Asymptotic analysis of singularly perturbed dynamical systems

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Contents

1 What are kinetic type equations

1.1 Taylor-Goldstein/telegraph equation

Functions f and g describe density of particles moving to the right/left with the flow of velocity v and randomly choosing the direction of motion at a rate α and lost from the system at a rate a:

$$\begin{aligned} \partial_t f &= -af - b\partial_x f + \alpha g - \alpha f, \\ \partial_t g &= -af + b\partial_x g - \alpha g + \alpha f, \end{aligned} \tag{1}$$

Introducing the total density $v = f + g$ and the net current to the right $w = f - g$, we have

$$\begin{aligned} \partial_t v &= av - b\partial_x w, \\ \partial_t w &= -b\partial_x v - dw, \end{aligned} \tag{2}$$

with $d = 2\alpha$.

The system (2) is supplemented by initial conditions

$$v(0) = \overset{o}{v}, \qquad w(0) = \overset{o}{w}, \tag{3}$$

and, say, the homogeneous Dirichlet conditions

$$v(-1, t) = v(1, t) = 0, \ t > 0. \tag{4}$$

1.2 Ageing and vertical migration of sole

A seemingly similar system is offered by

$$\partial_t \mathbf{n} = \mathcal{S}\mathbf{n} + \mathcal{M}\mathbf{n} + \mathcal{C}\mathbf{n}, \tag{5}$$

where $\mathbf{n} = (n_1, \ldots, n_N)$, $\mathcal{S} = diag\{-\partial_a, \ldots, -\partial_a\}$, $\mathcal{M} = diag\{-\mu_1, \ldots, -\mu_N\}$, $\mathcal{C} = \{k_{ij}\}_{1 \le i,j \le N}$. Here n_i is the population density of fish in patch i, a is the age, $\mu_i(a)$ is the mortality rate, and the coefficients k_{ij} represent the migration rates from patch j to patch i, $j \neq i$. The system was introduced to describe evolution of a continuous age-structured population of sole The characteristic feature of the population is daily vertical migration provoked by light intensity.

This system is supplemented by the boundary condition of the McKendrick-Von Foerster type

$$\mathbf{n}(0,t) = \int_0^\infty \mathcal{B}(a) n(a,t) da \tag{6}$$

where $\mathcal{B}(a) = diag\{\beta_1(a), \ldots, \beta_N(a)\}$ gives the fertility at age a and patches 1 to N. The initial condition is given by

$$\mathbf{n}(a,0) = \Phi(a). \tag{7}$$

The transition matrix $\mathcal{C}$ is a typical transition matrix (of a time-continuous process); that is off-diagonal entries are positive and columns sum up to 0.

1.3 Fokker-Planck equations

$$u_t(\mathbf{x},\xi) = -\xi \partial_{\mathbf{x}} u(\mathbf{x},\xi) + \mathcal{C}u(\mathbf{x},\xi). \tag{8}$$

Here u is the particle distribution function in the phase space, $\mathbf{x}$ denotes the position and ξ the velocity of the particle. $\mathcal{C}$ is the transition operator, which is, e.g.,

$$\mathcal{C}u(\mathbf{x},\xi) = \partial_\xi(\xi + \partial_\xi) u(\mathbf{x},\xi), \qquad \mathbf{x}, \xi \in \mathbb{R}^3.$$

for the Brownian motion,

$$\mathcal{C}u(\mathbf{x},\xi) = \partial_\xi((1-\xi^2)\partial_\xi u(\mathbf{x},\xi)), \qquad \mathbf{x} \in \mathbb{R}^3, \xi \in [-1,1],$$

for electron scattering in plasma, etc.

1.4 Linear Maxwell-Boltzmann equation

We are interested in modelling the motion of a gas of *test particles* through a background of *field particles*. The test particles are driven by an external force $\mathbf{F}$ that depends on the position vector $\mathbf{x}$ and on the velocity $\mathbf{v}$, but not on time t, and are scattered by localized in space and instantaneous collisions with field particles which are supposed to be fixed. This, together with the assumption of low density of the test particles, makes the problem linear and the time evolution of the one-particle distribution function f of test particles, depending on position $\mathbf{x}$, velocity $\mathbf{v}$, and time t, is described by the linear equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} + \nu f = \int_{\mathbb{R}^3} k(\mathbf{v}, \mathbf{v}') f(t, \mathbf{x}, \mathbf{v}') d\mathbf{v}'. \tag{9}$$

Here the independent variables $(\mathbf{x}, \mathbf{v})$ take values in a set $\Lambda \subseteq \mathbb{R}^3 \times \mathbb{R}^3$, which is called the phase space of the problem.

1.5 Semi-classical Linear Boltzmann equation

$$\begin{aligned} f_t(t, \mathbf{x}, \mathbf{v}) \;=\; & -\mathbf{v}\partial_{\mathbf{x}} f(t, \mathbf{x}, \mathbf{v}) \\ & -4\pi\lambda(v) f(t, \mathbf{x}, \mathbf{v}) + \lambda(v) \int_{S^2} f(t, \mathbf{x}, v\boldsymbol{\omega}') d\boldsymbol{\omega}' \\ & -f(t, \mathbf{x}, \mathbf{v}) 4\pi \left(n_2 \frac{v_+}{v} \nu(v_+) n_1 H(v^2 - 1) \nu(v) \right) \\ & +n_1 \frac{v_+}{v} \nu(v_+) \int_{S^2} f(t, \mathbf{x}, v_+\boldsymbol{\omega}') d\boldsymbol{\omega}' \\ & +n_2 \nu(v) H(v^2 - 1) \int_{S^2} f(t, \mathbf{x}, v_-\boldsymbol{\omega}') d\boldsymbol{\omega}' \end{aligned}$$

where S^2 is the unit sphere in $\mathbb{R}^3_v$, $\mathbf{v} = v\boldsymbol{\omega}$ with $v \in [0, \infty[$, $\boldsymbol{\omega} \in S^2$, where

H is the Heaviside function, $v_\pm = \sqrt{v^2 \pm 1}$. The Maxwell molecule assumption yields

$$0 < \lambda_{min} \leq \lambda(v) \leq \lambda_{max} < +\infty, v > 0.$$

and

$$0 < \nu_{min} \leq \nu(v) \leq \nu_{max} < +\infty \quad \text{for } v \in [1, \infty[.$$

The common theme of the models introduced above is that they can be written in a single form

$$\partial_t u = \mathcal{A}u + \mathcal{S}u + \mathcal{C}u \tag{10}$$

where u is some type of a 'particle' distribution function, the operator $\mathcal{A}$ (which may be zero), $\mathcal{S}$, $\mathcal{C}$ describe attenuation, transport in the phase space and transitions between states, of particles, respectively. Thus, the evolution is driven by an interplay of several different mechanisms which may be of different magnitude, or act on different time scales.

2 Various singular scalings

2.1 Taylor-Goldstein/telegraph equation

Case 1. Small relaxation time in Cattaneo model

$$\begin{aligned} \partial_t v &= av - b\partial_x w, \\ \epsilon\partial_t w &= -b\partial_x v - dw. \end{aligned} \tag{11}$$

Case 2. Random walk theory: very strong correlations, strong current.

$$\begin{aligned} \partial_t v &= av - \epsilon^{-1} b\partial_x w, \\ \epsilon\partial_t w &= -\epsilon^{-1} b\partial_x v - \epsilon^{-2} dw. \end{aligned} \tag{12}$$

Case 2. Random walk theory: strong correlations; vibrations: strong damping.

$$\begin{aligned} \partial_t v &= av - b\partial_x w, \\ \partial_t w &= -b\partial_x v - \epsilon^{-1} dw. \end{aligned} \tag{13}$$

2.2 Ageing and vertical migration of sole

$$\partial_t \mathbf{n} = \mathcal{S}\mathbf{n} + \mathcal{M}\mathbf{n} + \frac{1}{\epsilon}\mathcal{C}\mathbf{n}, \tag{14}$$

reflects the fact that vertical migration occur on a much faster time scale that the demographic processes.

2.3 Fokker-Planck equations

$$u_t(\mathbf{x}, \xi) = -\xi \partial_{\mathbf{x}} u(\mathbf{x}, \xi) + \frac{1}{\epsilon}\mathcal{C}u(\mathbf{x}, \xi), \tag{15}$$

expresses our interest what happens if the collisions are dominant.

2.4 Linear Maxwell-Boltzmann equation

Case 1. Weak external field.

$$\frac{\partial f}{\partial t} = -\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} + \frac{1}{\epsilon} \left(\nu f - \int\limits_{\mathbb{R}^3} k(\mathbf{v}, \mathbf{v}') f(t, \mathbf{x}, \mathbf{v}') d\mathbf{v}' \right). \tag{16}$$

Case 2. Strong external field.

$$\frac{\partial f}{\partial t} = -\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{1}{\epsilon} \left(\mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} - \nu f + \int\limits_{\mathbb{R}^3} k(\mathbf{v}, \mathbf{v}') f(t, \mathbf{x}, \mathbf{v}') d\mathbf{v}' \right). \tag{17}$$

2.5 Semi-classical Linear Boltzmann equation

$$
\begin{aligned}
f_t(t,\mathbf{x},\mathbf{v}) &= -\frac{1}{\epsilon^p}\mathbf{v}\partial_{\mathbf{x}} f(t,\mathbf{x},\mathbf{v}) \\
&\quad -\frac{1}{\epsilon^q}\left(4\pi\lambda(v) f(t,\mathbf{x},\mathbf{v}) + \lambda(v)\int\limits_{S^2} f(t,\mathbf{x},v\boldsymbol{\omega}')d\boldsymbol{\omega}'\right) \\
&\quad -\frac{1}{\epsilon^r}\left(f(t,\mathbf{x},\mathbf{v})4\pi\left(n_2\frac{v_+}{v}\nu(v_+)n_1 H(v^2-1)\nu(v)\right)\right. \\
&\quad +n_1\frac{v_+}{v}\nu(v_+)\int\limits_{S^2} f(t,\mathbf{x},v_+\boldsymbol{\omega}')d\boldsymbol{\omega}' \\
&\quad \left. + n_2\nu(v)H(v^2-1)\int\limits_{S^2} f(t,\mathbf{x},v_-\boldsymbol{\omega}')d\boldsymbol{\omega}'\right)
\end{aligned}
$$

The the models introduced above can be written in a general form

$$
\partial_t u = \mathcal{A}u + \mathcal{S}_1 u + \frac{1}{\epsilon^p}\mathcal{S}_2 u + \frac{1}{\epsilon^q}\mathcal{C}_1 u\frac{1}{\epsilon^r}\mathcal{C}_2 u \tag{18}
$$

3 Singular perturbations and asymptotic analysis

The goal of this section is to give a concise explanation of concepts of asymptotic analysis.

The aims of asymptotic analysis:

1. To provide link between different level description of the same phenomena (e.g. Newton versus Boltzmann versus Navier-Stokes description of gases);

2. To provide efficient numerical techniques for solving equations by identifying different regimes of evolution yielding to possibly simpler numerical codes.

In order to introduce the ideas of asymptotic analysis, let us consider a particular case of a singularly perturbed value problem

$$\begin{cases} \dfrac{\partial f_\epsilon}{\partial t} &= Sf_\epsilon + \dfrac{1}{\epsilon} Cf_\epsilon, \\ f_\epsilon(0) &= f_0, \end{cases} \tag{19}$$

where the presence of the small parameter ϵ indicates that the phenomenon modelled by the operator C is more relevant than that modelled by S or, in other words, they act on different time scales.

As elsewhere in these lectures, we are concerned with kinetic type problems and the operator S describes some form of transport, whereas C is an interaction/transition operator describing interstate transfers, e.g., they may be collision operator in the kinetic problems or a transition matrix in the structured population theory.

We are often interested in situations when the transition processes between structure states are dominant. If this is the case, the population quickly becomes homogenised with respect to the structure and starts to behave as an unstructured one, governed by suitable equations (which in analogy with the classical kinetic theory will be called *hydrodynamic* equations). It is expected that these equations should be the limit, or approximating, equation for (19) as $\epsilon \to 0$ (the parameter ϵ in such a case is related to the mean free time between state switches).

To put this in a mathematical framework, we can suppose to have on the right-hand side a family of operators $\{C_\epsilon\}_{\epsilon>0} = \{S + \frac{1}{\epsilon}C\}_{\epsilon>0}$ acting in a suitable Banach space X, and a given initial datum. The classical asymptotic analysis consists in looking for a solution in the form of a truncated power series

$$f_\epsilon^{(n)}(t) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + \cdots + \epsilon^n f_n(t),$$

and builds up an algorithm to determine the coefficients $f_0, f_1, f_2, \ldots, f_n$. Then $f_\epsilon^{(n)}(t)$ is an approximation of order n to the solution $f_\epsilon(t)$ of the original equation

in the sense that we should have

$$\|f_\epsilon(t) - f_\epsilon^{(n)}(t)\|_X = o(\epsilon^n)\,, \tag{20}$$

for uniformly for $0 \leq t \leq T$, where $T > 0$.

It is important to note that the zeroth-order approximation satisfies

$$Cf_0(t) = 0$$

which is the mathematical expression of the fact that the hydrodynamic approximation should be transition-free and that's why the null-space of the dominant collision operator is called the *hydrodynamic space* of the problem.

Another important observation pertains to the fact that in most cases the limit equation involves less independent variables than the original one. Thus the solution of the former cannot satisfy all boundary and initial conditions of the latter. Such problems are called *singularly perturbed*. If, for example, the approximation (20) does not hold in a neighbourhood of $t = 0$, then it is necessary to introduce an *initial layer* correction by repeating the above procedure with rescaled time to improve the convergence for small t. The original approximation which is valid only away from $t = 0$ is referred to as the *bulk approximation*.

Similarly, there approximation could fail close to the spatial boundary of the domain as well as close to the region where the spatial and temporal boundaries meet. To improve accuracy in such cases one introduces the so-called *boundary* and *corner layer* corrections which will be in detail discussed later.

A first way to look at the problem from the point of view of the approximation theory is to provide a systematic way of finding a new (simpler) family of operators, still depending on ϵ, say B_ϵ, generating new evolution problems

$$\frac{\partial \varphi_\epsilon}{\partial t} = B_\epsilon \varphi_\epsilon,$$

supplemented by appropriate initial conditions, such that the solutions $\varphi_\epsilon(t)$ of the new evolution problem satisfy

$$\|f_\epsilon(t) - \varphi_\epsilon(t)\|_X = o(\epsilon^n)\,, \tag{21}$$

for $0 \leq t \leq T$, where $T > 0$ and $n \geq 1$. In this case we say that B_ϵ is a hydrodynamic approximation of C_ϵ of order n.

This approach mathematically produces weaker results than solving system (19) for each ϵ and eventually taking the limit of the solutions as $\epsilon \to 0$. But in real situation, ϵ is small but not zero, and it is interesting to find simpler operators B_ϵ for modelling a particular regime of a physical system of interacting particles.

A slightly different point of view consists in requiring that the limiting equation for the approximate solution does not contain ϵ. In other words, the task is now to find a new (simpler) operator, say B, and a new evolution problem

$$\frac{\partial \varphi}{\partial t} = B\varphi,$$

with an appropriate initial condition, such that the solutions $\varphi(t)$ of the new evolution problem satisfy

$$\|f_\epsilon(t) - \varphi(t)\|_X \to 0, \;\; \text{as} \;\; \epsilon \to 0, \tag{22}$$

for $0 \leq t \leq T$, where $T > 0$.

In this case we say that B is the hydrodynamic limit of operators C_ϵ as $\epsilon \to 0$. This approach can be treated as (and in fact is) a particular version of the previous one as very often the operator B is obtained as the first step in the procedure leading eventually to the family $\{B_\epsilon\}_{\epsilon \geq 0}$. For instance, for the nonlinear Boltzmann equation with the original Hilbert scaling, B would correspond to the Euler system, whereas B_ϵ could correspond to the Navier-Stokes system with ϵ-dependent viscosity, or to Burnett equation at yet higher level.

In any case the asymptotic analysis, should consist of two main points:

- determining an algorithm which provides in a systematic way the approximating family B_ϵ (or the limit operator B),

- proving the convergence of f_ϵ in the sense of (21) (or of (22)).

Even if the formal part and the rigorous part of an asymptotic analysis seem not to be related, the formal procedure can be of great help in proving the convergence theorems.

We will focus on the modification of the classical Chapman-Enskog procedure. The advantage of it procedure is that the projection of the solution to the Boltzmann equation onto the null-space of the collision operator, that is, the hydrodynamic part of the solution, is not expanded in ϵ, and thus the whole information carried by this part is kept together. This is in contrast to the Hilbert type expansions, where, if applicable, only the zero order term of the expansion of the hydrodynamic part is recovered from the limit equation.

Hence the main feature of the modified Chapman-Enskog procedure is that the initial value problem is decomposed into two problems, for the kinetic and hydrodynamic parts of the solution, respectively. This decomposition consists in splitting the unknown function into the part belonging to the null space V of the operator C, which describes the dominant phenomenon, whereas the remaining part belongs to the complementary subspace W.

Thus the first step of the asymptotic procedure is finding the null-space of the dominant collision operator C; then the decomposition is performed using the (spectral) projection $\mathcal{P}$ onto the null-space V by applying $\mathcal{P}$ and the complementary projection $\mathcal{Q} = I - \mathcal{P}$ to equation (19). In this one obtains a system of evolution equations in the subspaces V and W. At this point the kinetic part of the solution is expanded in series of ϵ, but the hydrodynamic part of the solution is left unexpanded. In other words, we keep all orders of approximation of the hydrodynamic part compressed into a single function.

In many classical approaches the initial/boundary/corner layer contributions are neglected and thus transitional effects are not taken into account. To overcome this, two time and space scales are introduced in order to obtain the necessary corrections. In general, the compressed asymptotic algorithm permits to derive in a natural way the hydrodynamic equation, the initial condition to supplement it, and the initial layer corrections. Moreover the layers are matched to the bulk solution at the boundary of the domain thus providing an approximation which is valid over the whole domain.

4 First semigroups

Laws of physics and, increasingly, also those of other sciences are in many cases expressed in terms of differential or integro–differential equations. If one models systems evolving with time, then the variable describing time plays a special role, as the equations are built by balancing the change of the system in time against its 'spatial' behaviour. In mathematics such equations are called *evolution equations*.

