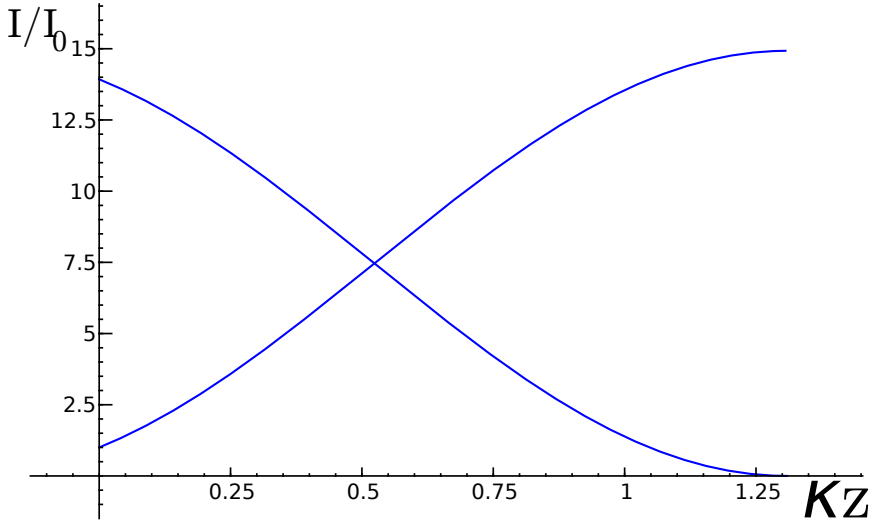

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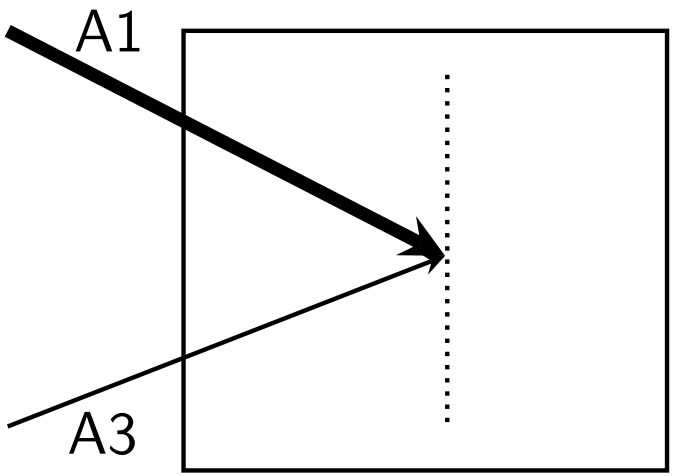
Four Wave Mixing is Real Time Holography

Write Hologram

$$T \propto \|A_1 + A_3\|^2 = \|A_1\|^2 + \|A_3\|^2 + A_1\overline{A_3} + A_3\overline{A_1}$$

Read Hologram with $A_2 = \overline{A_1}$

$$A_4 \propto TA_2 = \left(\|A_1\|^2 + \|A_3\|^2\right) A_2 + A_2 A_1 \overline{A_3} + A_2 A_3 \overline{A_1}$$



