Second Harmonic Generation and related second order Nonlinear Optics
Nicolas Fressengeas

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Second Harmonic Generation
and related second order Nonlinear Optics

N. Fressengeas

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November 12, 2012
Useful reading... [YY84, DGN91, LKW99]


Contents

1 Three-wave interaction
   - Assumptions framework
   - Three Wave propagation equation
   - Sum frequency generation
   - Scalar approximation

2 Non Linear Optics Application
   - Second Harmonic Generation
   - Optical Parametric Amplifier
   - Optical Parametric Oscillator

3 Phase matching
   - Phase matching conditions
   - Phase matching in uni-axial crystals
   - Quasi-phase matching
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A classical Maxwell framework

With standard assumptions: no charge, no current, no magnet and no conductivity

Maxwell Model

- \( \text{div} (D) = 0 \)
- \( \text{div} (B) = 0 \)
- \( \text{curl} (E) = -\frac{\partial B}{\partial t} \)
- \( \text{curl} (H) = \frac{\partial D}{\partial t} \)

Matter equations

- \( D = \varepsilon_0 E + P = \varepsilon E \)
- \( B = \mu_0 H \)
- \( P = \varepsilon_0 \chi_L E + P_{NL} \)

Wave equation in isotropic medium

\[
\Delta E - \mu_0 \varepsilon \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}
\]

Solutions to be found only in a specific framework:

We look here for:

- Quadratic non-linearity
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All waves are transverse plane waves propagating in the $z$ direction

- Transversal $E$: has $x$ and $y$ components
- Wave equation:
  \[
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  \]

Quadratic non-linearity

- $P_{NL}$ is transversal
- \[
  [P_{NL}]_i = \sum_{\{j,k\} \in \{x,y\}^2} [d]_{ijk} [E]_j [E]_k = [d]_{ijk} [E]_j [E]_k
  \]

Three wave interaction only

- Three waves only are present: $\omega_1$, $\omega_2$ and $\omega_3$
- Non linear interaction of two waves: sum, difference, doubling, rectification...
- We consider only those for which $\omega_1 + \omega_2 = \omega_3$
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Three wave interaction solution ansatz

Sum of three waves

\[ [E]_{x,y} (z, t) = \mathcal{R}e \left( \sum_{\nu=1}^{3} \exp (i k_{\nu} z - i \omega_{\nu} t) \right) \]

- Dispersion law: \( k_{\nu}^2 = \mu_0 \varepsilon_{\nu} \omega_{\nu}^2 \)
- \( \varepsilon \) dispersion: \( \varepsilon_{\nu} = \varepsilon (\omega_{\nu}) \)

How good is this ansatz?

- We have assumed \( \omega_1 + \omega_2 = \omega_3 \)
- Why would \( \omega_3 \) not be a source as well?
- OK if wave 3 is small enough

Separate investigation at each frequency

- At \( \omega_{\nu} \), consider only the part of \( P_{NL} \) oscillating at frequency \( \omega_{\nu} \): \( P_{NL}^{\omega_{\nu}} \)
- Three separate equations
Three wave interaction solution ansatz

**Sum of three waves**

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Wave propagation for frequency $\omega_\nu, \nu \in \{1, 2, 3\}$

3 Perturbed wave equations

$$
\frac{\partial^2 E^{(\omega_\nu)}(z,t)}{\partial z^2} - \mu_0 \varepsilon \omega_\nu \frac{\partial^2 E^{(\omega_\nu)}(z,t)}{\partial t^2} = \mu_0 \frac{\partial^2 P^{\omega_\nu}_{NL}(z,t)}{\partial t^2}
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Temporal harmonic notation

- $E^{(\omega_\nu)}$ is a plane wave
- Non linear polarization is a plane wave
- Considering only... the $\omega_\nu$ part $\Leftrightarrow$ the plane wave part
- $\frac{\partial^2 E^{(\omega_\nu)}(z,t)}{\partial z^2} + \mu_0 \varepsilon \omega_\nu \omega_\nu E^{(\omega_\nu)}(z,t) = -\mu_0 \omega_\nu^2 P^{\omega_\nu}_{NL}(z,t)$
Wave propagation for frequency $\omega_\nu$ \quad $\nu \in \{1, 2, 3\}$

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Non Linear Polarisation $P_{NL}$ in harmonic framework

Reminder

$[P_{NL}]_i = [d]_{ijk}[E]_j[E]_k$

Temporal harmonic framework for $\omega_1$

- Multiply complex fields
- Include Conjugates to take Real Part
- Select only the $\omega_1$ component

$[P_{NL}^{\omega_1}(z, t)]_i = \text{Re} \left( [d]_{ijk}[E^{(\omega_3)}(z)]_j[E^{(\omega_2)}(z)]_k e^{i(k_3-k_2)z-i(\omega_3-\omega_2)t} \right)$

Wave propagation equation

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The Slow Varying Approximation

The Slow Varying Approximation

- Beam envelope is assumed to vary slowly in the longitudinal direction
- Equivalent as assuming a narrow beam
- Second derivative with respect to \(z\) neglected compared to
  - the first one with respect to \(z\)
  - the others with respect to \(x\) and \(y\)

To put it in maths...

\[
\frac{\partial^2 E^{(\omega_1)}(z,t)}{\partial z^2} = \frac{\partial^2}{\partial z^2} \text{Re} \left( E^{(\omega_1)}(z) \exp \left( i \left( k_1 z - \omega_1 t \right) \right) \right)
\]

\[
\ldots = \text{Re} \left( \frac{\partial^2 E^{(\omega_1)}(z)}{\partial z^2} + 2ik_1 \frac{\partial E^{(\omega_1)}(z)}{\partial z} - k_1^2 E^{(\omega_1)}(z) \right) e^{i(k_1 z - \omega_1 t)} \right)
\]
The Slow Varying Approximation