Equations of the applied sciences are usually formulated pointwise; that is, all the operations, such as differentiation and integration, are understood in the classical (calculus) sense and the equation itself is supposed to be satisfied for all values of the independent variables in the relevant domain:

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= [\mathcal{A}u(t,\cdot)](x), \quad x \in \Omega \\ u(t,0) &= \overset{\circ}{u}, \end{aligned} \tag{23}$$

where $\mathcal{A}$ is a certain expression, differential, integral, or functional, that can be evaluated at any point $x \in \Omega$ for all functions from a certain subset S.

When we are trying to solve (23), we change its meaning by imposing various

a priori restrictions on the solution to make it amenable to particular techniques. Quite often (23) does not provide a complete description of the dynamics even if it looks complete from the modelling point of view. Then the obtained solution maybe be not what we have been looking for. This becomes particularly important if we cannot get our hands on the actual solution but use 'soft analysis' to find important properties of it.

We will use one particular way of looking at the evolution of a system in which we describe time changes as transitions from one state to another; that is, the evolution is described by a family of operators $(G(t))_{t\geq 0}$, parameterised by time, that map an initial state of the system to all subsequent states in the evolution; that is solutions are represented as

$$u(t) = G(t)u_0, \tag{24}$$

where $(G(t))_{t\geq 0}$ is the semigroup and u_0 is an initial state. Note that in this interpretation the function of several variables $(t, x) \to u(t, x)$ is interpreted as a function of one variable t but taking values in the space of functions of x: $t \to u(t, \cdot)$.

In other words, we forget about points in the state space Ω and 'points' are states represented by functions $u(x)$ in some abstract space X which is chosen partially for the relevance to the problem and partially for mathematical convenience. For example, if (23) describes the evolution of an ensemble of particles, then u is the particle density function and the natural space seems to be $L_1(\Omega)$ as in this case the norm of a nonnegative u, that is, the integral over Ω, gives the total number of particles in the ensemble.

It is important to note that this choice is not unique but is rather a mathematical intervention into the model, which could change it in a quite dramatic way.

For instance, in this case we could choose the space of measures on Ω with the same interpretation of the norm, but also, if we are interested in controlling the maximal concentration of particles, a more proper choice would be some rea-

sonable space with a supremum norm, such as, for example, the space of bounded continuous functions on Ω, $C_b(\Omega)$. Once we select our space, the right-hand side can be interpreted as an operator $A : D(A) \to X$ (we hope) defined on some subset $D(A)$ of X (not necessarily equal to X) such that $x \to [\mathcal{A}u](x) \in X$. With this, (23) can be written as an ordinary differential equation in X, as described in the definition below.

These considerations lead to the semigroup theory which is concerned with methods of finding solutions of the Cauchy problem.

Definition 1 *Given a complex Banach space and a linear operator A with $D(A)$, $ImA \subset X$ and given $u_0 \in X$, find a function $u(t) = u(t, u_0)$ such that*

1. $u \in C^0([0, \infty)) \cap C^1((0, \infty))$,
2. *for each $t > 0$, $u(t) \in D(A)$ and*

$$u_t = Au, \qquad t > 0, \tag{25}$$
$$u(0) = u_0 \in X. \tag{26}$$

are satisfied.

A function satisfying all conditions above is called the classical (or strict) solution of (25), (26).

If the solution to (25), (26) is unique, then, as suggested above, we can introduce a family of operators $(G(t))_{t\geq 0}$ such that $u(t, u_0) = G(t)u_0$. Ideally, $G(t)$ should be defined on the whole space for each $t > 0$, and the function $t \to G(t)u_0$ should be continuous for each $u_0 \in X$, leading to well-posedness of (25), (26). Moreover, uniqueness and linearity of A imply that $G(t)$ are linear operators. A fine-tuning of these requirements leads to the following definition.

Definition 2 *A family $(G(t))_{t\geq 0}$ of bounded linear operators on X is called a C_0-semigroup, or a strongly continuous semigroup, if*

(i) $G(0) = I$*;*

(ii) $G(t+s) = G(t)G(s)$ *for all* $t, s \geq 0$*;*

(iii) $\lim_{t\to 0^+} G(t)x = x$ *for any* $x \in X$*.*

A linear operator A is called the (infinitesimal) generator of $(G(t))_{t\geq 0}$ if

$$Ax = \lim_{h\to 0^+} \frac{G(h)x - x}{h}, \tag{27}$$

with $D(A)$ defined as the set of all $x \in X$ for which this limit exists. Typically the semigroup generated by A is denoted by $(G_A(t))_{t\geq 0}$.

If $(G(t))_{t\geq 0}$ is a C_0-semigroup, then the local boundedness and (ii) lead to the existence of constants $M > 0$ and ω such that for all $t \geq 0$

$$\|G(t)\|_X \leq Me^{\omega t}. \tag{28}$$

From (27) and the condition (iii) of Definition 2 we see that if A is the generator of $(G(t))_{t\geq 0}$, then for $x \in D(A)$ the function $t \to G(t)x$ is a classical solution of the following Cauchy problem,

$$\begin{aligned} \partial_t u(t) &= A(u(t)), \quad t > 0, & (29)\\ \lim_{t\to 0^+} u(t) &= x. & (30) \end{aligned}$$

We noted above that for $x \in D(A)$ the function $u(t) = G(t)x$ is a classical solution to (29), (30). For $x \in X \setminus D(A)$, however, the function $u(t) = G(t)x$ is continuous but, in general, not differentiable, nor $D(A)$-valued, and, therefore, not a classical solution.

4.1 Around the Hille–Yosida Theorem

Theorem 1 *A generates a strongly continuous semigroup $(G(t))_{t\geq 0}$ satisfying (28) if and only if*

(a) A is closed and densely defined,

(b) there exist $M > 0, \omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and for all $n \geq 1, \lambda > \omega$,

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}. \tag{31}$$

4.2 Dissipative Operators

Let X be a Banach space (real or complex) and X^* be its dual. From the Hahn–Banach theorem, for every $x \in X$ there exists $x^* \in X^*$ satisfying

$$<x^*, x>= \|x\|^2 = \|x^*\|^2.$$

Therefore the *duality set*

$$\mathcal{J}(x) = \{x^* \in X^*;\ <x^*, x>= \|x\|^2 = \|x^*\|^2\} \tag{32}$$

is nonempty for every $x \in X$.

Definition 3 *We say that an operator $(A, D(A))$ is* dissipative *if for every $x \in D(A)$ there is $x^* \in \mathcal{J}(x)$ such that*

$$\Re\ <x^*, Ax> \leq 0. \tag{33}$$

Combination of the Hille–Yosida theorem with the properties of dissipative operators gives a generation theorem for dissipative operators, known as the Lumer–Phillips theorem ([10, Theorem 1.43] or [7, Theorem II.3.15]).

Theorem 2 *For a densely defined dissipative operator $(A, D(A))$ on a Banach space X, the following statements are equivalent.*

(a) The closure $\overline{v}A$ generates a semigroup of contractions.

(b) $\overline{v}Im(\lambda I - A) = X$ for some (and hence all) $\lambda > 0$.

If either condition is satisfied, then A satisfies (33) for any $x^* \in \mathcal{J}(x)$.

In particular, if we know that A is closed then the density of $Im(\lambda I - A)$ is sufficient for A to be a generator. On the other hand, if we do not know a priori that A is closed then $Im(\lambda I - A) = X$ yields A being closed and consequently that it is the generator.

4.3 Nonhomogeneous Problems

Consider the problem of finding the solution to:

$$\begin{aligned} \frac{du}{dt}(t) &= Au(t) + f(t), \quad 0 < t < T \\ u(0) &= u_0, \end{aligned} \tag{34}$$

where $0 < T \leq \infty$, A is the generator of a semigroup, and $f : (0, T) \to X$ is a known function. For u to be a continuous solution, f must be continuous. However, this condition proves to be insufficient. We observe that if u is a classical solution of (34), then it must be given by

$$u(t) = G(t)u_0 + \int_0^t G(t-s)f(s)ds. \tag{35}$$

The integral is well defined even if $f \in L_1([0, T], X)$ and $u_0 \in X$. We call u defined by (35) the *mild solution* of (34). For an integrable f such u is continuous but not necessarily differentiable, and therefore it may be not a solution to (34).

The following theorem gives sufficient conditions for a mild solution to be a classical solution (see, e.g., [10, Corollary 4.2.5 and 4.2.6]).

Theorem 3 *Let A be the generator of a C_0-semigroup $(G(t))_{t\geq 0}$ and $x \in D(A)$. Then (35) is a classical solution of (34) if either*

(i) $f \in C^1([0,T],X)$, or

(ii) $f \in C([0,T],X) \cap L_1([0,T],D(A))$.

4.4 Bounded Perturbation Theorem and Related Results

Theorem 4 *Let $(A, D(A))$ is a generator of a semigroup and $B \in \mathcal{L}(X)$. Then $(K, D(K)) = (A+B, D(A))$ is a generator of a semigroup.*

Moreover, the semigroup $(G_{A+B}(t))_{t\geq 0}$ generated by $A+B$ satisfies either Duhamel equation:

$$G_{A+B}(t)x = G_A(t)x + \int_0^t G_A(t-s)BG_{A+B}(s)xds, \quad t \geq 0, x \in X \tag{36}$$

and

$$G_{A+B}(t)x = G_A(t)x + \int_0^t G_{A+B}(t-s)BG_A(s)xds, \quad t \geq 0, x \in X, \tag{37}$$

where the integrals are defined in the strong operator topology.

Moreover, $(G_{A+B}(t))_{t\geq 0}$ is given by the Dyson–Phillips series obtained by iterating (36):

$$G_{A+B}(t) = \sum_{n=0}^{\infty} G_n(t), \tag{38}$$

where $G_0(t) = G_A(t)$ *and*

$$G_{n+1}(t)x = \int_0^t G_A(t-s)BG_n(s)xds. \qquad t \geq 0, x \in X. \tag{39}$$

The series converges in the operator norm of $\mathcal{L}(X)$ *and uniformly for* t *in bounded intervals.*

4.5 Spectral Mapping Theorem and Long Time Behaviour of Semigroups

It is important to be able to determine the behaviour of solutions $u(t) = G(t)\overset{o}{u}$ as $t \to \infty$.

We know that always

$$\|G(t)\| \leq Me^{\omega t}, \tag{40}$$

for some $M \geq 1, \omega \in \mathbb{R}$.

If the generator A of $(G(t))_{t\geq 0}$ is dissipative, then $(G(t))_{t\geq 0}$ is a semigroup of contractions; that is,

$$\|G(t)\| \leq 1, \quad t \geq 0.$$

Let $\sigma(A)$ denote the spectrum of A and define the spectral bound

$$s(A) = \sup \Re\sigma(A).$$

If A is bounded then, by the Lyapunov theorem,

$$\|G(t)\| \leq Me^{\omega t},$$

for any $\omega > s(A)$. In particular, $s(A) < 0$ yields $\|G(t)\| \sim e^{-\omega t}$, $\omega > 0$, as $t \to \infty$. In general, however, the above is a delicate problem.

A useful observation is that if $(G(t))_{t\geq 0}$ generated by A satisfies (40), then the semigroup $(G_a(t))_{t\geq 0}$ generated by $A - aI$ satisfies

$$\|G_a(t)\| \leq Me^{(\omega-a)t}, \tag{41}$$

Recall that for singularly perturbed problems (of resonance type) we assumed $KerC \neq \{0\}$, hence $0 \in \sigma(A)$. Also typically we deal with dissipative C. Assumption/property of C which makes the asymptotic analysis work is that $\lambda = 0$ is an isolated dominant eigenvalue:

$$\sigma(C) = \{0\} \cup \sigma(QCQ)$$

where P is the spectral projection onto the (hydrodynamic) subspace V and Q is the complementary spectral projection onto the complementary (kinetic) subspace W. Hence

$$C = \begin{bmatrix} 0 & 0 \\ 0 & QCQ \end{bmatrix}, \quad e^{tC} = \begin{bmatrix} I & 0 \\ 0 & e^{tQCQ} \end{bmatrix}$$

and we assume that

$$\|e^{tQCQ}\| \leq Me^{-\omega t}$$

for some $M \geq 1, \omega < 0$.

5 Hydrodynamic limit–the sole equation

5.1 Age structured population model

Let us recall the model

$$\partial_t \mathbf{n} = \mathcal{S}\mathbf{n} + \mathcal{M}\mathbf{n} + \frac{1}{\epsilon}\mathcal{C}\mathbf{n}, \tag{42}$$

or in the expanded version

$$\begin{aligned}
\partial_t n_1 &= -\partial_a n_1 - \mu_1 n_1 + \frac{1}{\epsilon}(c_{11}n_1 + \ldots + c_{1N}n_N), \\
&\vdots \quad \vdots \quad \vdots, \\
\partial_t n_N &= -\partial_a n_N - \mu_1 n_N + \frac{1}{\epsilon}(c_{N1}n_1 + \ldots + c_{NN}n_N),
\end{aligned} \tag{43}$$

supplemented by the boundary condition of the McKendrick-Von Foerster type

$$\begin{aligned}
n_1(0,t) &= \int_0^\infty \beta_1(a) n_1(a,t) da \\
&\vdots \quad \vdots \quad \vdots \\
n_N(0,t) &= \int_0^\infty \beta_N(a) n_N(a,t) da
\end{aligned}$$

and initial condition

$$n_1(a,0) = \overset{\circ}{n}_1(a), \ldots, n_N(a,0) = \overset{\circ}{n}_N(a).$$

Recall, that the transition matrix $\mathcal{C}$ is a typical transition matrix (of a time-continuous process); that is off-diagonal entries are positive and columns sum up to 0. We further assume that it generates an irreducible (n-dimensional) semigroup. Thus, 0 is the dominant eigenvalue of $\mathcal{C}$ with a positive eigenvector $\mathbf{e}$ which will be fixed to satisfy $\mathbf{1} \cdot \mathbf{e} = 1$, where $\mathbf{1} = (1, 1 \ldots, 1)$.

The vector $\mathbf{e} = (e_1, \cdots, e_N)$ represents the so called stable patch structure; that is, the asymptotic distribution of the population into the patches. Thus, it is reasonable to approximate

$$e_i = \frac{n_i}{n}, \quad i = 1, \ldots, N$$

where $n = \sum_{i=1}^N n_i$. Adding together equations in (43) and using the above we obtain

$$\partial_t n = -\partial_a n - \mu^*(a) n \tag{44}$$

where $\mu^* = \mathbf{1} \cdot \mathcal{M}\mathbf{e} = \sum_{i=1}^{N} \mu_i e_i$ is the so-called 'aggregate' mortality. This model, supplemented with appropriate averaged boundary condition is called the aggregated model and is expected to provide averaged approximate description of the population.

Using general terminology, (44) is the macroscopic and (43) the mesoscopic description of the population.

Our aim is to show that the aggregated model (44) can be obtained as an asymptotic limit of (43) as $\epsilon \to 0$.

We recall that the method depends on the spectral properties of the operators $\mathcal{S}$ and $\mathcal{C}$. To be able to start, we must assume that $\lambda = 0$ is the dominant simple eigenvalue of the operator $\mathcal{C}$.

It is easy to see that this requirement is satisfied if $\mathcal{C}$ is the generator of a semigroup having AEG. The fact that $\lambda = 0$ needs to be dominant ensures an exponential decay of the initial layer. This assumption may, however, be relaxed if we are not that interested in the properties of the layer.

Remark 1 In many cases we have several state variables and the operator $\mathcal{C}$ only acts on some of them. Then the above requirement refers to the action of $\mathcal{C}$ in this restricted space.

The assumptions allow to perform the asymptotic analysis in the compressed (Chapman-Enskog) form. The spectral projections $\mathcal{P}, \mathcal{Q} : \mathbb{R}^n \to \mathbb{R}^n$ are given by

$$\mathcal{P}\mathbf{x} = (\mathbf{1} \cdot \mathbf{x})\mathbf{e}, \quad \mathcal{Q}\mathbf{x} = \mathbf{x} - (\mathbf{1} \cdot \mathbf{x})\mathbf{e} \tag{45}$$

which gives the hydrodynamical space $V := Span\{\mathbf{e}\}$ and the kinetic space

$$W = Im\mathcal{Q} = \{\mathbf{x};\ \mathbf{1} \cdot \mathbf{x} = 0\},$$

as well as the solution decomposition

$$\mathbf{n} = \mathcal{P}\mathbf{n} + \mathcal{Q}\mathbf{n} = v + w = p\mathbf{e} + w$$

where $p = p(a,t)$ is a scalar function.

To simplify notation we shall consider the problem with $\mathcal{M} = 0$ and with B and C being a independent. Thus, consider

$$\begin{cases} \partial_t \mathbf{n}_\epsilon &= \mathcal{S}\mathbf{n}_\epsilon + \dfrac{1}{\epsilon}\mathcal{C}\mathbf{n}_\epsilon, \\ \gamma \mathbf{n}_\epsilon &= \mathcal{B}\mathbf{n}_\epsilon, \\ \mathbf{n}_\epsilon(0) &= \overset{\circ}{\mathbf{n}}, \end{cases} \tag{46}$$

The hydrodynamic space of $\mathcal{C}$ is thus one dimensional; we denote by $\mathcal{P}$ the spectral projection of the state space onto this space. Let $\mathcal{Q} = \mathcal{I} - \mathcal{P}$ be the complementary projection. Accordingly, by $\mathcal{P}\mathbf{n} = v$ we denote the hydrodynamic part of the solution $\mathbf{n}$ and by $\mathcal{Q}\mathbf{n} = w$ its kinetic part.

