Analysis in weighted spaces : preliminary version

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Chapter 1

Weighted L^2 analysis on a punctured ball

1.1 A simple model problem

Let B_R (resp. $\bar{B}_R$) denote the open (closed) ball of radius $R > 0$ in $\mathbb{R}^n$ and $B_R^* = B_R - \{0\}$ (resp. $\bar{B}_R^*$) denote the corresponding punctured ball.

Given $\nu \in \mathbb{R}$ and a function

$$f : B_1^* \subset \mathbb{R}^n \longrightarrow \mathbb{R}$$

satisfying

$$\| |x|^{-\nu} f \|_{L^\infty(B_1)} \leq 1$$

we would like to study the solvability of the equation

$$\left\{ \begin{array}{rcll} |x|^2 \, \Delta u & = & f & \text{in} \quad B_1^* \\ u & = & 0 & \text{on} \quad \partial B_1 \end{array} \right. \tag{1.1}$$

A solution of this equation is understood in the sense of distributions, namely u is a solution of (1.1) if $u \in L^1(B_1 - \bar{B}_R)$, for all $R \in (0,1)$ and if

$$\int_{B_1} u \, \Delta v \, dx = \int_{B_1} f \, v \, |x|^{-2} \, dx$$

for all $\mathcal{C}^\infty$ functions v with compact support in $\bar{B}_1^*$.

We claim that :

Proposition 1.1.1. *Assume that $n \geq 3$ and $\nu \in (2-n, 0)$. Then there exists a constant $c = c(n,\nu) > 0$ and for all $f \in L^\infty_{loc}(B_1^*)$ there exists u a solution of (1.1) which satisfies*

$$\| |x|^{-\nu} \, u \|_{L^\infty(B_1)} \leq 1$$

The proof of this result is a simple consequence of the maximum principle. First, recall the expression of the Euclidean Laplacian in polar coordinates

$$\Delta = \partial_r^2 + \frac{n-1}{r} \, \partial_r + \frac{1}{r^2} \, \Delta_{S^{n-1}}$$

Using this expression we get at once

$$|x|^2 \, \Delta |x|^\nu = -\nu \, (2-n-\nu) \, |x|^\nu$$

away from the origin. Now, if $\nu \in (2-n, 0)$ (this is where we use the fact that $n \geq 3$!), we observe that the constant

$$c_{n,\nu} := \gamma \, (2-n-\nu) > 0$$

The existence of a solution of (1.1) can then be obtained arguing as follows : Given $R \in (0, 1/2)$, we first solve the problem

$$\left\{ \begin{array}{rclcl} |x|^2 \, \Delta u_R & = & f & \text{in} & B_1 - \bar{B}_R \\ u_R & = & 0 & \text{on} & \partial B_1 \cup \partial B_R \end{array} \right. \tag{1.2}$$

Since $f \in L^\infty(B_1 - \bar{B}_R)$, the existence of a solution $u_R \in W^{2,p}(B_1 - \bar{B}_R)$ for any $p \in (1, \infty)$ follows from the following classical result :

Proposition 1.1.2 ([**?**], Theorem 9.15)**.** *Given $p \in (1,\infty)$ and Ω a smooth bounded domain of $\mathbb{R}^n$, if $g \in L^p(\Omega)$ then there exists a unique solution of*

$$\left\{ \begin{array}{rclcl} \Delta v & = & g & \textit{in} & \Omega \\ v & = & 0 & \textit{on} & \partial\Omega \end{array} \right.$$

which belongs to $W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$.

In our case $\Omega = B_1 - \bar{B}_R$ and

$$f \in L^\infty(B_1 - \bar{B}_R) \subset L^p(B_1 - \bar{B}_R),$$

for all $p \in (1, \infty)$, and hence

$$u_R \in W^{2,p}(B_1 - \bar{B}_R) \cap W^{1,p}_0(B_1 - \bar{B}_R).$$

One can use the Sobolev Imbedding Theorem to show that $u_R \in \mathcal{C}^{1,\alpha}(\bar{B}_1 - B_R)$ for all $\alpha \in (0,1)$.

Proposition 1.1.3 ([**?**], Theorem 7.26). *If $\alpha = 1 - \frac{n}{p}$, then*

$$W^{2,p}(\Omega) \subset \mathcal{C}^{1,\alpha}(\bar{\Omega})$$

provided Ω is a smooth bounded domain of $\mathbb{R}^n$.