Closely related to the paraxial approximation

- Beam envelope is assumed to vary slowly in the longitudinal direction.
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\[
\frac{\partial^2 E^{(\omega_1)}(z,t)}{\partial z^2} = \partial^2 \text{Re} \left( E^{(\omega_1)}(z) \exp \left( i \left( k_1 z - \omega_1 t \right) \right) \right)
\]

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\cdots = \Re \left( \left[ \frac{\partial^2 E(\omega_1)(z)}{\partial z^2} + 2i k_1 \frac{\partial E(\omega_1)(z)}{\partial z} - k_1^2 E(\omega_1)(z) \right] e^{i(k_1 z - \omega_1 t)} \right)
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Wave Propagation Equation under SVA approximation

Obtaining an envelope equation, which is simpler

Non SVA wave propagation equation

\[
\frac{\partial^2 E^{(\omega_1)}(z, t)}{\partial z^2} + \mu_0 \varepsilon_1 \omega_1^2 E^{(\omega_1)}(z, t) = -\mu_0 \omega_1^2 P^{\omega_1}_{NL}(z, t)
\]

SVA equation

\[
\Re \left( \left[ 2ik_1 \frac{\partial E^{(\omega_1)}(z)}{\partial z} - k_1^2 E^{(\omega_1)}(z) \right] e^{(i(k_1 z - \omega_1 t))} \right) \\
+ \Re \left( \mu_0 \varepsilon_1 \omega_1^2 E^{(\omega_1)}(z) e^{(i(k_1 z - \omega_1 t))} \right) \\
= \Re \left( -\mu_0 \omega_1^2 [d]_{ijk} E^{(\omega_3)}(z)_j [E^{(\omega_2)}(z)_k] e^{(i(k_3 - k_2)z - i(\omega_3 - \omega_2)t)} \right)
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\]

SVA equation

- \(\Re \left( 2 ik_1 \frac{\partial E^{(\omega_1)}(z)}{\partial z} - k_1^2 E^{(\omega_1)}(z) \right) e^{i(k_1z-\omega_1t)} \)
- \(+ \Re \left( \mu_0 \varepsilon_\omega \omega_1^2 E^{(\omega_1)}(z) e^{i(k_1z-\omega_1t)} \right) \)
- \(= \Re \left( -\mu_0 \omega_1^2 [d]_{ijk} E^{(\omega_3)}(z) e^{i(k_3-k_2)z-i(\omega_3-\omega_2)t} \right) \)
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Obtaining an envelope equation, which is simpler

Non SVA wave propagation equation

\[ \frac{\partial^2 E^{(\omega_1)}(z, t)}{\partial z^2} + \mu_0 \varepsilon_\omega \omega_1^2 E^{(\omega_1)}(z, t) = -\mu_0 \omega_1^2 P_{NL}^{\omega_1}(z, t) \]

SVA equation

\[ \text{Re} \left( \left[ 2ik_1 \frac{\partial E^{(\omega_1)}(z)}{\partial z} - k_1^2 E^{(\omega_1)}(z) \right] e^{i(k_1 z - \omega_1 t)} \right) \]

\[ + \text{Re} \left( \mu_0 \varepsilon_\omega \omega_1^2 E^{(\omega_1)}(z) e^{i(k_1 z - \omega_1 t)} \right) \]

\[ = \text{Re} \left( -\mu_0 \omega_1^2 [d]_{ijk} [E^{(\omega_3)}(z)]_j [E^{(\omega_2)}(z)]_k e^{i(k_3 - k_2) z - i(\omega_3 - \omega_2) t} \right) \]
Wave Propagation Equation under SVA approximation

Obtaining an envelope equation, which is simpler

Non SVA wave propagation equation

\[
\frac{\partial^2 E^{(\omega_1)}(z, t)}{\partial z^2} + \mu_0 \varepsilon \omega_1 \omega_1^2 E^{(\omega_1)}(z, t) = -\mu_0 \omega_1^2 P_{NL}^{\omega_1}(z, t)
\]

SVA equation

1. \[ \Re \left( \left[ 2i k_1 \frac{\partial E^{(\omega_1)}(z)}{\partial z} - k_1^2 E^{(\omega_1)}(z) \right] e^{i(k_1 z - \omega_1 t)} \right) \]
2. \[ + \Re \left( \mu_0 \varepsilon \omega_1 \omega_1^2 E^{(\omega_1)}(z) e^{i(k_1 z - \omega_1 t)} \right) \]
3. \[ = \Re \left( -\mu_0 \omega_1^2 [d]_{ijk} \left[ E^{(\omega_3)}(z) \right]_j \left[ E^{(\omega_2)}(z) \right]_k e^{i(k_3 - k_2) z - i(\omega_3 - \omega_2) t} \right) \]
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Obtaining an envelope equation, which is simpler

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\]

SVA equation

\[
\mathcal{R}e \left( \left( 2ik_1 \frac{\partial E^{(\omega_1)}(z)}{\partial z} \right) e^{i(k_1 z - \omega_1 t)} \right)
\]

\[
= \mathcal{R}e \left( -\mu_0 \omega_1^2 [d]_{ijk} E^{(\omega_3)}(z) \frac{E^{(\omega_2)}(z)}{E^{(\omega_2)}(z)} k e^{i(k_3 - k_2)z - i(\omega_3 - \omega_2)t} \right)
\]
Wave Propagation Equation under SVA approximation