Applying these projections to both sides of the equation (46) and noting that

$$\begin{aligned} \mathcal{P}\mathcal{S}\mathcal{P}v &= -\partial_a p\mathbf{e}, \\ \mathcal{Q}\mathcal{S}\mathcal{P}v &= \mathcal{P}\mathcal{S}\mathcal{Q}w = 0, \end{aligned}$$

we get

$$\begin{aligned} \partial_t v &= \mathcal{P}\mathcal{S}\mathcal{P}v \\ \varepsilon\partial_t w &= \varepsilon\mathcal{Q}\mathcal{S}\mathcal{Q}w + \mathcal{Q}\mathcal{C}\mathcal{Q}w, \end{aligned} \tag{47}$$

with the initial conditions

$$v(0) = \overset{o}{v},\ w(0) = \overset{o}{w},$$

where $\overset{o}{v} = \mathcal{P}\overset{o}{\mathbf{n}}$, $\overset{o}{w} = \mathcal{Q}\overset{o}{\mathbf{n}}$, and the boundary conditions are transformed into

$$\begin{aligned} \gamma v &= \mathcal{P}\mathcal{B}\mathcal{P}v + \mathcal{P}\mathcal{B}\mathcal{Q}w, \\ \gamma w &= \mathcal{Q}\mathcal{B}\mathcal{P}v + \mathcal{Q}\mathcal{B}\mathcal{Q}w. \end{aligned} \tag{48}$$

Denoting $\mathbf{b} = (\beta_1, \ldots, \beta_N)$ (independent of a), we get

$$\begin{aligned} \mathcal{PBP}v &= \mathbf{e}(\mathbf{b}\cdot\mathbf{e})\int_0^\infty p(a)da, \\ \mathcal{PBQ}w &= \mathbf{e}\int_0^\infty ((\mathbf{b}\cdot\mathbf{n}(a)) - p(a)(\mathbf{b}\cdot\mathbf{e}))da, \end{aligned} \tag{49}$$

Let us consider the bulk part approximation

$$(v(t), w(t)) \approx (\overline{v}(t), \overline{w}(t)). \tag{50}$$

As noted earlier, we do not expand $\overline{v}(t)$ but put $\overline{w} = \overline{w}_0 + \epsilon\overline{w}_1 + \ldots$ and insert this expansion to the system above getting

$$\begin{aligned} \overline{v}_t &= \mathcal{PSP}\overline{v} + \ldots, \\ \overline{w}_{0,t} + \epsilon\overline{w}_{1,t} + \ldots &= \mathcal{QSQ}\overline{w}_0 + \epsilon\mathcal{QSQ}\overline{w}_1 + \ldots \\ &\quad + \frac{1}{\epsilon}\mathcal{QCQ}\overline{w}_0 + \mathcal{QCQ}\overline{w}_1 + \ldots, \\ \gamma\overline{v} &= \mathcal{PBP}\overline{v} + \mathcal{PBQ}\overline{w}_0 + \epsilon\mathcal{PBQ}\overline{w}_1 + \ldots, \\ \gamma\overline{w}_0 + \epsilon\gamma\overline{w}_1 + \ldots &= \mathcal{QBP}\overline{v} + \mathcal{QBQ}\overline{w}_0 + \epsilon\mathcal{PBQ}\overline{w}_1 + \ldots, \\ \overline{v}(0) &= \overset{o}{v}, \\ \overline{w}_0(0) + \epsilon\overline{w}_1(0) + \ldots &= \overset{o}{w} \end{aligned} \tag{51}$$

Comparing like powers of ϵ in the second equation, we get the following hierarchy

$$\begin{aligned} \mathcal{QCQ}\overline{w}_0 &= 0, \\ \mathcal{QSP}\overline{v} + \mathcal{QCQ}\overline{w}_1 &= 0 \end{aligned}$$

which yields $\overline{w}_0 = 0$ and $\overline{w}_1 = (\mathcal{QCQ})^{-1}\mathcal{QSP}\overline{v}$ on account of invertibility of $\mathcal{QCQ}$ in the kinetic subspace W. Then from the first equation, the first boundary

condition and the first initial condition we find a closed system for $\overline{v}$:

$$\begin{aligned} \overline{v}_t &= \mathcal{PSP}\overline{v}, \\ \gamma\overline{v} &= \mathcal{PBP}\overline{v}, \\ \overline{v}(0) &= \overset{o}{v}. \end{aligned} \tag{52}$$

By earlier calculations, in real terms this reads

$$\begin{aligned} \partial_t p &= -\partial_a p, \\ p(t,0) &= (\mathbf{b}\cdot\mathbf{e})\int_0^\infty p(t,a)da, \\ p(0,a) &= \overset{o}{n}_1(a)e_1 + \ldots + \overset{o}{n}_N(a)e_N. \end{aligned} \tag{53}$$

However, we can easily see that the initial condition for $\overline{w}_1$ (and thus $\overline{w}$) is determined by $\bar{v}$; also the boundary condition for $\overline{w}$ contains term $\mathcal{QBP}\overline{v}$ which, in general, is different from zero. Hence, we can expect that the approximation will be not of the required ϵ order. Let us ascertain this by introducing the error equation which also gives some ideas of how to move forward.

The error of the approximation (50) is defined as

$$\begin{aligned} E(t) &= (e(t), f(t)) \\ &:= (v(t)-\overline{v}(t), w(t)-\overline{w}(t)) = (v(t)-\overline{v}(t), w(t)-\epsilon\overline{w}_1(t)). \end{aligned} \tag{54}$$

One can note an apparently inconsistent inclusion of an ϵ order term into the approximation which should be of ϵ order and thus such terms should appear as error terms. However, inclusion of this term simplifies calculations below.

Hence, for the error we have the following equations

$$\begin{aligned} e_t &= v_t - \overline{v}_t = \mathcal{PSP}v - \overline{v}_t \\ &= \mathcal{PSP}e \end{aligned} \tag{55}$$

where we used the first equation of (52). Next

$$\begin{aligned} f_t &= w_t - \epsilon\overline{w}_{1,t} \\ &= \mathcal{QSQ}f + \frac{1}{\epsilon}\mathcal{QCQ}f - \epsilon\overline{w}_{1,t} + \epsilon\mathcal{QSQ}\overline{w}_1 \end{aligned} \tag{56}$$

The error satisfies the following side-conditions

$$\begin{aligned} e(0) &= 0, \\ f(0) &= \overset{o}{w} - \epsilon\overline{w}_1(0) = \overset{o}{w} + \epsilon(\mathcal{QCQ})^{-1}\mathcal{QSP}\ \overset{o}{v} \end{aligned}$$

and

$$\begin{aligned} \gamma e &= \gamma v - \gamma\overline{v} = \mathcal{PBP}v + \mathcal{PBQ}w - \gamma\overline{v} \\ &= \mathcal{PBP}e + \mathcal{PBQ}f + \epsilon\mathcal{PBQ}\overline{w}_1, \\ \gamma f &= \gamma w - \epsilon\gamma\overline{w}_1 = \mathcal{QBP}v + \mathcal{QBQ}w - \epsilon\gamma\overline{w}_1 \\ &= \mathcal{QBP}e + \mathcal{QBQ}f + \mathcal{QBP}\overline{v} + \epsilon\mathcal{QBQ}\overline{w}_1 - \epsilon\gamma\overline{w}_1. \end{aligned}$$

Combining the above, we see that the error E is a solution of the problem

$$\begin{aligned} E_t &= \mathcal{S}E + \frac{1}{\epsilon}\mathcal{C}E + \epsilon R_1, \\ E(0) &= R_2 + \epsilon R_3, \end{aligned} \tag{57}$$

$$\gamma E = BE + R_4 + \epsilon R_5 \tag{58}$$

where

$$R_1 = \begin{bmatrix} \mathcal{PSQ}\overline{w}_1 \\ -\overline{w}_{1,t} + \mathcal{QSQ}\overline{w}_1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 \\ \overset{o}{w} \end{bmatrix}, \quad R_4 = \begin{bmatrix} 0 \\ \mathcal{QBP}\overline{v} \end{bmatrix}$$

and

$$R_3 = \begin{bmatrix} 0 \\ (\mathcal{QCQ})^{-1}\mathcal{QSP}\ \overset{o}{v} \end{bmatrix}, \quad R_5 = \begin{bmatrix} \mathcal{PBQ}\overline{w}_1 \\ \mathcal{QBQ}\overline{w}_1 - \gamma\overline{w}_1. \end{bmatrix}.$$

We observe that the problem for error has the same structure as the original one but it is inhomogeneous, both in the equation and in the side conditions.

While the inhomogeneity of the equation can be in general dealt with by the Duhamel formula (35), the inhomogeneity in the 'boundary condition' is more complicated and there are various versions of trace theorems which can lift the inhomogeneity at the boundary to the interior of the boundary. Here we shall assume that there is a bounded solution operator $L_{\epsilon,\lambda}$ of the stationary problem

$$\begin{aligned} \lambda u &= Su + \frac{1}{\epsilon}Cu, \\ \gamma u &= g, \end{aligned} \tag{59}$$

which satisfies $L_{\epsilon,\lambda}g \in D(S) \cap D(K)$ and $\|L_{\epsilon,\lambda}\| \to 0$ as $\lambda \to \infty$ uniformly in $\epsilon \in (0, \epsilon_0)$ for some $\epsilon_0 > 0$.

We note that this assumption gives solvability of the following boundary value problem

$$\begin{aligned} \lambda u &= Su + \frac{1}{\epsilon}Cu, \\ \gamma u &= Bu + f, \end{aligned} \tag{60}$$

where B is a bounded operator between appropriate spaces, at least for sufficiently large λ. Indeed, consider $L_{\epsilon,\lambda}g$ for an unspecified, for the time being, function g. Then we obtain the boundary equation for g:

$$g = BL_{\epsilon,\lambda}g + f$$

and clearly, $\|BL_{\epsilon,\lambda}\| \leq q < 1$ provided λ is such that $\|L_{\epsilon,\lambda}\| \leq q/\|B\|$. By uniformity assumption, λ and q can be chosen independently of ϵ.

Then

$$g = (I - BL_{\epsilon,\lambda})^{-1} f$$

and, by the Neuman expansion, $\|H_{\epsilon,\lambda}\| \leq (1-q)^{-1}$. Hence, the solution u to (60) is given by

$$u = H_{\epsilon,\lambda}f = L_{\epsilon,\lambda}g = L_{\epsilon,\lambda}(I - BL_{\epsilon,\lambda})^{-1} f \tag{61}$$

with $\|H_{\epsilon,\lambda}\| \leq q/(\|B\|(1-q))$. The condition of uniformity with respect to ϵ is not as unrealistic as it seems due to the properties of K.

If we agree for this assumption, we can accept (as will be demonstrated below) that ϵ order contribution to the inhomogeneities result in the ϵ order contributions to the errors. If so, then (58) does not give us good estimates for the error due to contribution in the initial and boundary conditions which are of lower order. There are two ways to deal with this problem. One is to try to remove these low order terms by introducing appropriate initial and boundary layer corrections, and this will be the main topic of this lecture. The other is to adopt appropriate assumptions which will eliminate the troublesome terms. We see that if the initial condition satisfies $\overset{o}{u}= (\overset{o}{v}, 0)$; that is, if we start from hydrodynamic subspace, only ϵ order terms will be present in the initial condition. Eliminating the bad term from the boundary condition requires an additional assumption on the mechanism of the process, namely we need to assume that $\mathcal{B}$ commutes with $\mathcal{P}$ in which case $\mathcal{QBP} = \mathcal{BQP} = 0$.

Example 5 *In our case*

$$\mathcal{QBP}v = (\mathbf{b} \otimes \mathbf{e} - \mathbf{e}(\mathbf{b} \cdot \mathbf{e})) \int_0^\infty p(a)da$$

and we see that $\mathcal{QBP} = 0$ *if* $\beta_i = \mathbf{b} \cdot \mathbf{e}$ *for each* i *or, in other words, fecundity is constant in each patch.*

Under these assumptions, the equation for error takes the following form

$$\begin{aligned} E_t &= \mathcal{S}E + \frac{1}{\epsilon}\mathcal{C}E + \epsilon R_1, \\ E(0) &= \epsilon R_3, \\ \gamma E &= BE + \epsilon R_5 \end{aligned} \tag{62}$$

Using the operator $H_{\epsilon,\lambda}$ and its properties of linearity and boundedness, we define $V(t) = E(t) - \epsilon H_{\epsilon,\lambda} R_5$ so that

$$\begin{aligned} V_t &= E_t - \epsilon H_{\epsilon,\lambda} R_{5,t} = \mathcal{S}E + \frac{1}{\epsilon}\mathcal{C}E + \epsilon R_1 - \epsilon H_{\epsilon,\lambda} R_{5,t} \\ &= \mathcal{S}V + \frac{1}{\epsilon}\mathcal{C}V + \lambda\epsilon H_{\epsilon,\lambda} R_5 + \epsilon R_1(t) - \epsilon H_{\epsilon,\lambda} R_{5,t} \end{aligned} \tag{63}$$

and further

$$V(0) = \epsilon R_3 - \epsilon H_{\epsilon,\lambda} R_5(0),$$

and

$$\gamma V = \gamma E - \epsilon\gamma(B H_{\epsilon,\lambda} R_5) = BE + \epsilon R_5 - \epsilon B H_{\epsilon,\lambda} R_5 - R_5 = BV.$$

Since the semigroup $(G(t))_{t\geq 0}$ generated by $\mathcal{S} + \frac{1}{\epsilon}\mathcal{C}$ on the domain $\{Bu = 0\}$ is contractive, formally we have

$$\begin{aligned}
\|E(t)\| &\leq \epsilon\|H_{\epsilon,\lambda} R_5(t)\| + \epsilon\|G(t)(R_3 - H_{\epsilon,\lambda} R_5(0)) \\
&\quad + \int_0^t G(t-s)(R_1(s) + \lambda H_{\epsilon,\lambda} R_5(s) - H_{\epsilon,\lambda} R_{5,s}(s))ds\| \\
&\leq \epsilon\|H_{\epsilon,\lambda}\| \sup_{0\leq t\leq T} \|R_5(t)\| + \epsilon(\|R_3\| + \|H_{\epsilon,\lambda}\|\|R_5(0)\| \\
&\quad + \epsilon T(\sup_{0\leq t\leq T} \|R_1\| + \|H_{\epsilon,\lambda}\| \sup_{0\leq t\leq T} (\lambda\|R_5(t)\| + \|R_{5,t}(t)\|)
\end{aligned}$$

which gives convergence, uniform on any finite time interval.

This result is, however, unsatisfactory for at least two reasons. Firstly, the assumption that the kinetic part of the initial condition is zero means that we are only allowed to start from equilibrium (averaged) data and we miss the transient phenomena occurring when the system stabilizes. The second assumption imposes very stringent conditions on the system as demonstrated on the example pertaining to our main model.

To remedy the situation we have to introduce corrections which will take care of the transient phenomena occurring close to $t = 0$ and to the boundary $a = 0$. They should not 'spoil' the approximation away from spatial and temporal boundaries and thus should rapidly decrease to zero with increasing distance from both boundaries.

5.2 Initial layer

We start with the initial layer corrector. The idea is to blow up the neighbourhood of $t = 0$ by introducing new, fast, time

$$\tau = \frac{t}{\epsilon}$$

and the initial layer corrections by $\widetilde{u}(\tau) = (\widetilde{v}(\tau), \widetilde{w}(\tau))$. Thanks to linearity of the problem, we will try approximate the solution u as the sum of the bulk part obtained above and the initial layer which we construct below. We insert the formal expansion

$$\begin{aligned}
\widetilde{v}(\tau) &= \widetilde{v}_0(\tau) + \epsilon\widetilde{v}_1(\tau) + \dots, \\
\widetilde{w}(\tau) &= \widetilde{w}_0(\tau) + \epsilon\widetilde{w}_1(\tau) + \dots
\end{aligned}$$

into the system (47) getting

$$\begin{aligned}
\epsilon^{-1}(\widetilde{v}_{0,\tau} + \epsilon\widetilde{v}_{1,\tau} + \dots) &= \mathcal{P}\mathcal{S}\mathcal{P}(\widetilde{v}_0 + \epsilon\widetilde{v}_1 + \dots), \\
\epsilon^{-1}(\widetilde{w}_{0,\tau} + \epsilon\widetilde{w}_{1,\tau} + \dots) &= \mathcal{Q}\mathcal{S}\mathcal{Q}(\widetilde{w}_0 + \epsilon\widetilde{w}_1 + \dots) \\
&\quad + \frac{1}{\epsilon}\mathcal{Q}\mathcal{C}\mathcal{Q}\widetilde{w}_0 + \mathcal{Q}\mathcal{C}\mathcal{Q}\widetilde{w}_1 + \dots, \\
\gamma(\widetilde{v}_0 + \epsilon\widetilde{v}_1 + \dots) &= \mathcal{P}\mathcal{B}\mathcal{P}(\widetilde{v}_0 + \epsilon\widetilde{v}_1 + \dots) \\
&\quad + \mathcal{P}\mathcal{B}\mathcal{Q}\widetilde{w}_0 + \epsilon\mathcal{P}\mathcal{B}\mathcal{Q}\widetilde{w}_1 + \dots, \\
\gamma\widetilde{w}_0 + \epsilon\gamma\widetilde{w}_1 + \dots &= \mathcal{Q}\mathcal{B}\mathcal{P}(\widetilde{v}_0 + \epsilon\widetilde{v}_1 + \dots) \\
&\quad + \mathcal{Q}\mathcal{B}\mathcal{Q}\widetilde{w}_0 + \epsilon\mathcal{P}\mathcal{B}\mathcal{Q}\widetilde{w}_1 + \dots, \\
\widetilde{v}(0) &= 0, \\
\overline{w}_0(0) + \epsilon\overline{w}_1(0) + \dots &= \overset{o}{w}
\end{aligned} \tag{64}$$

where in the initial condition we have taken into account that the exact initial condition for the hydrodynamic part is already satisfied by the bulk hydrodynamic approximation but the bulk kinetic part cannot satisfy the exact initial condition.

Comparing coefficients at like powers of ϵ, from the first equation we immediately obtain $\widetilde{v}_{0,\tau} = 0$ which implies $\widetilde{v}_0$ on account of the decay to zero of the initial layer term. Then, at the same ϵ^{-1} level, we obtain

$$\widetilde{w}_{0,\tau} = \mathcal{Q}\mathcal{C}\mathcal{Q}\widetilde{w}_0$$

which yields

$$\widetilde{w}_0 = e^{\tau \mathcal{Q}\mathcal{C}\mathcal{Q}} \overset{o}{w} .$$

We note that due to the assumption that $\lambda = 0$ is the dominant eigenvalue of $\mathcal{K}$ and Q is the complementary spectral projection corresponding to $\lambda = 0$, $\widetilde{w}_0$ decays to 0 exponentially fast. We also note that the initial layer is fully determined by the initial condition $\overset{o}{w}$ and thus no corrections to the boundary conditions can be made at this level; on the contrary, as we shall see, the initial layer introduces an additional error on the boundary.