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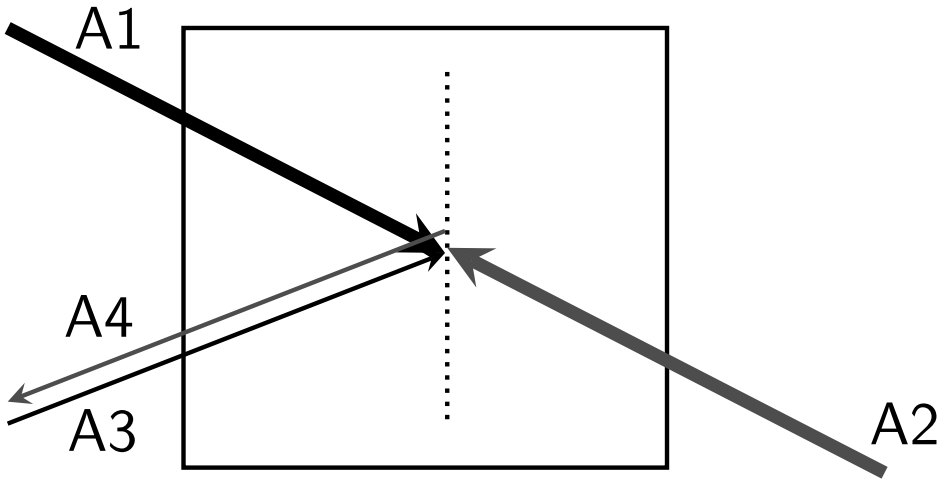
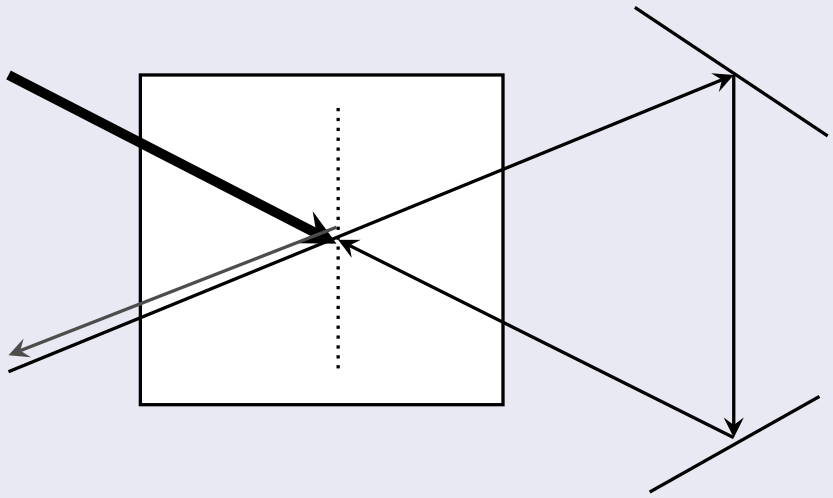