Obtaining an envelope equation, which is simpler

Non SVA wave propagation equation

\[
\frac{\partial^2 E^{(\omega_1)}(z, t)}{\partial z^2} + \mu_0 \varepsilon_{\omega_1} \omega_1^2 E^{(\omega_1)}(z, t) = -\mu_0 \omega_1^2 P_{NL}^{\omega_1}(z, t)
\]

SVA equation

\[
\left( \begin{array}{c} 2ik_1 \frac{\partial E^{(\omega_1)}(z)}{\partial z} e^{(i(k_1 z - \omega_1 t))} \\ -\mu_0 \omega_1^2 [d]_{ijk} E^{(\omega_3)}(z) J_k E^{(\omega_2)}(z) e^{(i(k_3 - k_2) z - i(\omega_3 - \omega_2) t)} \end{array} \right)
\]
Wave Propagation Equation under SVA approximation

Obtaining an envelope equation, which is simpler

Non SVA wave propagation equation

\[
\frac{\partial^2 E(\omega_1)(z, t)}{\partial z^2} + \mu_0 \varepsilon \omega_1 \omega_1^2 E(\omega_1)(z, t) = -\mu_0 \omega_1^2 P_{NL}^{\omega_1}(z, t)
\]

SVA equation

\[
\begin{align*}
&\left(2ik_1 \frac{\partial E(\omega_1)(z)}{\partial z}\right) \\
&= \left(-\mu_0 \omega_1^2 [d]_{ijk} E(\omega_3)(z) \overline{E(\omega_2)(z)} e^{i(k_3-k_2-k_1)z}\right)
\end{align*}
\]
The three waves

Phase mismatch and dispersion relationship

- Phase mismatch: \( \Delta k = k_1 + k_2 - k_3 \)
- Recall the dispersion relationship: \( k_1^2 = \mu_0 \varepsilon \omega_1^2 \)
- Wave impedance: \( \eta_\nu = \sqrt{\frac{\mu_0}{\varepsilon \omega_\nu}} \)

Three wave propagation, rotating \( i \rightarrow j \rightarrow k \) \((i, j, k) \in \{x, y\}^3\)

\[
\left[ \frac{\partial E(\omega_1)}{\partial z} \right]_i = \frac{i \omega_1}{2} \eta_1 [d]_{ijk} [E(\omega_3)]_j [E(\omega_2)]_k \exp(-i \Delta kz)
\]

\[
\left[ \frac{\partial E(\omega_2)}{\partial z} \right]_k = -\frac{i \omega_2}{2} \eta_2 [d]_{kij} [E(\omega_1)]_i [E(\omega_3)]_j \exp(-i \Delta kz)
\]

\[
\left[ \frac{\partial E(\omega_3)}{\partial z} \right]_j = \frac{i \omega_3}{2} \eta_3 [d]_{jk} [E(\omega_2)]_k [E(\omega_1)]_i \exp(i \Delta kz)
\]
The three waves

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Three wave propagation, rotating \( i \rightarrow j \rightarrow k \)  \((i, j, k) \in \{x, y\}^3\)

\[
\begin{align*}
\frac{\partial E(\omega_1)}{\partial z} & = \frac{i \omega_1}{2} \eta_1 [d]_{ijk} [E(\omega_3)]_j [E(\omega_2)]_k \exp(-i \Delta k z) \\
\frac{\partial E(\omega_2)}{\partial z} & = -\frac{i \omega_2}{2} \eta_2 [d]_{kij} [E(\omega_1)]_i [E(\omega_3)]_j \exp(-i \Delta k z) \\
\frac{\partial E(\omega_3)}{\partial z} & = \frac{i \omega_3}{2} \eta_3 [d]_{jki} [E(\omega_2)]_k [E(\omega_1)]_i \exp(i \Delta k z)
\end{align*}
\]
6 equations for various quadratic phenomena

All in one for: frequency sum and difference, second harmonic generation and optical rectification, parametric amplifier...

Six equations

\[
\begin{align*}
\left[ \frac{\partial E(\omega_1)}{\partial z} \right]_i & = + \frac{i \omega_1}{2} \eta_1 [d]_{ijk} [E(\omega_3)]_j \overline{E(\omega_2)}_k \exp (-i \Delta k z) \\
\left[ \frac{\partial E(\omega_2)}{\partial z} \right]_k & = - \frac{i \omega_2}{2} \eta_2 [d]_{kij} [E(\omega_1)]_i \overline{E(\omega_3)}_j \exp (-i \Delta k z) \\
\left[ \frac{\partial E(\omega_3)}{\partial z} \right]_j & = + \frac{i \omega_3}{2} \eta_3 [d]_{jki} [E(\omega_2)]_k \overline{E(\omega_1)}_i \exp (i \Delta k z)
\end{align*}
\]

Why all those names?

They differ by:

- The input frequencies and the generated ones
- The one that is the smallest and those which are large
- ...
6 equations for various quadratic phenomena
All in one for: frequency sum and difference, second harmonic generation and optical rectification, parametric amplifier...