Let us modify the approximation taking into account the initial layer:

$$u(t) = (v(t), w(t)) \approx (\overline{v}(t), \epsilon\overline{w}_1 + \widetilde{w}_0)$$

and define the new error

$$\begin{aligned} \widetilde{E}(t) &= (\widetilde{e}(t), \widetilde{f}(t)) = (v(t) - \overline{v}(t), w(t) - \epsilon\overline{w}_1(t) - \widetilde{w}_0(t/\epsilon)) \\ &= (e(t), f(t) - \widetilde{w}_0(t/\epsilon)) \end{aligned} \tag{65}$$

Using the fact that the problem is linear, we get from (55) and (56)

$$\widetilde{e}_t = \mathcal{P}\mathcal{S}\mathcal{P}\widetilde{e}.$$

Next

$$\widetilde{f}_t = \mathcal{Q}\mathcal{S}\mathcal{Q}\widetilde{f} + \frac{1}{\epsilon}\mathcal{Q}\mathcal{C}\mathcal{Q}\widetilde{f} + \mathcal{Q}\mathcal{S}\mathcal{Q}\widetilde{w}_0 - \epsilon\overline{w}_{1,t} + \epsilon\mathcal{Q}\mathcal{S}\mathcal{Q}\overline{w}_1$$

Similarly, we get

$$\begin{aligned} \widetilde{e}(0) &= 0, \\ \widetilde{f}(0) &= -\epsilon\overline{w}_1(0) = \epsilon(\mathcal{Q}\mathcal{K}\mathcal{Q})^{-1}\mathcal{Q}\mathcal{S}\mathcal{P} \overset{o}{v} \end{aligned}$$

and

$$\begin{aligned}\gamma\widetilde{e} &= \mathcal{PBP}\widetilde{e} + \mathcal{PBQ}\widetilde{f} + \mathcal{PBQ}\widetilde{w}_0 + \epsilon\mathcal{PBQ}\overline{w}_1,\\ \gamma\widetilde{f} &= \mathcal{QBP}\widetilde{e} + \mathcal{QBQ}\widetilde{f} + \mathcal{QBQ}\widetilde{w}_0 + \mathcal{QBP}\overline{v}\\ &\quad +\epsilon\mathcal{QBQ}\overline{w}_1 - \epsilon\gamma\overline{w}_1 - \gamma\widetilde{w}_0.\end{aligned}$$

5.3 Boundary layer correction

We have seen that, as expected, the troublesome term $\mathcal{QBP}\overline{v}$ in the boundary condition has been unaffected by addition of the initial layer. To get rid of it we introduce the boundary layer by blowing up the state variable a according to

$$\alpha = a/\alpha$$

and defining

$$\widehat{u}(t,\alpha) = (\widehat{v}(t,\alpha), \widehat{w}(t,\alpha)).$$

The operator S is a first order differentiation operator acting on the other state variable, hence with the change of variables $a \to \alpha = a/\epsilon$ we have

$$\mathcal{S}_a\widehat{u} = \frac{1}{\epsilon}\mathcal{S}_\alpha\widehat{u}, \tag{66}$$

where the subscripts denote the variable which S acts on.

Similarly, the nonlocal boundary operator has the rescaling property

$$\gamma\widehat{u} = \gamma u = \mathcal{B}\widehat{u} = \epsilon\mathcal{B}_\alpha\widehat{u} \tag{67}$$

Again, by the linearity of the problem we approximate the solution u by the sum of the bulk and parts, obtained above, and the boundary layer which we construct below. We insert the formal expansion of the boundary layer

$$\begin{aligned}\widehat{v}(t,\alpha) &= \widehat{v}_0(t,\alpha) + \epsilon\widehat{v}_1(t,\alpha) + \ldots,\\ \widehat{w}(t,\alpha) &= \widehat{w}_0(t,\alpha) + \epsilon\widehat{w}_1(t,\alpha) + \ldots\end{aligned}$$

into the system (47) getting

$$\begin{aligned}
\widehat{v}_{0,t} + \epsilon\widehat{v}_{1,t} + \ldots &= \frac{1}{\epsilon}\mathcal{P}\mathcal{S}_\alpha\mathcal{P}(\widehat{v}_0 + \epsilon\widehat{v}_1 + \ldots), \\
\widehat{w}_{0,t} + \epsilon\widehat{w}_{1,t} + \ldots &= \frac{1}{\epsilon}\mathcal{Q}\mathcal{S}_\alpha\mathcal{Q}(\widehat{w}_0 + \epsilon\widehat{w}_1 + \ldots) \\
&\quad + \frac{1}{\epsilon}\mathcal{Q}\mathcal{C}\mathcal{Q}\widehat{w}_0 + \mathcal{Q}\mathcal{C}\mathcal{Q}\widehat{w}_1 + \ldots.
\end{aligned} \tag{68}$$

This shows that the boundary layer satisfies

$$\begin{aligned}
0 &= \mathcal{P}\mathcal{S}_\alpha\mathcal{P}\widehat{v}_0, \\
0 &= \mathcal{Q}\mathcal{S}_\alpha\mathcal{Q}\widehat{w}_0 + \mathcal{Q}\mathcal{C}\mathcal{Q}\widehat{w}_0
\end{aligned}$$

which is simply the stationary original equation

$$\mathcal{S}_\alpha\widehat{u} + \mathcal{C}\widehat{u} = 0 \tag{69}$$

and we have freedom of choosing the boundary conditions which will help to eliminate the term $\mathcal{Q}\mathcal{B}\mathcal{P}\overline{v}$. To find the proper boundary conditions, let us assume that we have a solution to the above equation with for the moment unspecified boundary condition and, as before, find the error of the approximation

$$\begin{aligned}
u(t,a) &= (v(t,a), w(t,a)) \\
&\approx (\overline{v}(t,a) + \widehat{v}(t,a/\epsilon), \epsilon\overline{w}_1(t,a) + \widetilde{w}_0(t/\epsilon,a) + \widehat{w}(t,a/e)).
\end{aligned}$$

and define the new error

$$\begin{aligned}
\widehat{E}(t,a) &= (\widehat{e}(t,a), \hat{f}(t,a)) \\
&= (\widetilde{e}(t,a) - \widehat{v}_0(t,a/\epsilon), \widetilde{f}(t,a) - \widehat{w}_0(t,a/\epsilon)).
\end{aligned} \tag{70}$$

Hence

$$\widehat{e}_t = \mathcal{P}\mathcal{S}\mathcal{P}\widehat{e} - \widehat{v}_{0,t}$$

and

$$\hat{f}_t = \mathcal{Q}\mathcal{S}\mathcal{Q}\hat{f} + \frac{1}{\epsilon}\mathcal{Q}\mathcal{C}\mathcal{Q}\hat{f} + \mathcal{Q}\mathcal{S}\mathcal{Q}\widetilde{w}_0 - \epsilon\overline{w}_{1,t} + \epsilon\mathcal{Q}\mathcal{S}\mathcal{Q}\overline{w}_1 - \widehat{w}_{0,t}$$

where we used (66) to eliminate terms in the last line.

For the boundary conditions we obtain

$$\begin{aligned}
\gamma\widehat{e} &= \mathcal{P}\mathcal{B}\mathcal{P}\widehat{e} + \mathcal{P}\mathcal{B}\mathcal{Q}\hat{f} + \mathcal{P}\mathcal{B}\mathcal{Q}\widetilde{w}_0 + \epsilon\mathcal{P}\mathcal{B}\mathcal{Q}\overline{w}_1 + \mathcal{P}\mathcal{B}\mathcal{P}\widehat{v}_0 \\
&\quad + \mathcal{P}\mathcal{B}\mathcal{Q}\widehat{w}_0 - \gamma\widehat{v}_0, \\
\gamma\hat{f} &= \mathcal{Q}\mathcal{B}\mathcal{P}\widehat{e} + \mathcal{Q}\mathcal{B}\mathcal{Q}\hat{f} + \mathcal{Q}\mathcal{B}\mathcal{P}\overline{v} + \epsilon\mathcal{Q}\mathcal{B}\mathcal{Q}\overline{w}_1 - \epsilon\gamma\overline{w}_1 \\
&\quad + \mathcal{Q}\mathcal{B}\mathcal{P}\widehat{v}_0 + \mathcal{Q}\mathcal{B}\mathcal{Q}\widehat{w}_0 - \gamma\widehat{w}_0 + \mathcal{Q}\mathcal{B}\mathcal{Q}\widetilde{w}_0 - \gamma\widetilde{w}_0.
\end{aligned}$$

Thus, to eliminate the bad term $\mathcal{Q}\mathcal{B}\mathcal{P}\overline{v}$, we solve the full boundary layer problem

$$\begin{aligned}
\mathcal{P}\mathcal{S}_\alpha\mathcal{P}\widehat{v}_0 &= 0, \\
\mathcal{Q}\mathcal{S}_\alpha\mathcal{Q}\widehat{w}_0 + \mathcal{Q}\mathcal{C}\mathcal{Q}\widehat{w}_0 &= 0, \\
\gamma\widehat{v}_0 &= 0, \\
\gamma\widehat{w}_0 &= \mathcal{Q}\mathcal{B}\mathcal{P}\overline{v}.
\end{aligned}$$

With this boundary condition for the boundary layer equation, the boundary conditions for the error equation are

$$\begin{aligned}
\gamma\widehat{e} &= \mathcal{P}\mathcal{B}\mathcal{P}\widehat{e} + \mathcal{P}\mathcal{B}\mathcal{Q}\hat{f} + \mathcal{P}\mathcal{B}\mathcal{Q}\widetilde{w}_0 + \epsilon\mathcal{P}\mathcal{B}\mathcal{Q}\overline{w}_1 + \epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{P}\widehat{v}_0 \\
&\quad + \epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{Q}\widehat{w}_0, \\
\gamma\hat{f} &= \mathcal{Q}\mathcal{B}\mathcal{P}\widehat{e} + \mathcal{Q}\mathcal{B}\mathcal{Q}\hat{f} + \epsilon\mathcal{Q}\mathcal{B}\mathcal{Q}\overline{w}_1 - \epsilon\gamma\overline{w}_1 + \epsilon\mathcal{Q}\mathcal{B}_\alpha\mathcal{P}\widehat{v}_0 \\
&\quad + \epsilon\mathcal{Q}\mathcal{B}_\alpha\mathcal{Q}\widehat{w}_0 + \mathcal{Q}\mathcal{B}\mathcal{Q}\widetilde{w}_0 - \gamma\widetilde{w}_0.
\end{aligned}$$

We note that still we have terms depending on t/ϵ which, when lifted as in (63) will, upon differentiation with respect to t, produce $1/\epsilon$ on the right hand side.

For the initial condition we obtain

$$\begin{aligned}
\widehat{e}(0) &= -\widehat{v}_0(0, a/\epsilon), \\
\hat{f}(0) &= -\widehat{w}_0(0, a/e) - \epsilon\overline{w}_1(0, a)
\end{aligned}$$

5.4 Corner layer

As noted above, by (63), all terms apart from $\mathcal{PBQ}\widetilde{w}_0$ and $\gamma\widetilde{w}_0$, which depend on t/ϵ, give rise to ϵ order error. To eliminate this initial layer contribution on the boundary, we need to introduce the corner layer by simultaneously rescaling time and space:

$$\tau = \frac{t}{\epsilon}, \qquad \alpha = \frac{a}{\epsilon}.$$

As before we use linearity to approximate the solution u by the sum of the bulk, initial and boundary layer parts, obtained above, and the corner layer which we construct below. We insert the formal expansion of the corner layer

$$\begin{aligned} \breve{v}(\tau,\alpha) &= \breve{v}_0(\tau,\alpha) + \epsilon\breve{v}_1(\tau,\alpha) + \dots, \\ \breve{w}(\tau,\alpha) &= \breve{w}_0(\tau,\alpha) + \epsilon\breve{w}_1(\tau,\alpha) + \dots \end{aligned}$$

into the system (47) getting

$$\begin{aligned} \frac{1}{\epsilon}(\breve{v}_{0,\tau} + \epsilon\breve{v}_{1,\tau} + \dots) &= \frac{1}{\epsilon}\mathcal{PS}_\alpha\mathcal{P}(\breve{v}_0 + \epsilon\breve{v}_1 + \dots), \\ \frac{1}{\epsilon}(\breve{w}_{0,\tau} + \epsilon\breve{w}_{1,\tau} + \dots) &= \frac{1}{\epsilon}\mathcal{QS}_\alpha\mathcal{Q}(\breve{w}_0 + \epsilon\breve{w}_1 + \dots) \\ &\quad + \frac{1}{\epsilon}\mathcal{QCQ}\breve{w}_0 + \mathcal{QCQ}\breve{w}_1 + \dots. \end{aligned} \tag{71}$$

This shows that the corner layer satisfies

$$\breve{v}_{0,\tau} = \mathcal{PS}_\alpha\mathcal{P}\breve{v}_0, \tag{72}$$

$$\breve{w}_{0,\tau} = \mathcal{QS}_\alpha\mathcal{Q}\breve{w}_0 + \mathcal{QCQ}\breve{w}_0 \tag{73}$$

which is the unperturbed original equation

$$\breve{u}_{0,\tau} = \mathcal{S}_\alpha\breve{u}_0 + \mathcal{K}\breve{u}_0 \tag{74}$$

and we have freedom of choosing the boundary and initial conditions which will help to eliminate the problematic terms on the boundary. To find the proper side

conditions, let us assume that we have a solution to the above equation with, for the moment, unspecified boundary condition and, as before, find the error of the approximation

$$\begin{aligned} u(t,a) &= (v(t,a), w(t,a)) \\ &\approx (\overline{v}(t,a) + \widehat{v}(t,a/\epsilon) + \breve{v}(t/\epsilon, a/\epsilon), \\ &\quad \epsilon\overline{w}_1(t,a) + \widetilde{w}_0(t/\epsilon, a) + \widehat{w}(t,a/e) + \breve{w}(t/\epsilon, a/\epsilon)) \end{aligned}$$

and define the new error

$$\begin{aligned} \breve{E}(t,a) &= (\breve{e}(t,a), \breve{f}(t,a)) \\ &= (\widehat{e}(t,a) - \breve{v}_0(t/\epsilon, a/\epsilon), \hat{f}(t,a) - \breve{w}_0(t/\epsilon, a/\epsilon)). \end{aligned} \tag{75}$$

Hence

$$\breve{e}_t = \mathcal{P}\mathcal{S}\mathcal{P}\breve{e} - \widehat{v}_{0,t}$$

where we used (72). Similarly, for the kinetic part of the error, we get

$$\breve{f}_t = \mathcal{Q}\mathcal{S}\mathcal{Q}\breve{f} + \frac{1}{\epsilon}\mathcal{Q}\mathcal{C}\mathcal{Q}\breve{f} + \mathcal{Q}\mathcal{S}\mathcal{Q}\widetilde{w}_0 - \epsilon\overline{w}_{1,t} + \epsilon\mathcal{Q}\mathcal{S}\mathcal{Q}\overline{w}_1 - \widehat{w}_{0,t}$$

where we used (73) to eliminate the corner layer terms in the last line.

For the boundary conditions, we obtain

$$\begin{aligned} \gamma\breve{e} &= \mathcal{P}\mathcal{B}\mathcal{P}\breve{e} + \mathcal{P}\mathcal{B}\mathcal{Q}\breve{f} \\ &\quad + \epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{P}\breve{v}_0 + \epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{Q}\breve{w}_0 + \epsilon\mathcal{P}\mathcal{B}\mathcal{Q}\overline{w}_1 + \epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{P}\widehat{v}_0 \\ &\quad + \epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{Q}\widehat{w}_0 + \mathcal{P}\mathcal{B}\mathcal{Q}\widetilde{w}_0 - \gamma\breve{v}_0, \\ \gamma\breve{f} &= \mathcal{Q}\mathcal{B}\mathcal{P}\breve{e} + \mathcal{Q}\mathcal{B}\mathcal{Q}\breve{f} + \epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{P}\breve{v}_0 + \epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{Q}\breve{w}_0 + \epsilon\mathcal{Q}\mathcal{B}\mathcal{Q}\overline{w}_1 \\ &\quad + \epsilon\mathcal{Q}\mathcal{B}_\alpha\mathcal{P}\widehat{v}_0 + \epsilon\mathcal{Q}\mathcal{B}_\alpha\mathcal{Q}\widehat{w}_0 - \epsilon\gamma\overline{w}_1 \\ &\quad + \mathcal{Q}\mathcal{B}\mathcal{Q}\widetilde{w}_0 - \gamma\widetilde{w}_0 - \gamma\breve{w}_0. \end{aligned}$$

Hence, to eliminate entries of order 1 at the boundary, we solve the equation (74) subject to the boundary conditions

$$\begin{aligned}\gamma\breve{v}_0 &= \mathcal{P}\mathcal{B}\mathcal{Q}\widetilde{w}_0,\\ \gamma\breve{w}_0 &= \mathcal{Q}\mathcal{B}\mathcal{Q}\widetilde{w}_0 - \gamma\widetilde{w}_0.\end{aligned}$$

This gives on the boundary

$$\begin{aligned}\gamma\breve{e} &= \mathcal{P}\mathcal{B}\mathcal{P}\breve{e} + \mathcal{P}\mathcal{B}\mathcal{Q}\breve{f}\\ &\quad +\epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{P}\breve{v}_0 + \epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{Q}\breve{w}_0\\ &\quad +\epsilon\mathcal{P}\mathcal{B}\mathcal{Q}\overline{w}_1 + \epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{P}\widehat{v}_0 + \epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{Q}\widehat{w}_0,\\ \gamma\breve{f} &= \mathcal{Q}\mathcal{B}\mathcal{P}\breve{e} + \mathcal{Q}\mathcal{B}\mathcal{Q}\breve{f} + \epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{P}\breve{v}_0 + \epsilon\mathcal{P}\mathcal{B}_\alpha\mathcal{Q}\breve{w}_0\\ &\quad +\epsilon\mathcal{Q}\mathcal{B}\mathcal{Q}\overline{w}_1 + \epsilon\mathcal{Q}\mathcal{B}_\alpha\mathcal{P}\widehat{v}_0 + \epsilon\mathcal{Q}\mathcal{B}_\alpha\mathcal{Q}\widehat{w}_0 - \epsilon\gamma\overline{w}_1.\end{aligned}$$