The maximum principle also implies that

$$|u_R(x)| \leq \frac{1}{c_{n,\nu}} \, |x|^\nu \tag{1.3}$$

for all $x \in \bar{B}_1 - B_R$. Indeed, observe that the function

$$w(x) = \frac{1}{c_{n,\nu}} \, |x|^\nu - u_R(x)$$

is positive on $\partial B_1 \cup \partial B_R$. Moreover

$$\Delta w \leq 0$$

in $\bar{B}_1 - B_R$. Therefore one can apply the maximum principle

Proposition 1.1.4 ([**?**], Theorem 8.1). *Assume that $v \in W^{1,2}(\Omega)$ satisfies $\Delta v \leq 0$ in some smooth bounded domain $\Omega \subset \mathbb{R}^n$. Then*

$$\inf_{\Omega} v \geq \inf_{\partial\Omega} (\min(v, 0))$$

This result applies to the function w in $B_1 - \bar{B}_R$. We conclude that $w \geq 0$ and hence

$$u_R \leq \frac{1}{c_{n,\nu}} \, |x|^\nu .$$

Applying the same reasoning to $-u_R$ we obtain the desired inequality. Observe that, in the case where u_R is $\mathcal{C}^2$, one can simply invoke the classical maximum principle ([**?**], Theorem 3.1).

Now, we would like to pass to the limit, as R tends to 0. To this aim, we use the following estimates for solutions of (1.2)

Proposition 1.1.5 ([**?**], Theorem 9.13). *Given a smooth bounded domain $\Omega \subset \mathbb{R}^n$ whose boundary has two disjoint components T_1 and T_2, $\Omega' \subset\subset \Omega \cup T_1$ and $p \in (1, \infty)$. There exists a constant $c = c(n, p, \Omega, \Omega') > 0$ such that, if $g \in L^p(\Omega)$ and $v \in W^{2,p}(\Omega)$, satisfy*

$$\left\{ \begin{array}{rclcc} \Delta v & = & g & \text{in} & \Omega \\ v & = & 0 & \text{on} & T_1 \end{array} \right.$$

then

$$\|v\|_{W^{2,p}(\Omega')} \leq c \left(\|v\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} \right)$$

Using this result with $\Omega = B_1 - \bar{B}_R$, $T_1 = \partial B_1$ and $\Omega' = B_1 - \bar{B}_{2R}$, together with the *a priori* bound (1.3), we conclude that, for all $R \in (0, 1/2)$ there exists a constant $c = c(n, \nu, R) > 0$ such that

$$\|u_{R'}\|_{W^{2,p}(B_1 - \bar{B}_{2R})} \leq c$$

for all $R' \in (0, R)$. It is now enough to apply the Sobolev Imbedding Theorem

Proposition 1.1.6 ([**?**], Theorem 7.26)**.** *The imbedding*

$$W^{1,p}(\Omega) \longrightarrow \mathcal{C}^{0,\alpha}(\bar{\Omega})$$

is compact provided $0 < \alpha < 1 - \frac{n}{p}$ *and* Ω *is a smooth bounded domain.*

It is now easy to use these two results together with a standard diagonal argument to show that there exists a sequence $(R_i)_i$ tending to 0 such that the sequence of functions u_{R_i} converges to some continuous function u on compacts of $\bar{B}_1^*$. Obviously u will be a solution of (1.1) and, passing to the limit in (1.3), will satisfy

$$c_{n,\nu} \, \||x|^{-\nu} \, u\|_{L^\infty(B_1)} \leq 1 \tag{1.4}$$

We have thus obtained a solution of (1.1) satisfying (1.4), provided $\nu \in (2 - n, 0)$. This completes the proof of Proposition 1.1.1.

Exercise 1.1.1. *Given points* $x_1, \ldots, x_m \in \mathbb{R}^n$*, weights parameters* $\mu, \nu_1, \ldots, \nu_m \in \mathbb{R}$*, we define two positive smooth functions*

$$g : \mathbb{R}^n - \{x_1, \ldots, x_n\} \longrightarrow \mathbb{R} \qquad \text{and} \qquad h : \mathbb{R}^n - \{x_1, \ldots, x_n\} \longrightarrow \mathbb{R}$$

such that :

(i) For each $i = 1, \ldots, m$*,* $g(x) = |x - x_i|^{\nu_i}$ *and* $h(x) = |x - x_i|^{\nu_i - 2}$ *in a neighborhood of the point* x_i*.*

(ii) $g(x) = |x|^\mu$ *and* $h(x) = |x|^\mu$ *away from a compact subset of* $\mathbb{R}^n$*.*