Six equations

\[
\begin{align*}
\left[ \frac{\partial E^{(\omega_1)}}{\partial z} \right]_i &= + \frac{i\omega_1}{2} \eta_1 [d]_{ijk} [E^{(\omega_3)}]_j [E^{(\omega_2)}]_k \exp(-i\Delta kz) \\
\left[ \frac{\partial E^{(\omega_2)}}{\partial z} \right]_k &= - \frac{i\omega_2}{2} \eta_2 [d]_{kij} [E^{(\omega_1)}]_i [E^{(\omega_3)}]_j \exp(-i\Delta kz) \\
\left[ \frac{\partial E^{(\omega_3)}}{\partial z} \right]_j &= + \frac{i\omega_3}{2} \eta_3 [d]_{jki} [E^{(\omega_2)}]_k [E^{(\omega_1)}]_i \exp(i\Delta kz)
\end{align*}
\]

Why all those names?
They differ by:

- The input frequencies and the generated ones
- The one that is the smallest and those which are large
- ...

(i, j, k) ∈ \{x, y\}³
Three-wave interaction

- Assumptions framework
- Three Wave propagation equation
- **Sum frequency generation**
- Scalar approximation

Non Linear Optics Application

- Second Harmonic Generation
- Optical Parametric Amplifier
- Optical Parametric Oscillator

Phase matching

- Phase matching conditions
- Phase matching in uni-axial crystals
- Quasi-phase matching
Example: sum frequency generation

Input beams assumed constant  Undepleted pump approximation

Assumptions

- $\omega_1 + \omega_2 = \omega_3$
- Generated beam null at $z = 0$ : $[E(\omega_3)(z = 0)]_j = 0$

$$\frac{\partial [E(\omega_1)]_i}{\partial z} = \frac{\partial [E(\omega_2)]_k}{\partial z} = 0$$

One equation remains:

$$\left[\frac{\partial E(\omega_3)}{\partial z}\right]_j = + \frac{i \omega_3}{2} \eta_3 [d]_{jki} [E(\omega_2)]_k [E(\omega_1)]_i \exp (-i \Delta kz)$$
Example: sum frequency generation

Input beams assumed constant
Undepleted pump approximation

Assumptions

- $\omega_1 + \omega_2 = \omega_3$
- Generated beam null at $z = 0$: $[E^{(\omega_3)} (z = 0)]_j = 0$
- $\frac{\partial [E^{(\omega_1)}]}{\partial z}_i = \frac{\partial [E^{(\omega_2)}]}{\partial z}_k = 0$

One equation remains

$$[\frac{\partial E^{(\omega_3)}}{\partial z}]_j = + \frac{i\omega_3}{2} \eta_3 [d]_{jki} [E^{(\omega_2)}]_k [E^{(\omega_1)}]_i \exp (-i\Delta kz)$$
Solving the SVA wave propagation equation

Equation to solve

\[
\left[ \frac{\partial E(\omega_3)}{\partial z} \right]_j = + \frac{i\omega_3}{2} \eta_3 [d]_{jki} \left[ E(\omega_2) \right]_k \left[ E(\omega_1) \right]_i \exp(-i\Delta kz)
\]

\[
\Delta k \neq 0
\]

\[
y' = ae^{ibx} \Rightarrow y = \frac{ia}{b} \left( 1 - e^{ibx} \right)
\]

Wave solution

\[
\left[ E(\omega_3) \right]_j
\]

\[
\frac{i\omega_3}{2} \eta_3 [d]_{jki} \left[ E(\omega_2) \right]_k \left[ E(\omega_1) \right]_i \frac{e^{i\Delta kz} - 1}{i\Delta k}
\]

Intensity

\[
\propto \left[ E(\omega_3) \right]_j \left[ E(\omega_3) \right]_j
\]

\[
\omega_3^2 \eta_3^2 [d^2]_{jki} \left| E(\omega_2) \right|_k^2 \left| E(\omega_1) \right|_i^2 \frac{\sin^2 \left( \frac{\Delta kz}{2} \right)}{\Delta k^2}
\]
Solving the SVA wave propagation equation

Equation to solve

\[ \left[ \frac{\partial E(\omega_3)}{\partial z} \right]_j = + \frac{i\omega_3}{2} \eta_3 [d]_{jki} E(\omega_2)_k E(\omega_1)_i \exp(-i\Delta kz) \]

\[ \Delta k \neq 0 \]

\[ y' = ae^{(ibx)} \Rightarrow y = \frac{ia}{b} \left( 1 - e^{(ibx)} \right) \]

Wave solution

\[ [E(\omega_3)]_j \]

\[ \frac{i\omega_3}{2} \eta_3 [d]_{jki} E(\omega_2)_k E(\omega_1)_i \frac{e^{(i\Delta kz)} - 1}{i\Delta k} \]

Intensity

\[ \propto [E(\omega_3)]_j \overline{[E(\omega_3)]_j} \]

\[ \omega_3^2 \eta_3^2 [d^2]_{jki} |E(\omega_2)_k|^2 |E(\omega_1)_i|^2 \frac{\sin^2(\frac{\Delta kz}{2})}{\Delta k^2} \]
Solving the SVA wave propagation equation

Equation to solve

\[
\left[ \frac{\partial E(\omega_3)}{\partial z} \right]_j = + \frac{i \omega_3}{2} \eta_3 [d]_{jki} [E(\omega_2)]_k [E(\omega_1)]_i \exp(-i \Delta k z)
\]

\[\Delta k \neq 0\]

\[y' = ae^{(ibx)} \Rightarrow y = \frac{ia}{b} (1 - e^{(ibx)})\]