In compact form, this reads

$$\begin{aligned}\breve{E}_t &= \mathcal{S}\breve{E} + \frac{1}{\epsilon}\mathcal{C}\breve{E} + \mathcal{S}\begin{bmatrix}0\\ \widetilde{w}_0(t/\epsilon)\end{bmatrix}\\ &\quad +\epsilon\mathcal{S}\begin{bmatrix}0\\ \overline{w}_1(t)\end{bmatrix} - \epsilon\begin{bmatrix}0\\ \overline{w}_{1,t}(t)\end{bmatrix} - \begin{bmatrix}\widehat{v}_0(t)\\ \widehat{w}_{0,t}(t)\end{bmatrix}\\ \gamma\breve{E}(t) &= \mathcal{B}\breve{E}(t) + \epsilon\mathcal{B}_\alpha\breve{u}_0(t/\epsilon) + \epsilon\mathcal{B}_\alpha\widehat{u}_0(t)\\ &\quad +\epsilon\mathcal{B}\begin{bmatrix}0\\ \overline{w}_1(t)\end{bmatrix} - \epsilon\begin{bmatrix}0\\ \gamma\overline{w}_1(t)\end{bmatrix}\\ &= \mathcal{B}\breve{E}(t) + \epsilon R_6(t/\epsilon) + \epsilon R_7(t).\end{aligned}$$

The terms coming from the corner layer are of great importance. Let us look

closer at the corner layer problem. We are to solve

$$\begin{aligned}
\breve{v}_{0,\tau} &= \mathcal{P}\mathcal{S}_\alpha\mathcal{P}\breve{v}_0, && (76)\\
\breve{w}_{0,\tau} &= \mathcal{Q}\mathcal{S}_\alpha\mathcal{Q}\breve{w}_0 + \mathcal{Q}\mathcal{C}\mathcal{Q}\breve{w}_0 && (77)\\
\gamma\breve{v}_0 &= \mathcal{P}\mathcal{B}\mathcal{Q}\widetilde{w}_0, && (78)\\
\gamma\breve{w}_0 &= \mathcal{Q}\mathcal{B}\mathcal{Q}\widetilde{w}_0 - \gamma\widetilde{w}_0 && (79)\\
\breve{v}_0(0) &= \breve{w}_0(0) = 0. && (80)
\end{aligned}$$

It would be tempting to believe that since the right hand sides behave as $e^{\mathcal{Q}\mathcal{K}\mathcal{Q}\tau}$, which is of negative type, the corner layer $(\breve{v}_0, \breve{w}_0)$ have the same negative type. However, this is not always the case (as we shall see in the example). The reason for this is that the first equation is, in general, only dissipative with no exponential decay and lifting of the inhomogeneities (78) and (79) produces a nontrivial initial condition which can evolve in a non-exponentially decaying way. Fortunately, often one can prove that the term $R_6(\tau)$ (and its derivative with respect to τ) is indeed of the form $A + O(e^{-\omega\tau})$ for some $\omega > 0$ with A independent of τ. Lifting this term according to (63)

$$V_t = \mathcal{S}V + \frac{1}{\epsilon}\mathcal{K}V + \lambda\epsilon H_{\epsilon,\lambda}R_6(\tau) - H_{\epsilon,\lambda}R_5'(t/\epsilon) \qquad (81)$$

so we lost one ϵ due to differentiation.

However, consider the integral

$$\|\int_0^t G(t-s)H_{\epsilon,\lambda}R_5'(s/\epsilon))ds\| \le \epsilon\|H_{\epsilon,\lambda}\|\int_0^{t/\epsilon}\|R_5'(\sigma))\|d\sigma \le \epsilon M$$

where M is finite on account of the exponential decay of R_5'. In fact, it suffices that R_5' be integrable.

In our situation equation (76) is decoupled from (77) and the problem for $\breve{v}_0$

is of the form

$$\begin{aligned}\breve{v}_{0,\tau} &= -\breve{v}_{0,\alpha},\\ \breve{v}_0(0,\alpha) &= 0,\\ \breve{v}_0(\tau,0) &= F(\tau).\end{aligned}$$

We immediately find the solution as

$$\breve{v}_0(\tau,\alpha) = \begin{cases} F(\tau-\alpha) & \text{for} \quad \tau > \alpha,\\ 0 & \text{for} \quad \tau < \alpha,\end{cases}$$

which clearly does not have exponential growth even if F has.

Using the formulae for projections (45) we see that

$$\mathcal{PBQ}\widetilde{w}_0 = \mathcal{PB}\widetilde{w}_0 - \mathcal{PBP}\widetilde{w}_0 = \mathbf{e}\int_0^\infty \mathbf{1}\cdot[\mathbf{b}\otimes e^{\tau\mathcal{QCQ}}\ \overset{o}{w}\ (a)]da$$

as $\mathcal{P}\widetilde{w}_0 = 0$. Hence,

$$\breve{v}_0(\tau,\alpha) = \begin{cases} \int_0^\infty \mathbf{1}\cdot[\mathbf{b}\otimes e^{(\tau-\alpha)\mathcal{QCQ}}\ \overset{o}{w}\ (a)]da & \text{for} \quad \tau > \alpha,\\ 0 & \text{for} \quad \tau < \alpha,\end{cases}$$

The term containing $\breve{v}_0$ in R_6 is therefore

$$\begin{aligned}&\int_0^\infty [\mathbf{b}\cdot\mathbf{e}]\breve{v}_0(\tau,\alpha)d\alpha\\ =\ &\int_0^\tau [\mathbf{b}\cdot\mathbf{e}]\left(\int_0^\infty \mathbf{1}\cdot[\mathbf{b}\otimes e^{(\tau-\alpha)\mathcal{QCQ}}\ \overset{o}{w}\ (a)]da\right)d\alpha\\ =\ &\epsilon[\mathbf{b}\cdot\mathbf{e}]\int_0^\infty \mathbf{1}\cdot[\mathbf{b}\otimes\left(\int_0^\tau e^{(\tau-\alpha)\mathcal{QCQ}}\ \overset{o}{w}\ (a)d\alpha\right)]da\\ =\ &\epsilon[\mathbf{b}\cdot\mathbf{e}]\int_0^\infty \mathbf{1}\cdot[\mathbf{b}\otimes\Big((\mathcal{QCQ})^{-1}(e^{\tau\mathcal{QCQ}}\ \overset{o}{w}\ (a)-\ \overset{o}{w}\ (a))\Big)]da\end{aligned}$$

and, since $e^{\tau \mathcal{Q}\mathcal{C}\mathcal{Q}}$ is of negative type, we see that this term satisfies the assumption on the behaviour of R_6.

Let us consider the term $\mathcal{B}\breve{w}_0$. The kinetic part of the corner layer satisfies

$$\begin{aligned} \breve{w}_{0,\tau} &= \mathcal{Q}\mathcal{S}_\alpha \mathcal{Q}\breve{w}_0 + \mathcal{Q}\mathcal{C}\mathcal{Q}\breve{w}_0 && (82)\\ \gamma\breve{w}_0 &= \mathcal{Q}\mathcal{B}\mathcal{Q}\widetilde{w}_0 - \gamma\widetilde{w}_0 && (83)\\ \breve{w}_0(0) &= 0. && (84) \end{aligned}$$

We have

$$\begin{aligned} \mathcal{Q}\mathcal{B}\mathcal{Q}\widetilde{w}_0 &= \mathcal{Q}\int_0^\infty (\mathbf{b}\otimes(\widetilde{w}_0 - \mathcal{P}\widetilde{w}_0)da = \mathcal{Q}\int_0^\infty \mathbf{b}\otimes\widetilde{w}_0 da \\ &= \int_0^\infty \mathbf{b}\otimes\widetilde{w}_0 da - \mathcal{P}\int_0^\infty \mathbf{b}\otimes\widetilde{w}_0 da \\ &= \int_0^\infty \mathbf{b}\otimes\widetilde{w}_0 da - \mathbf{e}\int_0^\infty \mathbf{1}\cdot[\mathbf{b}\otimes\widetilde{w}_0]da \end{aligned}$$

that is

$$[\mathcal{Q}\mathcal{B}\mathcal{Q}\widetilde{w}_0](\tau) = \int_0^\infty \mathbf{b}\otimes e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\ \overset{o}{w}\ (a)da - \mathbf{e}\int_0^\infty \mathbf{1}\cdot[\mathbf{b}\otimes e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\ \overset{o}{w}\ (a)]da,$$

so the boundary value can be written as

$$\gamma\breve{w}_0(\tau) = \mathcal{L}(e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\ \overset{o}{w})$$

where $\mathcal{L}$ is a bounded linear operator from interior to the boundary.

To convert the problem into homogeneous boundary problem, we introduce $Z(\tau,\alpha) = r(\alpha)\mathcal{L}(e^{\tau\mathcal{Q}\mathcal{K}\mathcal{Q}}\ \overset{o}{w})$ where r is a scalar sufficiently regular function with

$r(0) = 1$. Then $W = \breve{w}_0 - Z$ satisfies

$$\begin{aligned} W_\tau &= \mathcal{Q}\mathcal{S}_\alpha\mathcal{Q}W + \mathcal{Q}\mathcal{C}\mathcal{Q}W \\ &\quad +\mathcal{Q}\mathcal{S}_\alpha\mathcal{Q}(r(\alpha)\mathcal{L}(e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\overset{o}{w})) + \mathcal{Q}\mathcal{C}\mathcal{Q}(r(\alpha)\mathcal{L}(e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\overset{o}{w})) \\ &\quad -r(\alpha)\mathcal{L}(\mathcal{Q}\mathcal{C}\mathcal{Q}e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\overset{o}{w}) \\ \gamma W &= 0 \\ W(0,\alpha) &= -r(\alpha)\mathcal{L}(\overset{o}{w}). \end{aligned}$$

We note that, since $\mathcal{L}(e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\overset{o}{w}))$ does not depend on α, $\mathcal{Q}\mathcal{S}_\alpha\mathcal{Q}$ acts only on r which is supposed to be sufficiently smooth and hence this term remains a bounded operator on the exponential. In fact, in our case we have

$$S_\alpha(r(\alpha)\mathcal{L}(e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\overset{o}{w})) = -r'(\alpha)\mathcal{L}(e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\overset{o}{w}).$$

Denote $\mathcal{A}_\alpha = \mathcal{Q}\mathcal{S}_\alpha\mathcal{Q} + \mathcal{Q}\mathcal{C}\mathcal{Q}$ and $(\mathcal{G}_\alpha(t))_{t\geq 0}$ the semigroup generated by it. Thus, the inhomogeneity is a bounded linear operator given by

$$\mathcal{L}_1(e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\overset{o}{w})) = \mathcal{A}_\alpha(r(\alpha)\mathcal{L}(e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\overset{o}{w})) - r(\alpha)\mathcal{L}(\mathcal{Q}\mathcal{C}\mathcal{Q}e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\overset{o}{w}).$$

Thus we have

$$\begin{aligned} \breve{w}_0(\tau,\alpha) &= r(\alpha)\mathcal{L}(e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\overset{o}{w}) - \mathcal{G}_\alpha(\tau)[r(\alpha)\mathcal{L}(\overset{o}{w})] \\ &\quad + \int_0^\tau \mathcal{G}_\alpha(\tau-\sigma)\mathcal{L}_1(e^{\tau\mathcal{Q}\mathcal{C}\mathcal{Q}}\overset{o}{w})d\sigma \end{aligned}$$

and the contribution to R_6 due to $\breve{w}_0$ is $\epsilon\int_0^\infty \mathbf{b}\otimes\breve{w}_0(\tau,\alpha)d\alpha$. To get estimates of the derivative with respect to t required in the error calculations, we have

$$
\begin{aligned}
&\epsilon \int_0^\infty \mathbf{b} \otimes \breve{w}_{0,t}(\tau,\alpha)d\alpha = \int_0^\infty \mathbf{b} \otimes \breve{w}_{0,\tau}(\tau,\alpha)d\alpha \\
= &\int_0^\infty \mathbf{b} \otimes \mathcal{A}_\alpha \breve{w}_0(\tau,\alpha)d\alpha \\
= &\int_0^\infty B\mathcal{A}_\alpha \Big(r(\alpha)\mathcal{L}(e^{\tau \mathcal{Q}\mathcal{C}\mathcal{Q}} \overset{o}{w}) - \mathcal{G}_\alpha(\tau)[r(\alpha)\mathcal{L}(\overset{o}{w})] \\
&+ \int_0^\tau \mathcal{G}_\alpha(\tau-\sigma)\mathcal{L}_1(e^{\tau \mathcal{Q}\mathcal{C}\mathcal{Q}} \overset{o}{w})d\sigma \Big) d\alpha \\
= &\int_0^\infty \mathbf{b} \otimes \Big(\mathcal{A}_\alpha \Big(r(\alpha)\mathcal{L}(e^{\tau \mathcal{Q}\mathcal{C}\mathcal{Q}} \overset{o}{w}) \Big) - \mathcal{G}_\alpha(\tau)[\mathcal{A}_\alpha(r(\alpha)\mathcal{L}(\overset{o}{w}))] \\
&+ \int_0^\tau \mathcal{G}_\alpha(\tau-\sigma) \Big(\mathcal{A}_\alpha \mathcal{L}_1(e^{\tau \mathcal{Q}\mathcal{K}\mathcal{Q}} \overset{o}{w})d\sigma \Big) \Big) d\alpha.
\end{aligned}
$$

Again, the action of the unbounded operator $\mathcal{A}_\alpha$ is absorbed by the auxiliary function $r(\alpha)$, and thus all the terms remain bounded with respect to the argument, which is exponential term $e^{\tau \mathcal{Q}\mathcal{C}\mathcal{Q}} \overset{o}{w}$, and we obtain the estimate

$$
\begin{aligned}
&\epsilon \| H_{\epsilon,\lambda} \int_0^\infty B\breve{w}_{0,t}(\tau,\alpha)d\alpha \| \\
&\leq \epsilon \| H_{\epsilon,\lambda} \| \| B \| \Big(C_1 \| e^{\tau \mathcal{Q}\mathcal{C}\mathcal{Q}} \| \| \overset{o}{w} \| + C_2 \| \mathcal{G}_\alpha(\tau) \| \overset{o}{w} \| \\
&+ C_3 \| \overset{o}{w} \| \int_0^\tau \| \mathcal{G}_\alpha(\tau-\sigma) \| \| e^{\sigma \mathcal{Q}\mathcal{C}\mathcal{Q}} \| d\sigma \Big) \\
&\epsilon e^{-\omega\tau}(C_4 + C_5\tau),
\end{aligned}
$$

where we used for both $(\mathcal{G}_\alpha(t))_{t\geq 0}$ and $e^{\tau \mathcal{Q}\mathcal{K}\mathcal{Q}}$ the negative type estimate by $e^{-\omega t}$.

If the types are different, say ω_1 and ω_2 with $\omega_2 < \omega_1$, we obtain

$$\int_0^\tau e^{-\omega_1(\tau-\sigma)} e^{-\omega_2\sigma} d\sigma = \frac{e^{-\omega_1\tau}}{\omega_1 - \omega_2}(e^{(\omega_1-\omega_2)\tau} - 1) = \frac{e^{-\omega_2\tau} - e^{-\omega_1\tau}}{\omega_1 - \omega_2}$$

which again is of negative exponential growth. Hence, we see that in this case our assumption for justifying the error estimates are satisfied.

6 Diffusion approximation

Consider the singularly perturbed telegraph equation

$$\partial_t \begin{bmatrix} v \\ w \end{bmatrix} = \mathcal{S} \begin{bmatrix} v \\ w \end{bmatrix} + \frac{1}{\varepsilon}\mathcal{C} \begin{bmatrix} v \\ w \end{bmatrix}, \tag{85}$$

where

$$\mathcal{S} = \begin{bmatrix} 0 & -b\partial_x \\ -c\partial_x & 0 \end{bmatrix}, \ \mathcal{C} = \begin{bmatrix} 0 & 0 \\ 0 & -d \end{bmatrix}.$$

or

$$\begin{aligned} \partial_t v + b\partial_x w &= 0, \\ \partial_t w + c\partial_x v + \frac{d}{\varepsilon} w &= 0, \end{aligned} \tag{86}$$

with constant coefficients b, c, d and a small parameter $\varepsilon > 0$, supplemented by the initial conditions

$$v(0) = \overset{o}{v}, \qquad w(0) = \overset{o}{w}, \tag{87}$$

and the homogeneous Dirichlet conditions

$$v(-1, t) = v(1, t) = 0, \ t > 0. \tag{88}$$

Clearly $\mathcal{C}$, considered on $\mathbb{C}^2$, has a one dimensional hydrodynamic space spanned by $\mathbf{e} = (0, 1)$. We have obvious spectral projections

$$\mathcal{P}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix},$$

$$\mathcal{Q}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ w \end{bmatrix}.$$

Then

$$\mathcal{P}\mathcal{S}\mathcal{P}\begin{bmatrix} v \\ 0 \end{bmatrix} = \mathcal{P}\begin{bmatrix} 0 & -b\partial_x \\ -c\partial_x & 0 \end{bmatrix}\begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and, recalling the candidate for the limit equation as in (52), we obtain

$$\overline{v}_t = \mathcal{P}\mathcal{S}\mathcal{P}\overline{v} = 0, \tag{89}$$

which gives trivial limit dynamics. Clearly, we do not have limit diffusion but let us consider second order approximation and go one level higher using the compressed expansion.

Recall that we are working with

$$\partial_t u = \mathcal{S}u + \frac{1}{\varepsilon}\mathcal{C}u, \tag{90}$$

in a Banach space $\mathcal{X}$ (which typically is $L^p_x(\Omega) \otimes X'$). Here we also assume that the setting is such that no boundary (and thus corner) layer phenomena occur so that we focus on temporal transient effects captured by the initial layer. However, the analysis can be extended to more general situations.

By applying the spectral projections to both sides of (90) we get

$$\begin{aligned} \partial_t v &= \mathcal{P}\mathcal{S}\mathcal{P}v + \mathcal{P}\mathcal{S}\mathcal{Q}w \\ \varepsilon\partial_t w &= \varepsilon\mathcal{Q}\mathcal{S}\mathcal{Q}w + \varepsilon\mathcal{Q}\mathcal{S}\mathcal{P}v + \mathcal{Q}\mathcal{C}\mathcal{Q}w, \end{aligned} \tag{91}$$

with the initial conditions

$$v(0) = \overset{o}{v},\ w(0) = \overset{o}{w},$$

where $\overset{o}{v} = \mathcal{P}\ \overset{o}{u}$, $\overset{o}{w} = \mathcal{Q}\ \overset{o}{u}$.