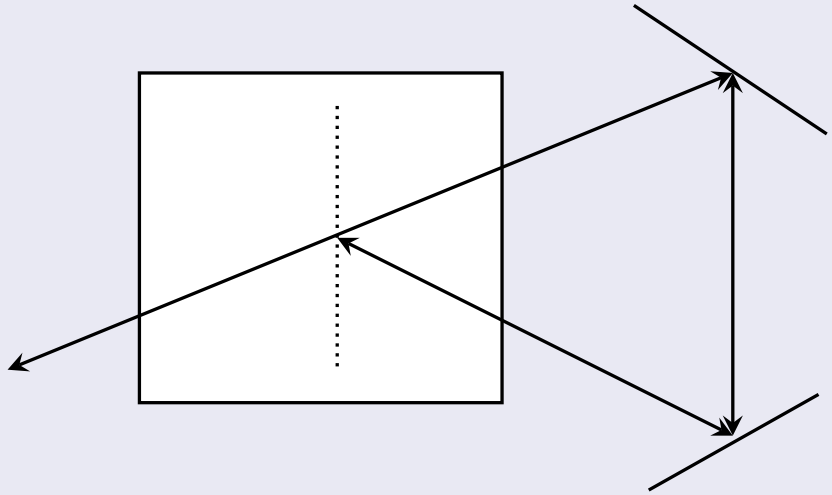
CAT Conjugator using Beam Fanning

Use Mirrors



CAT Conjugator using Beam Fanning

Beam fanning for pump



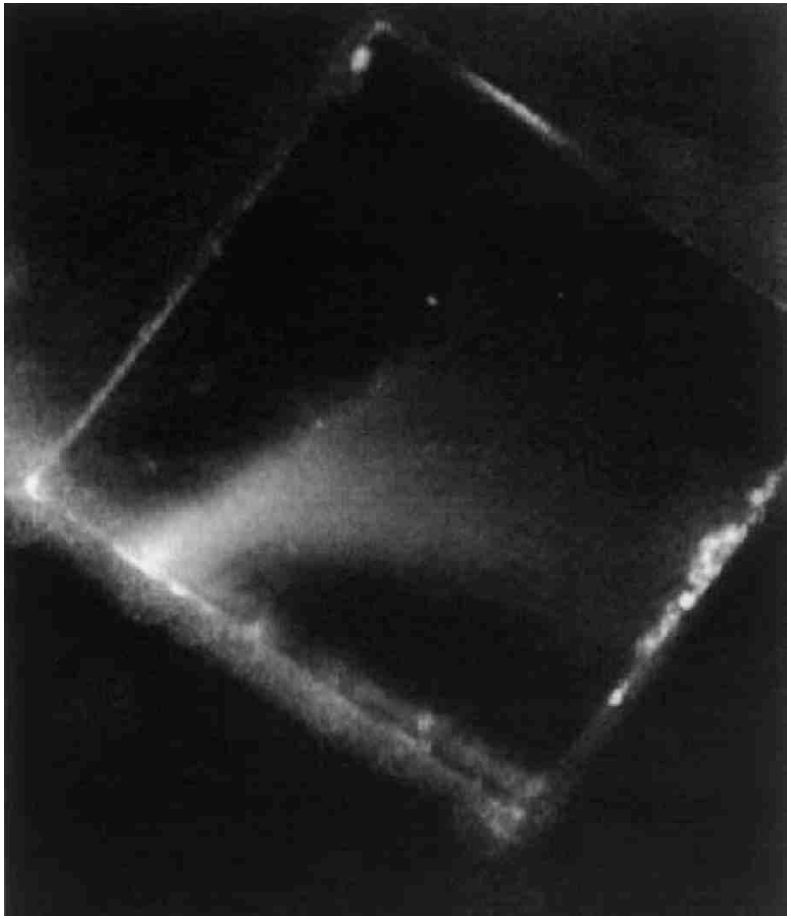
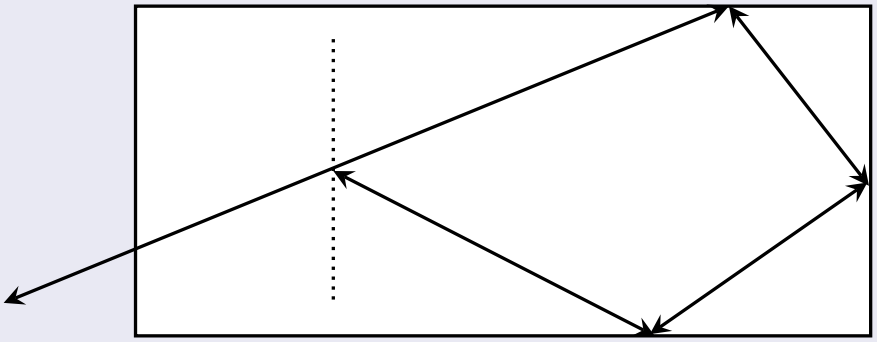


Figure: Beam fanning in photorefractive Baryum Titanate

CAT Conjugator using Beam Fanning

Total internal reflection



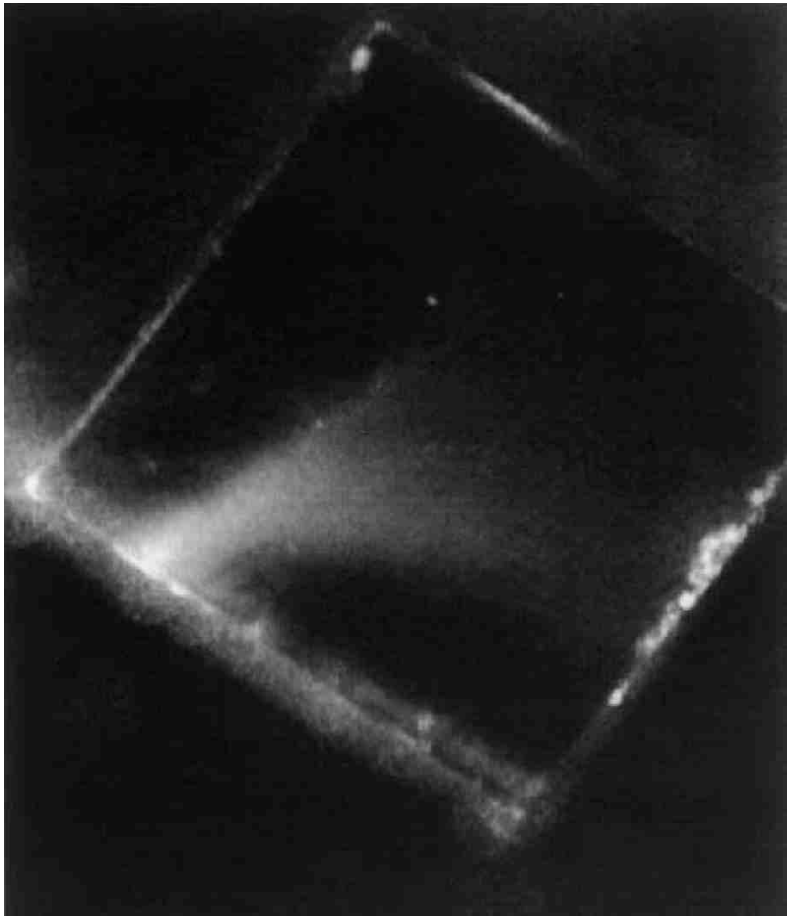


Figure: Beam fanning in photorefractive Baryum Titanate