Wave solution

\[
\frac{i \omega_3}{2} \eta_3 [d]_{jki} [E(\omega_2)]_k [E(\omega_1)]_i \frac{e^{(i \Delta k z)} - 1}{i \Delta k}
\]

\[\text{Intensity} \propto [E(\omega_3)]_j [E(\omega_3)]_j \]

\[
\omega_3^2 \eta_3^2 [d^2]_{jki} |E(\omega_2)|^2_k |E(\omega_1)|^2_i \frac{\sin^2(\frac{\Delta k z}{2})}{\Delta k^2}
\]
Solving the SVA wave propagation equation

Equation to solve

\[ \frac{\partial E(\omega_3)}{\partial z} = + \frac{i \omega_3^2 \eta_3 [d]_{jki} [E(\omega_2)]_k [E(\omega_1)]_i \exp(-i \Delta k z)}{2} \]

\[ \Delta k \neq 0 \]

\[ y' = a e^{(ibx)} \Rightarrow y = \frac{ia}{b} (1 - e^{(ibx)}) \]

Wave solution

\[ [E(\omega_3)]_j = \frac{i \omega_3^2 \eta_3 [d]_{jki} [E(\omega_2)]_k [E(\omega_1)]_i e^{(i \Delta k z)} - 1}{i \Delta k} \]

Intensity

\[ \propto \left| [E(\omega_3)]_j [E(\omega_3)]_j \right| \]

\[ \omega_3^2 \eta_3^2 \left| d^2 \right|_{jki} \left| E(\omega_2) \right|_k^2 \left| E(\omega_1) \right|_i^2 \sin^2 \left( \frac{\Delta k z}{2} \right) \]
Solving the SVA wave propagation equation

Equation to solve

\[
\left[ \frac{\partial E^{(\omega_3)}}{\partial z} \right]_j = + \frac{i \omega_3}{2} \eta_3 [d]_{jki} [E^{(\omega_2)}]_k [E^{(\omega_1)}]_i \exp (-i \Delta k z) \]

\( \Delta k \neq 0 \)

\[ y' = ae^{(ibx)} \Rightarrow y = \frac{ia}{b} \left( 1 - e^{(ibx)} \right) \]

Wave solution

\[ \left[ E^{(\omega_3)} \right]_j \]

Intensity

\[ \propto \left[ E^{(\omega_3)} \right]_j \left[ \left| E^{(\omega_3)} \right|_j \right] \]

\[ \omega_3^2 \eta_3^2 [d^2]_{jki} \left| E^{(\omega_2)} \right|_k^2 \left| E^{(\omega_1)} \right|_i^2 \sin^2 \left( \frac{\Delta k z}{2} \right) / \Delta k^2 \]

\( \Delta k = 0 \)

\[ y' = a \Rightarrow y = ax \]

Wave solution

\[ \left[ E^{(\omega_3)} \right]_j \]

Intensity

\[ \propto \left[ E^{(\omega_3)} \right]_j \left[ \left| E^{(\omega_3)} \right|_j \right] \]

\[ \omega_3^2 \eta_3^2 [d^2]_{jki} \left| E^{(\omega_2)} \right|_k^2 \left| E^{(\omega_1)} \right|_i^2 z^2 \]
Solving the SVA wave propagation equation

Equation to solve
\[
\left[ \frac{\partial E(\omega_3)}{\partial z} \right]_j = + \frac{i\omega_3}{2} \eta_3 [d]_{jki} \left[ E(\omega_2) \right]_k \left[ E(\omega_1) \right]_i \exp (-i\Delta kz)
\]

\[\Delta k \neq 0\]
\[y' = ae^{(ibx)} \Rightarrow y = \frac{ia}{b} \left( 1 - e^{(ibx)} \right)\]

Wave solution
\[
\left[ E(\omega_3) \right]_j \propto \frac{i\omega_3}{2} \eta_3 [d]_{jki} \left[ E(\omega_2) \right]_k \left[ E(\omega_1) \right]_i \exp \left( \frac{e^{i\Delta kz}}{i\Delta k} - 1 \right)
\]

Intensity
\[
\propto \left[ E(\omega_3) \right]_j \left[ E(\omega_3) \right]_j \propto \omega_3^2 \eta_3^2 \left[ d^2 \right]_{jki} \left| E(\omega_2) \right|^2 \left| E(\omega_1) \right|_i \frac{\sin^2 \left( \frac{\Delta kz}{2} \right)}{\Delta k^2}
\]

\[\Delta k = 0\]
\[y' = a \Rightarrow y = ax\]

Wave solution
\[
\left[ E(\omega_3) \right]_j \propto \frac{i\omega_3}{2} \eta_3 [d]_{jki} \left[ E(\omega_2) \right]_k \left[ E(\omega_1) \right]_i z
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\]
Solving the SVA wave propagation equation

Equation to solve

\[ \left[ \frac{\partial E(\omega_3)}{\partial z} \right]_j = + \frac{i\omega_3}{2} \eta_3 [d]_{jki} \left[ E(\omega_2) \right]_k \left[ E(\omega_1) \right]_i \exp(-i\Delta kz) \]

\( \Delta k \neq 0 \)

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Wave solution

\[ \left[ E(\omega_3) \right]_j \]

\[ \frac{i\omega_3}{2} \eta_3 [d]_{jki} \left[ E(\omega_2) \right]_k \left[ E(\omega_1) \right]_i \frac{e^{(i\Delta kz)} - 1}{i\Delta k} \]

Intensity

\[ \propto \left[ E(\omega_3) \right]_j \left[ E(\omega_3) \right]_j \]

\[ \omega_3^2 \eta_3^2 [d^2]_{jki} \left| E(\omega_2) \right|^2_k \left| E(\omega_1) \right|^2_i \frac{\sin^2 \left( \frac{\Delta k z}{2} \right)}{\Delta k^2} \]