The projected operator $\mathcal{PSP}$ vanishes for the telegraph equation and for many other (but not all!) types of linear equations. Hence, for simplicity of presentation, we perform analysis for such a case. Thus, we obtain the following form of (91)

$$\begin{aligned} \partial_t v &= \mathcal{PSQ}w \\ \partial_t w &= \mathcal{QSP}v + \mathcal{QSQ}w + \frac{1}{\varepsilon}\mathcal{QCQ}w \qquad (92) \\ v(0) &= \overset{o}{v},\ w(0) = \overset{o}{w}\ . \qquad (93) \end{aligned}$$

As before, we represent the solution of (92) as a sum of the bulk and the initial layer parts:

$$\begin{aligned} v(t) &= \bar{v}(t) + \tilde{v}(\tau), \qquad (94) \\ w(t) &= \bar{w}(t) + \tilde{w}(\tau), \qquad (95) \end{aligned}$$

where, in this case, the variable τ in the initial layer part is given by $\tau = t/\varepsilon$. Other scalings may require different formulae for τ.

Let us recall the algorithm describing main features of the compressed asymptotic procedure:

Algorithm 1

1. The bulk approximation $\bar{v}$ is not expanded into powers of ε.
2. The bulk approximation $\bar{w}$ is explicitly written in terms of $\bar{v}$ and expanded in powers of ε.
3. The time derivative $\partial_t \bar{v}$ and the initial value $\bar{v}(0)$ are expanded into powers of ε.

Thus

$$\begin{aligned} \bar{w} &= \bar{w}_0 + \varepsilon \bar{w}_1 + O(\varepsilon^2), \\ \tilde{v} &= \tilde{v}_0 + \varepsilon \tilde{v}_1 + O(\varepsilon^2), \\ \tilde{w} &= \tilde{w}_0 + \varepsilon \tilde{w}_1 + O(\varepsilon^2). \end{aligned} \tag{96}$$

Substituting the expansion for $\bar{w}$ into (92) and comparing terms of the same powers of ε yields

$$\partial_t \bar{v} = \mathcal{PSQ}(\bar{w}_0 + \varepsilon \bar{w}_1 + O(\varepsilon^2)). \tag{97}$$

and

$$\bar{w}_0 \equiv 0,$$

$$\bar{w}_1 = -(\mathcal{QCQ})^{-1}\mathcal{QSP}\bar{v}.$$

Inserting the expressions for $\bar{w}_0$ and $\bar{w}_1$ into (97) gives the equation

$$\partial_t \bar{v} = -\varepsilon \mathcal{PSQ}(\mathcal{QCQ})^{-1}\mathcal{QSP}\bar{v} + O(\varepsilon^2), \tag{98}$$

which, as we shall see, is the approximate diffusion equation.

For the initial layer a similar procedure yields

$$\begin{aligned} \tilde{v}_0(\tau) &\equiv 0, \\ \partial_\tau \tilde{w}_0 &= \mathcal{QCQ}\tilde{w}_0, && (99) \\ \partial_\tau \tilde{v}_1 &= \mathcal{PSQ}\tilde{w}_0, && (100) \\ \partial_\tau \tilde{w}_1 &= \mathcal{QCQ}\tilde{w}_1 + \mathcal{QSP}\tilde{v}_0 + \mathcal{QSQ}\tilde{w}_0. && (101) \end{aligned}$$

We observe that, due to $\bar{w}_0 \equiv 0$, the initial condition for $\tilde{w}_0$ is $\tilde{w}_0(0) = \overset{o}{w}$. Solving (99) with this initial value allows to integrate (100) which gives

$$\tilde{v}_1(\tau) = \mathcal{PSQ}(\mathcal{QCQ})^{-1} e^{\tau \mathcal{QCQ}} \overset{o}{w}, \tag{102}$$

upon which $\tilde{v}_1(0) = \mathcal{PSQ}(\mathcal{QCQ})^{-1}\ \overset{o}{w}$. This in turn allows one to determine the initial condition for the diffusion equation: we have from (95) that $\overset{o}{v} = \bar{v}(0) + \varepsilon\tilde{v}_1(0) + O(\varepsilon^2)$ so that

$$\bar{v}(0) = \overset{o}{v} - \varepsilon\mathcal{PSQ}(\mathcal{QCQ})^{-1}\ \overset{o}{w}\ + O(\varepsilon^2). \tag{103}$$

In what follows we adopt a uniform notation valid for all discussed examples. In general, by ρ we shall denote a solution of the 'diffusion' equation determined by discarding the $O(\varepsilon^2)$ terms in (98), that is,

$$\partial_t\rho = -\varepsilon\mathcal{PSQ}(\mathcal{QCQ})^{-1}\mathcal{QSP}\rho. \tag{104}$$

Thus, ρ is expected to provide an approximation of $\bar{v}$. By $\hat{\rho}$ we denote the solution of this equation with uncorrected initial condition $\hat{\rho}(0) = \overset{o}{v}$ and by $\bar{\rho}$ the solution with the corrected initial value obtained by discarding the $O(\varepsilon^2)$ terms in (103), that is, $\bar{\rho}(0)$ is given by

$$\bar{\rho}(0) = \overset{o}{v}\ -\varepsilon\mathcal{PSQ}(\mathcal{QCQ})^{-1}\ \overset{o}{w}\ . \tag{105}$$

We shall see that for many cases the above formulae lead to the initial value problem for the diffusion equation.

6.0.1 Can we prove the convergence?

As before, we present a formal equation for the error to show that under reasonable regularity we get $O(\epsilon^2)$ approximation. The equation satisfied by the error, which is defined as

$$\begin{aligned} y(t) &= v(t) - [\bar{v}(t) + \epsilon\tilde{v}_1(t/\epsilon)], \\ z(t) &= w(t) - [\tilde{w}_0(t/\epsilon) + \epsilon(\bar{w}_1(t) + \tilde{w}_1(t/\epsilon))], \end{aligned} \tag{106}$$

is

$$\begin{aligned}\partial_t y &= \mathcal{PSP}y + \mathcal{PSQ}z + \epsilon\mathcal{PSP}\tilde{v}_1 + \epsilon\mathcal{PSQ}\tilde{w}_1,\\ \partial_t z &= \mathcal{QSP}y + \mathcal{QSQ}z + \frac{1}{\epsilon}\mathcal{QCQ}z + \epsilon\mathcal{QSQ}\tilde{w}_1\\ &\quad +\epsilon\mathcal{QSP}\tilde{v}_1 + \epsilon\mathcal{QSQ}\bar{w}_1 - \epsilon\partial_t\bar{w}_1, \end{aligned} \tag{107}$$

that is, denoting $E(t) = y(t) + z(t)$, the error system (107) can be written as

$$\partial_t E = \left(\mathcal{S} + \frac{1}{\epsilon}\mathcal{C}\right) E + \epsilon\bar{F} + \epsilon\tilde{F}$$

Denoting by $(G_\epsilon(t))_{t\geq 0}$ the contractive semigroup generated by $\mathcal{S} + \epsilon^{-1}\mathcal{C}$, we get

$$\|E(t)\| \leq \|E(0)\| + \epsilon\int_0^t \|\bar{F}(s)\|ds + \epsilon\int_0^t \|\tilde{F}(s)\|ds.$$

It is easy to see that $E(0) = O(\epsilon^2)$ and so this equation yields the error of approximation to be $O(\epsilon)$, which is not good as we have ϵ order terms in the approximation. A closer look at the term involving $\tilde{F}$ shows that it contains $e^{-t/\epsilon}$ which, upon integration, produces another ϵ so that the initial condition and the initial layer contribution to the error are $O(\epsilon^2)$. The fact that the contribution of $\bar{F}$ is also $O(\epsilon^2)$ is highly nontrivial but can be proved for a large class of problems.

It is important to note that the above considerations show that the presented asymptotic procedure **potentially** produces the convergence of the expected order. Since in most cases we work with unbounded operators, every step must be carefully justified.

6.1 Telegraph equation–final result

To avoid the effect of a boundary layer here, it is enough to assume that $\overset{o}{v}$ and $\overset{o}{w}$ are three times differentiable and

$$\partial_x \overset{o}{v}(\pm 1) = 0\ ,\ \overset{o}{w}(\pm 1) = 0\ ,\ \partial_{xx}\overset{o}{w}(\pm 1) = 0. \tag{108}$$

To get explicit expression for the operator $\mathcal{PSQ}(\mathcal{QCQ})^{-1}\mathcal{QSP}$ we evaluate

$$\mathcal{QSP}\begin{bmatrix} v \\ 0 \end{bmatrix} = \mathcal{Q}\begin{bmatrix} 0 & -b\partial_x \\ -c\partial_x & 0 \end{bmatrix}\begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -c\partial_x v \end{bmatrix},$$

$$\mathcal{PSQ}\begin{bmatrix} 0 \\ w \end{bmatrix} = \mathcal{P}\begin{bmatrix} 0 & -b\partial_x \\ -c\partial_x & 0 \end{bmatrix}\begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} -b\partial_x w \\ 0 \end{bmatrix},$$

The inverse $(\mathcal{QCQ})^{-1}$ is given by

$$(\mathcal{QCQ})^{-1}\begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ -w/d \end{bmatrix}.$$

Then

$$\mathcal{PSQ}(\mathcal{QCQ})^{-1}\mathcal{QSP}\begin{bmatrix} v \\ 0 \end{bmatrix} = \mathcal{PSQ}\begin{bmatrix} 0 \\ \frac{c}{d}\partial_x v \end{bmatrix} = \begin{bmatrix} -\frac{bc}{d}\partial_{xx}\, v \\ 0 \end{bmatrix}.$$

Hence the approximating diffusion equation, as given by (104), is

$$\partial_t \rho = \varepsilon \frac{bc}{d}\partial^2_{xx}\rho. \tag{109}$$

The uncorrected initial condition is $\rho(0) = \hat{\rho}(0) = \overset{o}{v}$, whereas the corrected one can be derived from (105) using $\mathcal{PSQ}$ and $(\mathcal{QCQ})^{-1}$ as calculated above, which gives

$$\bar{\rho}(0) = \overset{o}{v}\ -\varepsilon\mathcal{PSQ}(\mathcal{QCQ})^{-1}\ \overset{o}{w} = \overset{o}{v} - \varepsilon\frac{b}{d}\partial_x \overset{o}{w}\ . \tag{110}$$

The initial layer is derived from (102) and is given by

$$\tilde{v}_1(\tau) = \mathcal{PSQ}(\mathcal{QCQ})^{-1}e^{\tau\mathcal{QCQ}}\ \overset{o}{w} = \frac{b}{d}e^{-d\tau}\partial_x \overset{o}{w}, \tag{111}$$

where $\tau = t/\varepsilon$.

Let us denote

$$D_3 = \{u \in W_2^3([-1,1]);\ u|_{x=\pm 1} = 0,\ \partial^2_{xx}u|_{x=\pm 1} = 0\}$$

The following theorem is true.

Theorem 6 *If* $\overset{o}{v}, \overset{o}{w} \in D_3$ *and the compatibility conditions (108) are satisfied. Then there is a constant* C *such that*

$$\|v(t) - \rho(t) - \varepsilon\tilde{v}_1(t/\varepsilon)\| \leq C\varepsilon^2$$

uniformly on $[0, \infty)$

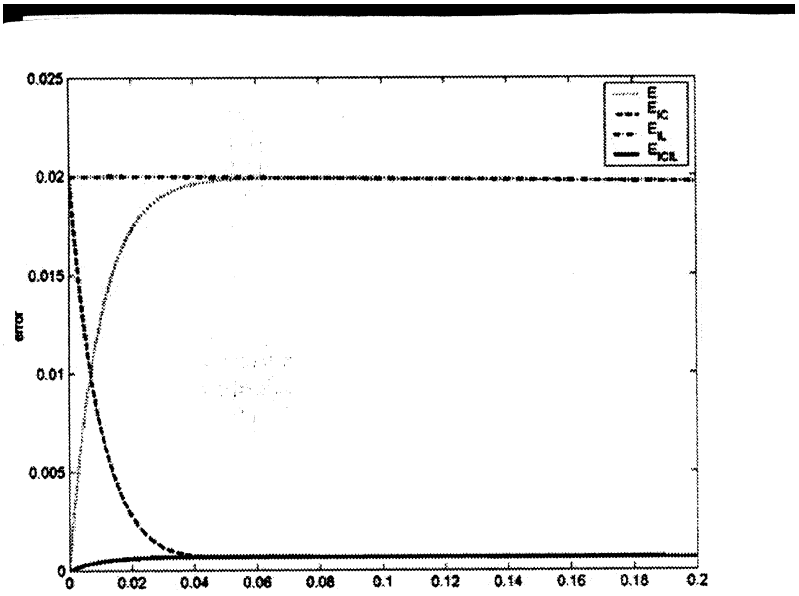

Fig 1. Error for various approximations for the telegraph equation

6.2 Higher approximation for the sole equation

Performing the second order analysis of the sole equation, we shall not get a diffusion equation but

$$\partial_t \bar{v} = \mathcal{P}(\mathcal{S} + \mathcal{M})\mathcal{P}\bar{v} - \varepsilon\mathcal{P}(\mathcal{S} + \mathcal{M})\mathcal{Q}(\mathcal{Q}\mathcal{C}\mathcal{Q})^{-1}\mathcal{Q}(\mathcal{H} + \mathcal{M})\mathcal{P}\bar{v}.$$

The explicit expressions for the involved operators can be calculated as

$$\begin{aligned}\mathcal{P}(\mathcal{S}+\mathcal{M})\mathcal{P}\bar{v} &= -(\partial_a p - p(\mathbf{1}\cdot\mathcal{M}\mathbf{e})\mathbf{e},\\ \mathcal{Q}(\mathcal{S}+\mathcal{M})\mathcal{P}\bar{v} &= -p(\mathbf{1}\cdot\mathcal{M}\mathbf{e}-\mathcal{M})\mathbf{e},\\ \mathcal{P}(\mathcal{S}+\mathcal{M})\mathcal{Q}\mathbf{x} &= -(\mathbf{1}\cdot\mathcal{M}\mathbf{x}-\mathbf{1}\cdot\mathcal{M}\mathbf{x})\mathbf{e},\end{aligned}$$

and, denoting by $\mathbf{h}$ the unique solution in $W=\mathcal{Q}X$ of

$$\mathcal{C}\mathbf{h} = -(\mathbf{1}\cdot\mathcal{M}\mathbf{e}-\mathcal{M})\mathbf{e}$$

we obtain

$$\mathcal{P}(\mathcal{S}+\mathcal{M})\mathcal{Q}(\mathcal{Q}\mathcal{C}\mathcal{Q})^{-1}\mathcal{Q}(\mathcal{S}+\mathcal{M}\mathcal{P}\bar{v} = p(\mathbf{1}\cdot\mathcal{M}\mathbf{h}).$$

Therefore

$$\partial_t p = -\partial_a p + p(\mathbf{1}\cdot\mathcal{M}\mathbf{e}+\epsilon\mathbf{1}\cdot\mathcal{M}\mathbf{h})$$

or, taking into account the form of $\mathcal{M}$, we obtain

$$\partial_t p = -\partial_a p - \mu^* p + \epsilon(\mathbf{1}\cdot\mathcal{M}\mathbf{h})p.$$

6.2.1 Fokker-Planck equation of Brownian motion

We conclude with a brief discussion of a more complicated example of the Fokker-Planck equation describing n-dimensional Brownian motion. The collision operator $\mathcal{C}$ now is given by the three-dimensional differential operator

$$(\mathcal{C}u)(x,\xi) = \partial_\xi(\xi+\partial_\xi)u(x,\xi), \tag{1}$$

$x,\xi\in\mathbb{R}^n$ and the streaming operator $\mathbf{S}$ is of the form

$$(\mathcal{S}u)(x,\xi) = \xi\partial_x u(x,\xi). \tag{2}$$

Here u is the particle distribution function in the phase space, x denotes the position and ξ the velocity of the particle.

The Fokker-Planck operator can be transformed to the well-known harmonic oscillator operator: for the function $u(\xi)$ we define $\xi = \sqrt{2}\zeta \in \mathbb{R}^n$ and

$$y(\zeta) = (\mathsf{A}_n u)(\zeta) := (\sqrt{2})^{\frac{n}{2}} e^{\frac{|\zeta|^2}{2}} u(\sqrt{2}\zeta). \tag{3}$$

This is an isometry of the space $L_2(\mathbb{R}^n, e^{\frac{|\xi|^2}{2}} d\xi)$ onto $L_2(\mathbb{R}^n, d\zeta)$ which transforms the Fokker-Planck collision operator $\mathcal{C}$ into

$$\tilde{C}y = \frac{1}{2(\sqrt{2})^{n/2}} e^{-\frac{|\zeta|^2}{2}} \left(\partial_\zeta^2 y - |\zeta|^2 y + ny\right). \tag{4}$$

Dropping the normalizing factor we arrive at the harmonic oscillator operator in $L_2(\mathbb{R}^n)$, denoted hereafter by H,

$$(Hy)(\zeta) = \partial_\zeta^2 y(\zeta) - |\zeta|^2 y(\zeta) + ny(\zeta). \tag{5}$$

To analyse this operator we introduce the sesquilinear form

$$h(\phi, \psi) = \int\limits_{\mathbb{R}^n} \left(\partial_\zeta \phi \partial_\zeta \bar{\psi} + |\zeta|^2 \phi \bar{\psi} + \phi \bar{\psi}\right) d\zeta, \tag{6}$$

defined originally on $C_0^\infty(\mathbb{R}^n)$ and the Hilbert space H_1 defined as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\phi\|_{H_1} = \sqrt{h(\phi, \phi)}$. Let A_h denote the operator associated with h It follows that the spectrum of A_h consists only of eigenvalues and the operator itself can be expressed in terms of the series of its eigenfunctions. Using the separation of variables and the one-dimensional theory of the harmonic oscillator we obtain the following expression for the eigenfunctions of A_h:

$$H_\alpha^{(n)}(\zeta) = \frac{(-1)^{|\alpha|}}{(2^{|\alpha|}\pi^{n/2}\alpha!)^{1/2}} e^{\frac{|\zeta|^2}{2}} \partial^\alpha e^{|\zeta|^2} = \prod_{i=1}^{n} H_{\alpha_i}^{(1)}(\zeta_i), \tag{7}$$

where $\zeta \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots \alpha_n)$ is a multi-index.