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Wave solution

\[ \left[ E(\omega_3) \right]_j \]

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Intensity

\[ \propto \left[ E(\omega_3) \right]_j \left[ E(\omega_3) \right]_j \]

\[ \omega_3^2 \eta_3^2 [d^2]_{jki} \left| E(\omega_2) \right|^2_k \left| E(\omega_1) \right|^2_i z^2 \]
Phase match or not phase match

Phase matching is a key issue to sum frequency generation

Phase mismatch \( \Delta k \neq 0 \)

- Oscillating intensity
- Max intensity \( \propto \frac{1}{\Delta k^2} \)

Intensity

\[ \Delta k z/2 \]

\[ 1/\Delta k^{-2} \]
Phase match or not phase match

Phase matching is a key issue to sum frequency generation

**Phase mismatch** \( \Delta k \neq 0 \)
- Oscillating intensity
- Max intensity \( \propto \frac{1}{\Delta k^2} \)

**Phase match** \( \Delta k = 0 \)
- Intensity quadratic increase
- Approximations do not hold long
Three-wave interaction

- Assumptions framework
- Three Wave propagation equation
- Sum frequency generation
- Scalar approximation

Non Linear Optics Application

- Second Harmonic Generation
- Optical Parametric Amplifier
- Optical Parametric Oscillator

Phase matching

- Phase matching conditions
- Phase matching in uni-axial crystals
- Quasi-phase matching
A Scalar Three Wave Interaction model

Further approximations to remove vectors

Simplifying notations

- Set indexes equal for polarization and frequency: \( A_{\nu} = \left[ E(\omega_{\nu}) \right]_{\nu} \)
- Consider \( \varepsilon_{\nu} = n_{\nu}^2 \varepsilon_0 \)
- Abbreviate \( C = \sqrt{\frac{\mu_0}{\varepsilon_0}} \sqrt{\frac{\omega_1 \omega_2 \omega_3}{n_1 n_2 n_3}} \)
- For lossless media, \( d \) is isotropic.
- Assume \( d \) is frequency independent
- Let \( K = dC/2 \)

Scalar three wave interaction

\[
\begin{align*}
\frac{\partial A_1}{\partial z} &= +iKA_2A_3 \exp(-i\Delta kz) \\
\frac{\partial A_2}{\partial z} &= -iKA_1A_3 \exp(+i\Delta kz) \\
\frac{\partial A_3}{\partial z} &= +iKA_2A_1 \exp(+i\Delta kz)
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Scalar three wave interaction

\[
\begin{align*}
\frac{\partial A_1}{\partial z} &= +iK \overline{A}_2 A_3 \exp (-i \Delta kz) \\
\frac{\partial A_2}{\partial z} &= -iK A_1 \overline{A}_3 \exp (+i \Delta kz) \\
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Scalar three wave interaction

\[
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\frac{\partial A_1}{\partial z} &= +iK \overline{A_2} A_3 \exp(-i\Delta kz) \\
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\]

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\]

\[
\frac{\partial A_3}{\partial z} = +iK A_2 A_1 \exp(+i\Delta kz)
\]
A Scalar Three Wave Interaction model

Further approximations to remove vectors

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- For lossless media, \( d \) is isotropic.
- Assume \( d \) is frequency independent
- Let \( K = dC/2 \)

Scalar three wave interaction

\[
\frac{\partial A_1}{\partial z} = +iK \overline{A}_2 A_3 \exp(-i\Delta k z)
\]
\[
\frac{\partial A_2}{\partial z} = -iK A_1 \overline{A}_3 \exp(+i\Delta k z)
\]
\[
\frac{\partial A_3}{\partial z} = +iK A_2 A_1 \exp(+i\Delta k z)
\]
1. Three-wave interaction
   - Assumptions framework
   - Three Wave propagation equation
   - Sum frequency generation
   - Scalar approximation

2. Non Linear Optics Application
   - Second Harmonic Generation
   - Optical Parametric Amplifier
   - Optical Parametric Oscillator

3. Phase matching
   - Phase matching conditions
   - Phase matching in uni-axial crystals
   - Quasi-phase matching
Second Harmonic Generation (SHG)

Sum frequency generation of two equal frequencies from the same source.

One input beam counts for two:
- $\omega_1 = \omega_2$, $A_1 = A_2$, $k_1 = k_2$,
- $\omega_3 = 2\omega_1$
- $\Delta k = 2k_1 - k_3$

2 remaining equations:
- $\frac{\partial A_1}{\partial z} = +iK\overline{A_1}A_3 \exp(-i\Delta kz)$
- $\frac{\partial A_3}{\partial z} = +iK A_1^2 \exp(+i\Delta kz)$

Phase matching:
- $\Delta k = 0$
- $\frac{\partial A_1}{\partial z} = +iK\overline{A_1}A_3$
- $\frac{\partial A_3}{\partial z} = +iK A_1^2$

Figure: Closeup of a BBO crystal inside a resonant build-up ring cavity for frequency doubling 461 nm blue light into the ultraviolet. (source: flickr)
Second Harmonic Generation (SHG)

Sum frequency generation of two equal frequencies from the same source.

One input beam counts for two
- $\omega_1 = \omega_2$, $A_1 = A_2$, $k_1 = k_2$,
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- $\Delta k = 2k_1 - k_3$

2 remaining equations
- $\frac{\partial A_1}{\partial z} = +iK A_1 A_3 \exp (-i\Delta k z)$
- $\frac{\partial A_3}{\partial z} = +iK A_1^2 \exp (+i\Delta k z)$

Phase matching $\Delta k = 0$
- $\frac{\partial A_1}{\partial z} = +iK A_1 A_3$
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Figure: Closeup of a BBO crystal inside a resonant build-up ring cavity for frequency doubling 461 nm blue light into the ultraviolet. (source flickr)
Non Linear Optics Application

Second Harmonic Generation

Sum frequency generation of two equal frequencies from the same source

One input beam counts for two

- $\omega_1 = \omega_2$, $A_1 = A_2$, $k_1 = k_2$,
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2 remaining equations

- $\frac{\partial A_1}{\partial z} = +iK\overline{A_1}A_3 \exp(-i\Delta k z)$
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Phase matching

- $\Delta k = 0$
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Figure: Closeup of a BBO crystal inside a resonant build-up ring cavity for frequency doubling 461 nm blue light into the ultraviolet. (source flickr)
Non Linear Optics Application

Second Harmonic Beam Generation

Remember...

\[ \Delta k = 0 \]

\[ \frac{\partial A_1}{\partial z} = +iK\overline{A_1}A_3 \]

\[ \frac{\partial A_3}{\partial z} = +iKA_1^2 \]

\[ A_1 \in \mathbb{R} \]

\[ A_3 \in i\mathbb{R} \]

\[ A_3 = i\tilde{A}_3 \Rightarrow \tilde{A}_3 \in \mathbb{R} \]

\[ A_1 = \overline{A_1} \]

Real equations

\[ \frac{\partial A_1}{\partial z} = -K\overline{A_1}\tilde{A}_3 \]

\[ \frac{\partial \tilde{A}_3}{\partial z} = KA_1^2 \]

Multiply by \( A_1 \) and \( \tilde{A}_3 \)

\[ \frac{\partial (A_1^2 + \tilde{A}_3^2)}{\partial z} = 0 \]

This is Energy Conservation

Start with no harmonic

\[ (A_1^2(z) + \tilde{A}_3^2(z)) = A_1^2(0) \]

\[ \tilde{A}_3 \text{ equation} \]

\[ \frac{\partial \tilde{A}_3}{\partial z} = KA_1^2 = K \left( A_1^2(0) - \tilde{A}_3^2(z) \right) \]
Second Harmonic Beam Generation

Remember...

\[ \Delta k = 0 \]

\[ \frac{\partial A_1}{\partial z} = +iKA_1A_3 \]
\[ \frac{\partial A_3}{\partial z} = +iKA_1^2 \]

\[ A_1 \in \mathbb{R} \quad \text{and} \quad A_3 \in i\mathbb{R} \]

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Multiply by \( A_1 \) and \( \tilde{A}_3 \)

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Non Linear Optics Application

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N. Fressengeas (LMOPS)
\[ \frac{\partial \tilde{A}_3}{\partial z} = K \left( A_1^2(0) - \tilde{A}_3^2(z) \right) \]

\[ \tilde{A}_3(z) = A_1(0) \tanh (KA_1(0)z) \]

\[ I_3(z) = I_1(0) \tanh^2 (KA_1(0)z) \]

\[ I_1(z) = I_1(0) - I_3(z) \]

\[ I_1(z) = I_1(0) \text{sech}^2 (KA_1(0)z) \]
Second Harmonic Beam Evolution

\[ \tilde{A}_3 \] equation

\[ \frac{\partial \tilde{A}_3}{\partial z} = K \left( A_{1}^2 (0) - \tilde{A}_3^2 (z) \right) \]

\[ \tilde{A}_3 \] expression

\[ \tilde{A}_3 (z) = A_1 (0) \tanh (KA_1 (0) z) \]

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Second Harmonic Beam Evolution

\[ \frac{d\tilde{A}_3}{dz} = K \left( A_1^2 (0) - \tilde{A}_3^2 (z) \right) \]

\[ \tilde{A}_3 (z) = A_1 (0) \tanh (KA_1 (0) z) \]

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**Second Harmonic Beam Evolution**

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\[ I_1(z) = I_1(0) \text{sech}^2 \left( KA_1(0) z \right) \]

\[ \frac{I_3(z)}{I_1(0)} \quad I_1(0) = \text{constant} \]
Second Harmonic Beam Evolution

$\tilde{A}_3$ equation

$$\frac{\partial \tilde{A}_3}{\partial z} = K \left( A_1^2(0) - \tilde{A}_3^2(z) \right)$$

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$\frac{I_3(z)}{I_1(0)}$, $z = L = \text{constant}$
Suprisingly...

- It is possible to convert 100% of a beam, with large interaction length or intensity.
- The process has no threshold and does not need noise to start.
- We have retrieved Energy Conservation in spite of drastic approximations.
SHG conclusion

Second Harmonic Beam Evolution

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Optical Parametric Amplifier (OPA)

Optical Amplification of a weak signal beam thanks to a powerful pump beam

Signal beam amplification

- $\omega_1$: weak signal to be amplified
- $\omega_3$: intense pump beam
- $\omega_2 = \omega_3 - \omega_1$: difference frequency generation (idler)

Undepleted pump approximation

$$A_3(z) = A_3(0) = \frac{K_p}{K}$$

Phase matched equations

$$\frac{\partial A_1}{\partial z} = +iK_p\overline{A_2}$$
$$\frac{\partial A_2}{\partial z} = -iK_pA_1$$