$H_m^{(1)}$ is the normalized one-dimensional Hermite function corresponding to the eigenvalue $\lambda_m = 2m + 1$

$$H_m(\zeta) := \frac{(-1)^m}{\sqrt{\sqrt{\pi} 2^m m!}} e^{\frac{\zeta^2}{2}} \partial_\zeta^m e^{-\zeta^2}. \tag{8}$$

Let C denote the Fokker-Planck collision operator obtained from A_h by the inverse transformation (3), and thus corresponding to the differential expression (1). For $k = 1, \ldots, n$ and the multi-index $\beta = (\beta_1, \ldots, \beta_k)$ we define

$$\Phi_\beta^{(k)} := \mathsf{A}_k^{-1} H_\beta^{(k)},$$

that is,

$$\Phi_\alpha^{(n)}(\xi) = \frac{(-1)^{|\alpha|}}{(2\pi)^{n/4}\sqrt{\alpha!}} \partial^\alpha e^{-\frac{|\xi|^2}{2}} = \prod_{i=1}^{n} \Phi_{\alpha_i}^{(1)}(\xi_i). \tag{9}$$

Since A_k is an isometric isomorphism, the family $\left\{\Phi_\alpha^{(n)}\right\}_{\alpha \in \mathbb{N}^n}$ forms an orthonormal basis in $L_2(\mathbb{R}^n, e^{\frac{|\xi|^2}{2}} d\xi)$.

We have therefore

$$u = \sum_{|\alpha|=0}^{\infty} u_\alpha \Phi_\alpha^{(n)} \tag{10}$$

and

$$\mathsf{C}u = -\sum_{|\alpha|=1}^{\infty} |\alpha| u_\alpha \Phi_\alpha^{(n)}, \tag{11}$$

so that it is clear that C is dissipative and satisfies all assumptions of the general theory.

To conclude we derive the form of the diffusion equation. To this end we express operator $\mathcal{S}$ in terms of eigenfunctions $\Phi_\alpha^{(n)}$. Let us adopt the following convention

$$\alpha(i, \pm 1) = (\alpha_1, \ldots, \alpha_i \pm 1, \ldots, \alpha_n).$$

The Hermite functions satisfy the following recurrence formula for $\Phi_\alpha^{(n)}$. Let $i = 1, \ldots, n$, then

$$\xi_i \Phi_\alpha^{(n)} = \sqrt{\alpha_i + 1} \Phi_{\alpha(i,+1)}^{(n)}(\xi) + \sqrt{\alpha_i} \Phi_{\alpha(i,-1)}^{(n)}(\xi). \tag{12}$$

If some $\alpha_i = 0$, then naturally the second summand vanishes. By Eq. (12) we obtain formally

$$\mathcal{S}u = -\sum_{k=1}^{n} \partial_k \left(\sum_{|\alpha|=0}^{\infty} \left(\sqrt{\alpha_k} u_{\alpha(k,-1)} + \sqrt{\alpha_k + 1} u_{\alpha(k,+1)} \right) \Phi_\alpha^{(n)} \right). \tag{13}$$

The hydrodynamic space is clearly spanned by $\Phi_0^{(n)}$. Hence we denote $\bar{v} = \bar{\rho}\Phi_0^{(n)}$ and $\tilde{v}_1 = \tilde{\rho}\Phi_0^{(n)}$. Introducing the notation

$$\mathbf{0}(i;l) = (0, \ldots, l, \ldots, 0)$$

and

$$\mathbf{0}(i,j;k,l) = (0, \ldots, k, \ldots, l, \ldots, 0),$$

where l (resp. (k,l)) are in the i-th (resp. i-th and j-th) place, we get

$$\mathcal{S}\bar{v} = -\sum_{k=1}^{n} \partial_k \Phi_{\mathbf{0}(k;1)}^{(n)} \bar{\rho}$$

and further

$$\begin{aligned} &\mathcal{S}\mathcal{Q}(\mathcal{Q}\mathcal{C}\mathcal{Q})^{-1}\mathcal{Q}\mathcal{S}\mathcal{P}\bar{v} \\ &= -\sum_{k=1}^{n} \partial_k \left(\sum_{l=1, l\neq k}^{n} \partial_l \Phi_{\mathbf{0}(k,l;1,1)}^{(n)} + \partial_k \left(\sqrt{2}\Phi_{\mathbf{0}(k,2)}^{(n)} + \Phi_0^{(n)} \right) \right) \bar{\rho}. \end{aligned}$$

Projecting this onto $\Phi_0^{(n)}$ we get the diffusion operator in the form

$$\mathcal{P}\mathcal{S}\mathcal{Q}(\mathcal{Q}\mathcal{C}\mathcal{Q})^{-1}\mathcal{Q}\mathcal{S}\mathcal{P}\bar{v} = -\Delta_x \bar{v}.$$

Similarly for the corrector of the initial value we obtain

$$\mathcal{P}\mathcal{S}\mathcal{Q}(\mathcal{Q}\mathcal{C}\mathcal{Q})^{-1} \overset{\circ}{w} = \Phi_0^{(n)} \sum_{k=1}^{n} \partial_k \overset{\circ}{u}_{\mathbf{0}(k;1)}$$

and the initial layer corrector $\tilde{v}_1$ will have the form

$$\tilde{v}_1\left(\frac{t}{\epsilon}\right) = e^{-\frac{t}{\epsilon}}\Phi_0^{(n)}\sum_{k=1}^{n}\partial_k \overset{\circ}{u}_{\mathbf{0}(k;1)},$$

where $\overset{\circ}{u}_{\mathbf{0}(k;1)}$ is the first moment of the initial value for u.

To formulate the final result of this section we introduce

$$\varrho(t,x) := \int_{\mathbb{R}} u(t,x,\xi)d\xi,$$

where u is the solution of the initial value problem for the Fokker-Planck equation of the Brownian motion.

Let $\overset{\circ}{u} \in W_1^3(\mathbb{R}^n, L_2(\mathbb{R}^n, e^{\frac{|\xi|^2}{2}}d\xi))$, then

$$\left\|\varrho(t) - \bar{\rho}(t) - \epsilon\tilde{\rho}\left(\frac{t}{\epsilon}\right)\right\|_{L_2(\mathbb{R}^n\times\mathbb{R}^n, e^{\frac{|\xi|^2}{2}}dxd\xi)} = O(\epsilon^2) \tag{14}$$

uniformly for t in bounded intervals of $[0,\infty[$. Here $\bar{\rho}$ is the solution of the following initial value problem

$$\begin{aligned}\partial_t\bar{\rho} &= \epsilon\partial_x^2\bar{\rho},\\ \bar{\rho}(0) &= \overset{\circ}{u}_0 - \epsilon\sum_{k=1}^{n}\partial_{x_k}\overset{\circ}{u}_{\mathbf{0}(k;1)},\end{aligned}$$

and the function $\tilde{\rho}$ in the initial layer corrector $\tilde{v}_1 = \tilde{\rho}\Phi_0$ is given by

$$\tilde{\rho}\left(\frac{t}{\epsilon}\right) = e^{-t/\epsilon}\sum_{k=1}^{n}\partial_{x_k}\overset{\circ}{u}_{\mathbf{0}(k;1)}.$$

7 Interplay of elastic and inelastic scattering operators in extended kinetic models and their hydrodynamic limits – reference manual

7.1 The Physical Model

We assume that the host medium is at thermodynamical equilibrium with temperature T and we consider the

$$\boxed{\text{Lorentz gas limit}}\ m/M << 1,$$

in other words, the test particles collide with something like a rigid net – they can be deflected (elastic collisions) or exchange quanta of energy with the background (inelastic collisions). Furthermore, the background particles can occur in two energy states:

- a ground level and
- an excited level,

spaced by an $\boxed{\text{energy gap } \Delta E}$ (which throughout the paper will be normalized to one). The number densities of the particles in the ground and in the excited state are assumed to be constant; we denote them by n_1 and n_2, respectively. They are related through

$$\boxed{\text{the Boltzmann factor } b := n_2/n_1 = e^{-\frac{\Delta E}{KT}},}$$

where K is the Boltzmann constant.

The elastic scattering operator is given by

$$(C^e f)(\mathbf{x}, \mathbf{v}) = -4\pi\lambda(v) f(\mathbf{x}, \mathbf{v}) + \lambda(v) \int_{S^2} f(\mathbf{x}, v\boldsymbol{\omega}') d\boldsymbol{\omega}'.$$

S^2 is the unit sphere in $\mathbb{R}^3_v$, $\mathbf{v} = v\boldsymbol{\omega}$ with $v \in [0, \infty[$, $\boldsymbol{\omega} \in S^2$.

The inelastic scattering operator is given by

$$\begin{aligned}(C^i f)(\mathbf{x}, \mathbf{v}) = -f(\mathbf{x}, \mathbf{v}) 4\pi \Big(& b\frac{v_+}{v}\nu(v_+) \\ & + H(v^2 - 1)\nu(v)\Big) \\ & + \frac{v_+}{v}\nu(v_+) \int_{S^2} f(\mathbf{x}, v_+\boldsymbol{\omega}') d\boldsymbol{\omega}' \\ & + b\nu(v) H(v^2 - 1) \int_{S^2} f(\mathbf{x}, v_-\boldsymbol{\omega}') d\boldsymbol{\omega}'\end{aligned}$$

where

- H is the Heaviside function,
- $v_\pm = \sqrt{v^2 \pm 1}$
- by Maxwell molecules assumption

$$0 < \lambda_{min} \leq \lambda(v) \leq \lambda_{max} < +\infty, v > 0.$$

$$0 < \nu_{min} \leq \nu(v) \leq \nu_{max} < +\infty \quad \text{for } v \in [1, \infty[.$$

Using the operators introduced above, we shall consider the kinetic equation in the form:

$$\frac{\partial f}{\partial t} + \frac{1}{Sh} Sf = \frac{1}{Kn_e} C^e f + \frac{1}{Kn_i} C^i f$$

where $Su = -\mathbf{v} \cdot \nabla_{\mathbf{x}} \mathbf{u}$.

The numbers $\boxed{Sh,\ Kn_e, \text{ and } Kn_i}$ measure the relative importance of the streaming, elastic collisions, and inelastic collisions in the balance equation for the test particle distribution function.

We shall further simplify our considerations by requiring that these three numbers are functions of a single parameter ϵ, which might represent the chosen smallness parameter.

We assume that the three numbers are power functions of ϵ.

The time evolution of the distribution function $\boxed{f = f(\mathbf{x}, \mathbf{v}, t)}$ of the test particles is governed by the linear Boltzmann in the adimensionalized form

$$\frac{\partial f}{\partial t} = \frac{1}{\epsilon^p} Sf + \frac{1}{\epsilon^q} C^e f + \frac{1}{\epsilon^r} C^i f,$$

where p, q, r are integers.

All the reasonable hydrodynamic limits for this equation, when $\epsilon \to 0$, will be discussed in the sequel.

The collision operators are defined, with reference to the kinetic variables only, by

$$\begin{aligned} C^e f &= -f(\xi, \boldsymbol{\omega}) \int_{S^2} \lambda(\xi, \boldsymbol{\omega} \cdot \boldsymbol{\omega}')\, d\boldsymbol{\omega}' \\ &+ \int_{S^2} \lambda(\xi, \boldsymbol{\omega} \cdot \boldsymbol{\omega}') f(\xi, \boldsymbol{\omega}')\, d\boldsymbol{\omega}' \end{aligned}$$

and

$$
\begin{aligned}
C^i f = -f(\xi,\boldsymbol{\omega}) \Bigg[& H(\xi-1)\int_{S^2} \nu(\xi,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')d\boldsymbol{\omega}' + \\
& + b\left(\frac{\xi+1}{\xi}\right)^{1/2}\int_{S^2}\nu(\xi+1,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')\Bigg] \\
& + \left(\frac{\xi+1}{\xi}\right)^{1/2}\int_{S^2}\nu(\xi+1,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')f(\xi+1,\boldsymbol{\omega}')\,d\boldsymbol{\omega}' \\
& + bH(\xi-1)\int_{S^2}\nu(\xi,\boldsymbol{\omega}\cdot\boldsymbol{\omega}')f(\xi-1,\boldsymbol{\omega}')\,d\boldsymbol{\omega}'.
\end{aligned}
$$

7.2 Formal Asymptotic Expansion

Let us consider the scaled equation

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon^p} S f_\epsilon + \frac{1}{\epsilon^q} C^e f_\epsilon + \frac{1}{\epsilon^r} C^i f_\epsilon.$$

We are looking for the diffusive/hydrodynamic limits of this equation.

There are two possible hydrodynamic spaces: $N(C^e)$ and $N(C) = N(C^i) = N(C^i + C^e)$. We can expect evolution

- $\boxed{\text{in } N(C^e)}$ if the elastic collisions are dominant, and
- $\boxed{\text{in } N(C)}$ if either inelastic collisions are dominant or both elastic and inelastic collisions are much stronger than the free-streaming.

Crucial point is to determine the hydrodynamic subspaces in all cases and the complementary subspaces.

Theorem 7 *Under the adopted assumptions*

a) The null-space of C^e is given by

$$N(C^e) = \{f \in L_1(\mathbb{R}^6_{x,v});\ f \text{ is independent on } \boldsymbol{\omega}\}. \tag{15}$$

and the spectral projection onto $N(C^e)$ is given by

$$(\mathbb{P}f)(\mathbf{x}, v) = \frac{1}{4\pi} \int\limits_{S^2} f(\mathbf{x}, v\boldsymbol{\omega}) d\boldsymbol{\omega}. \tag{16}$$

b) The null-spaces of C^i and $C = C^i + C^e$ coincide and are given by

$$\begin{aligned} N(C) \ &= \ N(C^i) = \{f \in L_1(\mathbb{R}^6_{x,v}, (1+v^{-1}) d\mathbf{v} d\mathbf{x}); \\ &\quad f \text{ is independent of } \boldsymbol{\omega} \text{ and satisfies} \\ &\quad f(\mathbf{x}, v_+) = bf(\mathbf{x}, v) \text{ for a.a. } \mathbf{x} \in \mathbb{R}^3_x, v \in \mathbb{R}_+\}. \end{aligned} \tag{17}$$

The spectral projections onto $N(C)$ and $N(C^i)$ coincide and are given by

$$(Pf)(\mathbf{x}, v^2 + n) = b^n \psi_0(\mathbf{x}, v^2), \quad \boldsymbol{x} \in \mathbb{R}^3_x, v \in [0, 1[, n \in \mathbb{N} \cup \{0\}, \tag{18}$$

where, for $v \in [0, 1[$,

$$\psi_0(\mathbf{x}, v^2) = \frac{\sum\limits_{j=0}^{\infty} \sqrt{v^2 + j} \int\limits_{S^2} f(\mathbf{x}, \sqrt{v^2 + j}\boldsymbol{\omega}) d\boldsymbol{\omega}}{4\pi \sum\limits_{j=0}^{\infty} b^j \sqrt{v^2 + j}}. \tag{19}$$

Thus

$$N(C^e) \supset N(C) = N(C^i),$$

and, in particular,

$$C^e P = 0.$$

To find the possible limiting equations we use the compressed Chapman-Enskog procedure.

Hence the idea we shall pursue is to separate the hydrodynamic part of the solution to Boltzmann equation by means of the appropriate spectral projection and then, by expanding the remaining part into a series of ϵ, to find and finally discard terms of higher order in ϵ, getting (at least formally) the limit equation satisfied by the hydrodynamic part.

Accordingly,

- in the first case we will be looking for the situations when the limit is the projection $\boxed{\text{onto } N(C^e)}$, and
- in the second when the limit is the projection $\boxed{\text{onto } N(C).}$

The asymptotic expansion is

$$\begin{aligned} f_\epsilon(t,\tau) &= \bar{f}(t) + \tilde{f}(\tau) \\ &= \rho(t) \\ &\quad +\bar{w}_0(t) + \epsilon\bar{w}_1(t) + \dots \\ &\quad +\tilde{\rho}_0(\tau) + \epsilon\tilde{\rho}_1(\tau) + \dots \\ &\quad +\tilde{w}_0(\tau) + \epsilon\tilde{w}_1(\tau) + \dots, \end{aligned}$$

where $\tau = t/\epsilon^2$, (or $\tau = t/\epsilon$). Terms

$$\rho, \tilde{\rho}_0, \tilde{\rho}_1 \dots \in N(C^e)$$

are the *hydrodynamic* part of the expansion, whereas

$$\bar{w}_0, \bar{w}_1, \dots, \tilde{w}_0, \tilde{w}_1, \dots \in N(C^e)^\perp$$

and are the *kinetic* part of the expansion.

• Terms depending on t are the *bulk* part of the expansion;

• terms depending on τ are the *initial layer*; they are to be determined independently of each other.

7.3 Evolution in $N(C^e)$: elastic collision dominance

To find possible limiting evolutions $\boxed{\text{in } N(C^e)}$ we shall use the projection $\mathbb{P}$ defined by

$$\mathbb{P}f = \frac{1}{4\pi}\int\limits_{S^2} f d\boldsymbol{\omega}.$$

Denoting $\mathbb{Q} = I - \mathbb{P}$, we operate with these projections onto our equation, and denoting

$$v_\epsilon = \mathbb{P}f_\epsilon \quad \text{and} \quad w_\epsilon = \mathbb{Q}f_\epsilon,$$

we obtain

$$\begin{aligned}
\partial_t v_\epsilon &= \frac{1}{\epsilon^p}\mathbb{P}S\mathbb{Q}w_\epsilon + \frac{1}{\epsilon^r}\mathbb{P}C^i\mathbb{P}v_\epsilon + \frac{1}{\epsilon^r}\mathbb{P}C^i\mathbb{Q}w_\epsilon \\
\partial_t w_\epsilon &= \frac{1}{\epsilon^p}\mathbb{Q}S\mathbb{P}v_\epsilon + \frac{1}{\epsilon^p}\mathbb{Q}S\mathbb{Q}w_\epsilon + \frac{1}{\epsilon^r}\mathbb{Q}C^i\mathbb{P}v_\epsilon \\
&+ \frac{1}{\epsilon^r}\mathbb{Q}C^i\mathbb{Q}w_\epsilon + \frac{1}{\epsilon^q}\mathbb{Q}C^e\mathbb{Q}w_\epsilon, \qquad (20)
\end{aligned}$$

Since we assumed that the elastic collisions are dominant, we must assume that $q > max\{p, r\}$. Since we are looking for the limiting equations, the equation for the approximation of v_ϵ cannot contain ϵ. This yields $r \leq 0$ and shows that p must be less or equal to the index k of the first nonzero term in the expansion of $w_\epsilon = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \ldots$. Consider first the case when

$$\boxed{\text{p=k}}.$$

Inserting the expansion into the second equation (20) we get

$$\begin{aligned}\epsilon^q\left(\partial_t w_0+\epsilon\partial_t w_1+\ldots\right) &= \epsilon^{q-p}\mathbb{Q}S\mathbb{P}v_\epsilon+\epsilon^{q-p}\mathbb{Q}S\mathbb{Q}(w_0+\epsilon w_1+\ldots)\\ &\quad+\epsilon^{q-r}\mathbb{Q}C^i\mathbb{P}v_\epsilon+\epsilon^{q-r}\mathbb{Q}C^i\mathbb{Q}(w_0+\epsilon w_1\ldots)\\ &\quad+\mathbb{Q}C^e\mathbb{Q}(w_0+\epsilon w_1+\ldots),\end{aligned}$$

Since $q>r$ and $q>p$, we obtain

$$\mathbb{Q}C^e\mathbb{Q}w_0=0$$

which yields $w_0=0$, because $\mathbb{Q}$ is the complementary spectral projection.

Clearly, the first nonzero term in the expansion of w will be w_k with k satisfying $k=\min\{q-p,q-r\}$. However, if $q-p\geq q-r$, then $r\geq p$, but $r\leq 0$ yielding $p\leq 0$ which contradicts the assumption that $p=k$. Thus $k=q-p$ and $q=2p$. In any case we obtain $w_k=-(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}v_\epsilon$ (provided the inverse exists). Changing now the notation from v_ϵ into ρ to emphasize the fact that the forthcoming equation is an approximating (limiting) equation for v_ϵ, we obtain the limiting equations, independent of ϵ, in the form

$$\frac{\partial\rho}{\partial t}=-\mathbb{P}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho+\mathbb{P}C^i\mathbb{P}\rho,\qquad\text{if}\quad r=0,\tag{21}$$

and

$$\frac{\partial\rho}{\partial t}=-\mathbb{P}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho,\qquad\text{if}\quad r<0.\tag{22}$$

A typical scaling for which (21) is the limiting equation is

$$\frac{\partial f_\epsilon}{\partial t}=\frac{1}{\epsilon}Sf_\epsilon+\frac{1}{\epsilon^2}C^ef_\epsilon+C^if_\epsilon\,,$$

and for (22):

$$\frac{\partial f_\epsilon}{\partial t}=\frac{1}{\epsilon}Sf_\epsilon+\frac{1}{\epsilon^2}C^ef_\epsilon+C^if_\epsilon\,.$$

Consider next

$$\boxed{p < k},$$

then the power of the coefficient multiplying $\mathbb{P}S\mathbb{Q}(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho$ is positive and therefore this term is negligible when ϵ tends to zero.

Then, the possible limiting equations are

$$\frac{\partial \rho}{\partial t} = \mathbb{P}C^i\mathbb{P}\rho, \qquad \text{if} \quad \boxed{\text{r=0}},$$

for the scaling

$$\frac{\partial f_\epsilon}{\partial t} = Sf_\epsilon + \frac{1}{\epsilon}C^e f_\epsilon + C^i f_\epsilon\,,$$

and

$$\frac{\partial \rho}{\partial t} = 0, \qquad \text{if} \quad \boxed{r < 0}\,.$$

for the scaling

$$\frac{\partial f_\epsilon}{\partial t} = Sf_\epsilon + \frac{1}{\epsilon}C^e f_\epsilon + \epsilon C^i f_\epsilon\,.$$

7.4 Evolution in $N(C)$: inelastic collision dominance

The cases when either C^i, or $C^i + C^e$ dominate have the same hydrodynamic subspace $N(C)$ and the same projectors onto it, ($N(C^e) \supset N(C^i)$):

$$(Pf)_n(\mathbf{x}, \xi) = b^{-n}G^{-1}(\xi)\sum_{j=0}^{\infty}\xi_j(\mathbb{P}f)_j(\mathbf{x}, \xi),$$

where $G(\xi) = \sum_{j=0}^{\infty} b^j\xi_j$.

Operating with P and $Q = I - P$ and denoting

$$v_\epsilon = Pf_\epsilon \quad \text{and} \quad w_\epsilon = Qf_\epsilon,$$

we obtain

$$\begin{aligned}\frac{\partial v_\epsilon}{\partial t} &= \frac{1}{\epsilon^p} PSQw_\epsilon \\ \frac{\partial w_\epsilon}{\partial t} &= \frac{1}{\epsilon^p} QSPv_\epsilon + \frac{1}{\epsilon^p} QSQw_\epsilon + \frac{1}{\epsilon^r} QC^iQw_\epsilon + \frac{1}{\epsilon^q} QC^eQw_\epsilon.\end{aligned}$$

As before, we observe that p must be less or equal to the index k of the first non-zero term w_k in the expansion of w_ϵ.

$$\text{If } \boxed{p < k},$$

then the equation for ρ (the approximation to v_ϵ) will be trivially reduced to

$$\frac{\partial \rho}{\partial t} = 0, \tag{23}$$

independently of what is happening in the second equation (though the initial layer corrector complementing the hydrodynamic equation changes slightly depending on whether C^e and C^i are of the same or different magnitude).

Thus typical cases will be

$$\frac{\partial f_\epsilon}{\partial t} = Sf_\epsilon + \frac{1}{\epsilon^q} C^e f_\epsilon + \frac{1}{\epsilon} C^i f_\epsilon \,.$$

with $\boxed{q \leq 1.}$

$$\text{If } \boxed{p = k > 0},$$

Here we have to distinguish two cases: $\boxed{r > q}$ and $\boxed{r = q}$ (in both cases we must have of course $\boxed{r > p}$).

Let us consider first the case $\boxed{r > q}$.

The limiting equation for the approximation ρ of v_ϵ is of the form

$$\frac{\partial \rho}{\partial t} = -PSQ(QC^iQ)^{-1}QSP\rho,$$

provided the inverse exists.

Thus typical cases will be

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon}Sf_\epsilon + \frac{1}{\epsilon^q}C^e f_\epsilon + \frac{1}{\epsilon^2}C^i f_\epsilon\,,$$

with $\boxed{q < 2.}$

The case $\boxed{r = q}$ is similar.

Due to the definition of Q we have as before $w_0 = 0$. Let w_k be the first non-zero term of the expansion of w. w_k is to be determined from

$$QSP\varrho = -Q(C^i + C^e)Qw_k$$

when $r - p = k$, that is $r = q = 2p$.

Consequently, the limiting equation is of the form

$$\frac{\partial \rho}{\partial t} = -PSQ(Q(C^i + C^e)Q)^{-1}QSP\rho,$$

provided the inverse exists.

Here a typical cases is

$$\frac{\partial f_\epsilon}{\partial t} = \frac{1}{\epsilon}Sf_\epsilon + \frac{1}{\epsilon^2}C^e f_\epsilon + \frac{1}{\epsilon^2}C^i f_\epsilon\,.$$

7.5 Reference manual

All the results are valid if the initial data are smooth with respect to $\mathbf{x}$ and decay sufficiently fast to zero as $v \to \infty$.

The only additional assumption is that the kinetic part of the initial datum annihilates constants over S^2 in the case of dominant inelastic or elastic and inelastic scattering.

We are dealing with the following Boltzmann equation

$$\begin{aligned}
\frac{\partial f_\epsilon}{\partial t} &= \frac{1}{\epsilon^p} S f_\epsilon + \frac{1}{\epsilon^q} C^e f_\epsilon + \frac{1}{\epsilon^r} C^i f_\epsilon \\
&= \frac{1}{\epsilon^p} v\boldsymbol{\omega} \cdot \frac{\partial f_\epsilon}{\partial \mathbf{x}} + \frac{1}{\epsilon^q} \left(-4\pi\lambda f + \lambda \int\limits_{S^2} f d\boldsymbol{\omega}' \right) \\
&\quad + \frac{1}{\epsilon^r} \left(-4\pi \left(H\nu + b\frac{v_+}{v}\nu_+ \right) f \right. \\
&\quad \left. + \frac{v_+}{v}\nu_+ \int\limits_{S^2} f_+ d\omega' + b\nu H \int\limits_{S^2} f_- d\omega' \right), \\
f_\epsilon(0) &= \overset{\circ}{f}
\end{aligned} \tag{24}$$

where $v_\pm = \sqrt{v^2 \pm 1}$, $g_\pm = g(v_\pm)$ and H is the Heaviside function. Functions λ and ν are functions of v variable only and $\nu_+ = \nu(v_+)$.

In all the cases the $O(\epsilon)$ approximation in $L_1(\mathbb{R}^6_{x,v})$, uniform on finite intervals $[0, t_0]$, is given by

$$f_\epsilon(t, \mathbf{x}, \mathbf{v}) = \rho(t, \mathbf{x}, \mathbf{v}) + \tilde{w}_0(t/\epsilon^k, \mathbf{x}, \mathbf{v}) + O(\epsilon),$$

where k is equal to the highest power of ϵ in (24).

7.5.1 Dominant elastic scattering: $q > \max\{p, r\}$

$$\mathbb{P}f = \frac{1}{4\pi}\int_{S^2} f d\boldsymbol{\omega}, \quad \mathbb{Q} = I - \mathbb{P}.$$

1. $p = \min\{q-p, q-r\}$ **($q = 2p$),** $r = 0$

The case $p = 1, q = 2, r = 0$

Hydrodynamic limit.

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \xi d \Delta \rho \\
&- \left(H(\xi - 1) m(\xi) + b\sqrt{\frac{\xi+1}{\xi}} m(\xi+1) \right) \rho \\
&+ \sqrt{\frac{\xi+1}{\xi}} m(\xi+1)\rho(\xi+1) \\
&+ bH(\xi-1)m(\xi)\rho(\xi-1), \\
\rho(0) &= \mathbb{P}\overset{\circ}{f},
\end{aligned}$$

where $m(\xi) = 4\pi\nu(\xi)$, $d = \frac{4\pi}{3\lambda(v)}$.

Initial layer corrector.

$$\tilde{w}_0(t/\epsilon^q, \mathbf{x}, \mathbf{v}) = e^{-\lambda(v)t/\epsilon^q}\mathbb{Q}\overset{\circ}{f}(\mathbf{x}, \mathbf{v}).$$

2. $p = \min\{q-p, q-r\}$ **($q = 2p$),** $r < 0$

The case $p = 1, q = 2, r = -1$

Hydrodynamic limit.

$$\begin{cases} \dfrac{\partial \rho}{\partial t} &= \xi d \Delta \rho, \\ \rho(0) &= \mathbb{P} \overset{\circ}{f}, \end{cases}$$

where $d = \frac{4\pi}{3\lambda(v)}$.

Initial layer corrector.

$$\tilde{w}_0(t/\epsilon^q, \mathbf{x}, \mathbf{v}) = e^{-\lambda(v)t/\epsilon^q} \mathbb{Q} \overset{\circ}{f} (\mathbf{v}, \mathbf{x}).$$

3. $p < \min\{q-p, q-r\}$**,** $r = 0$

The case $p = 0, q = 1, r = 0$

Hydrodynamic limit.

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\left(H(\xi - 1) m(\xi) + b \sqrt{\frac{\xi + 1}{\xi}} m(\xi + 1) \right) \rho \\ &+ \sqrt{\frac{\xi + 1}{\xi}} m(\xi + 1) \rho(\xi + 1) \\ &+ b H(\xi - 1) m(\xi) \rho(\xi - 1), \\ \rho(0) &= \mathbb{P} \overset{\circ}{f}, \end{aligned}$$

where $m(\xi) = 4\pi\nu(\xi)$.

Initial layer corrector.

$$\tilde{w}_0(t/\epsilon^q, \mathbf{x}, \mathbf{v}) = e^{-\lambda(v)t/\epsilon^q} \mathbb{Q} \overset{\circ}{f} (\mathbf{x}, \mathbf{v}).$$

4. $p < \min\{q-p, q-r\}$**,** $r < 0$

The case $p = 0, q = 1, r = -1$

Hydrodynamic limit.

$$\begin{cases} \dfrac{\partial \rho}{\partial t} &= 0, \\ \rho(0) &= \mathbb{P}\overset{\circ}{f}, \end{cases}$$

thus

$$\rho(t) = \mathbb{P}\overset{\circ}{f}\,.$$

Initial layer corrector.

$$\tilde{w}_0(t/\epsilon^q, \mathbf{x}, \mathbf{v}) = e^{-\lambda(v)t/\epsilon^q}\mathbb{Q}\overset{\circ}{f}\,(\mathbf{x}, \mathbf{v}).$$

7.5.2 Dominant inelastic scattering: $r > \max\{p, q-1\}$

We use the sequential notation for functions: $f = (f_n)_{n\geq 0}$, where $\boxed{f(\xi + n) = f_n(\xi)}$ for $\xi \in [0, 1[$ and $n = 0, 1, \ldots$.

The spectral projection on either $N(C^i)$ or $N(C^i + C^e)$ is given by

$$(Pf)_n(\mathbf{x}, \xi) = b^{-n}G^{-1}(\xi)\sum_{j=0}^{\infty}\xi_j(\mathbb{P}f)_j(\mathbf{x}, \xi),$$

where

$b = n_2/n_1 < 1, \quad \xi_j = \sqrt{\xi + j}$,

$G(\xi) = \sum_{j=0}^{\infty} b^j\xi_j$ and

$\mathbb{P}$ is the projection defined above.

1. $p = \min\{r - p, r - q\}$ **(**$r = 2p$**),** $r = q$

The case $p = 1, q = r = 2$

Hydrodynamic limit.

We have $\rho = (\rho_n)_{n\geq 0}$ where

$$\rho_n = b^n G^{-1} \varrho,$$

and ϱ is the solution to

$$\begin{cases} \dfrac{\partial \varrho}{\partial t} &= \dfrac{B(\xi)}{3G(\xi)} \Delta_x \varrho, \\ \varrho(0) &= P\mathring{f}, \end{cases}$$

where

$$\begin{aligned} B(\xi) &= \frac{\xi_0^4}{\lambda_0(\xi)\xi_0 + b\xi_1\nu_1(\xi)} \\ &\quad + \sum_{j=1}^{\infty} \frac{b^j \xi_j^4}{\lambda_j(\xi)\xi_j + b\xi_{j+1}\nu_{j+1}(\xi) + \xi_j \nu_j(\xi)}. \end{aligned}$$

Initial layer.

In all cases below we require that $\mathring{w}$ satisfy $\mathbb{P}\mathring{w} = 0$,and we use the standard notation

$$p_n(\xi) = \sqrt{\frac{\xi + n + 1}{\xi + n}} \nu(\xi + n + 1).$$

With these

$\tilde{w}_{0,n}(t/\epsilon^r, \mathbf{x}, \xi\boldsymbol{\omega}) =$

$$\begin{cases} e^{-(\lambda_0(\xi) + bp_0(\xi))t/\epsilon^r} \ \mathring{w}_0 \ (\mathbf{x}, \xi, \boldsymbol{\omega}), \\ e^{-(\lambda_n(\xi) + bp_n(\xi) + \nu_n(\xi))t/\epsilon^r} \ \mathring{w}_n \ (\mathbf{x}, \xi, \boldsymbol{\omega}) \\ \qquad \text{for } n \geq 1. \end{cases}$$

2. $p = \min\{r-p, r-q\}$ **(**$r = 2p$**),** $r > q$

The case $p = 1, q = 1, r = 2$

Hydrodynamic limit.

We have $\rho = (\rho_n)_{n\geq 0}$ where

$$\rho_n = b^n G^{-1} \varrho,$$

and ϱ is the solution to

$$\begin{cases} \dfrac{\partial \varrho}{\partial t} &= \dfrac{B(\xi)}{3G(\xi)} \Delta_x \varrho, \\ \varrho(0) &= \overset{\circ}{Pf}, \end{cases}$$

where

$$B(\xi) = \frac{\xi_0^4}{b\xi_1\nu_1} + \sum_{j=1}^{\infty} \frac{b^j \xi_j^4}{b\xi_{j+1}\nu_{j+1} + \xi_j \nu_j}.$$

Initial layer.

$\tilde{w}_{0,n}(t/\epsilon^r, \mathbf{x}, \xi\boldsymbol{\omega}) =$

$$\begin{cases} e^{-bp_0(\xi))t/\epsilon^r} \ \overset{\circ}{w}_0 \ (\mathbf{x}, \xi, \boldsymbol{\omega}), \\ e^{-(bp_n(\xi)+\nu_n(\xi))t/\epsilon^r} \ \overset{\circ}{w}_n \ (\mathbf{x}, \xi, \boldsymbol{\omega}) \\ \qquad \text{for } n \geq 1. \end{cases}$$

3. $p < \min\{r-p, r-q\}$**,** $q = r$

The case $p = 0, q = r = 1$

Hydrodynamic limit.

We have

$$\begin{cases} \dfrac{\partial \rho}{\partial t} &= 0, \\ \rho(0) &= \overset{\circ}{Pf}, \end{cases}$$

thus

$$\rho(t,x,\mathbf{v}) = P\overset{\circ}{f}\ .$$

Initial layer.

$$\tilde{w}_{0,n} = \begin{cases} e^{-(\lambda_0(\xi)+bp_0(\xi))t/\epsilon^r}\ \overset{\circ}{w}_0, \\ e^{-(\lambda_n(\xi)+bp_n(\xi)+\nu_n(\xi))t/\epsilon^r}\ \overset{\circ}{w}_n \quad \text{for} \quad n \geq 1. \end{cases}$$

4. $p < \min\{r-p, r-q\}$**,** $q < r$

The case $p = 0, q = 0, r = 1$

Hydrodynamic limit.

We have

$$\begin{cases} \dfrac{\partial \rho}{\partial t} &= 0, \\ \rho(0) &= P\overset{\circ}{f}, \end{cases}$$

thus

$$\rho(t,x,\mathbf{v}) = P\overset{\circ}{f}\ .$$

Initial layer.

$$\tilde{w}_{0,n} = \begin{cases} e^{-bp_0(\xi))t/\epsilon^r}\ \overset{\circ}{w}_0, \\ e^{-(bp_n(\xi)+\nu_n(\xi))t/\epsilon^r}\ \overset{\circ}{w}_n \quad \text{for} \quad n \geq 1. \end{cases}$$

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