Figure: White light continuum seeded optical parametric amplifier (OPA) able to generate extremely short pulses. (source Freie Universität Berlin)
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Solving the OPA equations

Phase matched equations

\[
\begin{align*}
\frac{\partial A_1}{\partial z} & = +iK_p \overline{A_2} \\
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\]

Initial conditions

- A weak signal: \( A_1 (0) \neq 0 \)
- No idler: \( A_2 (0) = 0 \)

Amplitude solution

- Amplified signal: \( A_1 (z) = A_1 (0) \cosh (K_p z) \)
- Idler: \( A_2 (z) = -iA_1 (0) \sinh (K_p z) \)
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**Intensities**

- Amplified signal: \( I_1(z) = I_1(0) \cosh^2(K_p z) \)
- Idler: \( I_2(z) = I_1(0) \sinh^2(K_p z) \)

**Amplification**

![Graph showing signal and idler intensities](image-url)
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Optical Parametric Oscillator (OPO)

Use Optical Parametric Amplification to make a tunable laser

**OPA pumped with** $\omega_3$

- Amplifier for $\omega_1$ and $\omega_2$
- With $\omega_1 + \omega_2 = \omega_3$
- Phase matching: $k_1 + k_2 = k_3$
- $\omega_1$ and $\omega_2$ initiated from noise

**Frequency tunable laser**

- Get Non Linear Medium
- Adjust Cavity for $\omega_1$ and $\omega_2$
- Pump with $\omega_3$
- You got it!
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Figure: Optical Parametric Oscillator (source Cristal Laser)
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Phase matching

Colinear (scalar) phase matching

Phase matching for co-propagation waves

- \( k_1 + k_2 = k_3 \Rightarrow \omega_1 n_1 + \omega_2 n_2 = \omega_3 n_3 \)
- for SHG : \( 2k_1 = k_3 \Rightarrow n_1 = n_3 \)
- The last is never achieved, due to normal dispersion: \( n_1 < n_3 \)

One and only solution

Use birefringent crystals and different polarizations
Colinear (scalar) phase matching

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One and only solution

Use birefringent crystals and different polarizations
Non colinear phase matching
Non colinear phase matching

Use clever geometries

With reflections
Non colinear phase matching

Use clever geometries

With reflections
Or even more clever...
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**SHG Type I Phase Matching**

**Waves polarization**
- 1 incident wave counts for 2
- They share the same polarization
- Second Harmonic polarization is orthogonal

**Type I phase matching**
- One refraction index for Fundamental
- The other for Second Harmonic
- They must be equal
- Propagate in the right direction

\[ n_0 \text{ and } n_e \text{ function of propagation direction: index ellipsoid cross-section} \]
Phase matching in uni-axial crystals

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\[ n_0 \text{ and } n_e \text{ function of propagation direction: index ellipsoid cross-section} \]
Phase matching in uni-axial crystals

**SHG Type I Phase Matching**

**Waves polarization**
- 1 incident wave counts for 2
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- Second Harmonic polarization is orthogonal

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- One refraction index for Fundamental
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SHG Type I phase matching: a few numbers

Fundamental index ellipsoid section
\[ \frac{1}{n_e^2(\theta)} = \frac{\cos^2(\theta)}{n_o^2} + \frac{\sin^2(\theta)}{n_e^2} \]

Harmonic index ellipsoid section
\[ \frac{1}{\tilde{n}_o^2(\theta)} = \frac{1}{n_o^2} \]

Solve the equation
\[ \sin^2(\theta) = \frac{\tilde{n}_o^{-2} - n_o^{-2}}{\tilde{n}_e^{-2} - \tilde{n}_o^{-2}} \]
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Type II phase matching

In the three beam interaction, Type I was

- Both input beams $\omega_1$ and $\omega_2$ share the same polarization
- The generated beam $\omega_3$ polarization is orthogonal

Another solution: Type II

- Input beams polarization are orthogonal
- Generated beam share one of them
- Not possible for SHG
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A hard task

- Phase matching is seldom colinear
- Vector phase matching in a complex index ellipsoid
- I will let you think on it

Paper by Bœuf can help

Calculating characteristics of noncolinear phase matching in uniaxial and biaxial crystals.
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1. Three-wave interaction
   - Assumptions framework
   - Three Wave propagation equation
   - Sum frequency generation
   - Scalar approximation

2. Non Linear Optics Application
   - Second Harmonic Generation
   - Optical Parametric Amplifier
   - Optical Parametric Oscillator

3. Phase matching
   - Phase matching conditions
   - Phase matching in uni-axial crystals
   - Quasi-phase matching
Quasi phase matching in layered media

Periodically Poled Lithium Niobate
- Periodic Domain Reversal
- $d$ sign reversal

Wave solution

$$\frac{i\omega_3}{2\eta_3}[d]_{jki} |E^{(\omega_1)}|^2 e^{(i\Delta k \Lambda)} \frac{1}{\Delta k} \sum_{n=1}^{N} (-1)^n e^{(i\Delta k \Lambda)}$$
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**Wave solution**

$$\frac{i\omega_3}{2} \eta_3 [d]_{jki} |E(\omega_1) |^2 i \frac{e^{(i\Delta k \Lambda)}}{\Delta k} - 1 \sum_{n=1}^{N} (-1)^n e^{(i\Delta k \Lambda)}$$

$$\left[ \frac{\partial E(\omega_3)}{\partial z} \right]_j$$
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Intensity Solution

$$\left| \frac{i\omega_3}{2} \eta_3 [d]_{jki} |E|_{\omega_1}^2 \right|^2 4\Lambda^2$$