Show that, provided $n \geq 3$ *and* $\mu, \nu_1, \ldots, \nu_m \in (2 - n, 0)$*, given a function*

$$f : \mathbb{R}^n - \{x_1, \ldots, x_n\} \longrightarrow \mathbb{R}$$

satisfying

$$|f| \leq h$$

it is possible to find a solution of the equation

$$\gamma^2 \, \Delta u = f$$

which satisfies

$$|u| \leq g$$

1.2 Analysis in weighted spaces in the punctured unit ball

Given $\delta \in \mathbb{R}$ we define the space

$$L^2_\delta(B_1^*) := |x|^{\delta+1}\, L^2(B_1)$$

This space is endowed with the norm

$$\|u\|_{L^2_\delta(B_1^*)} := \left(\int_{B_1} |u|^2\, |x|^{-2\delta-2}\, dx\right)^{1/2}$$

It is easy to check that

Lemma 1.2.1. *The space* $(L^2_\delta(B_1^*), \|\cdot\|_{L^2_\delta(B_1^*)})$ *is a Banach space.*

Exercise 1.2.1. *Provide a proof of Lemma 1.2.1.*

We define the unbounded operator A_δ by

$$\begin{array}{rccc} A_\delta : & L^2_\delta(B_1^*) & \longrightarrow & L^2_\delta(B_1^*) \\ & u & \longmapsto & |x|^2\, \Delta u \end{array}$$

The domain of this operator is the set of functions $u \in L^2_\delta(B_1^*)$ such that $A_\delta u = f \in L^2_\delta(B_1^*)$ in the sense of distributions : This means that $u \in W^{2,2}(B_1 - \bar{B}_R)$, for all $R \in (0, 1/2)$ and

$$\int_{B_1} u\, \Delta v\, dx = \int_{B_1} f\, v\, |x|^{-2}\, dx$$

for all $\mathcal{C}^\infty$ functions v with compact support in B_1^*.

We start with some properties of A_δ which are inherited from the corresponding classical properties for elliptic operators.

Proposition 1.2.1. *Assume that $\delta \in \mathbb{R}$ is fixed. There exists a constant $c = c(n, \delta) > 0$ such that for all $u, f \in L^2_\delta(B_1^*)$ satisfying $|x|^2\, \Delta u = f$ in B_1^* we have*

$$\|\nabla u\|_{L^2_{\delta-1}(B^*_{1/2})} + \|\nabla^2 u\|_{L^2_{\delta-2}(B^*_{1/2})} \leq c\,(\|f\|_{L^2_\delta(B_1^*)} + \|u\|_{L^2_\delta(B_1^*)})$$

The proof of this result follows from the :

Proposition 1.2.2 ([?], Theorem 9.11)**.** *Given a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $\Omega' \subset\subset \Omega$ and $p \in (1, \infty)$. There exists a constant $c = c(n, p, \Omega, \Omega') > 0$ such that, if $g \in L^p(\Omega)$ and $v \in W^{2,p}(\Omega)$, satisfy*

$$\Delta v = g \qquad in \qquad \Omega$$

then

$$\|v\|_{W^{2,p}(\Omega')} \leq c\, \left(\|v\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}\right)$$

The proof of the Proposition 1.2.1 goes as follows : Given $R \in (0, 1/2)$ we define the functions

$$v(x) := u(R\,x) \qquad \text{and} \qquad g(x) := f(R\,x)$$

Obviously, we have

$$|x|^2\,\Delta v = g$$

in $B_2 - \bar{B}_{1/2}$. We can then apply the result of Proposition 1.2.2 with $\Omega = B_2 - \bar{B}_{1/2}$ and $\Omega' = B_{3/2} - \bar{B}_1$, we conclude that

$$\|v\|^2_{W^{2,2}(B_{3/2}-\bar{B}_1)} \leq c\,\left(\|v\|^2_{L^2(B_2-\bar{B}_{1/2})} + \|g\|^2_{L^2(B_2-\bar{B}_{1/2})} \right)$$

Performing the change of variables backward, we conclude that

$$R^{2-n}\,\|\nabla u\|^2_{L^2(B_{3R/2}-\bar{B}_R)} + R^{4-n}\,\|\nabla^2 u\|^2_{L^2(B_{3R/2}-\bar{B}_R)} \leq$$
$$c\,\left(R^{-n}\,\|u\|^2_{L^2(B_{2R}-\bar{B}_{R/2})} + R^{-n}\,\|f\|^2_{L^2(B_{2R}-\bar{B}_{R/2})} \right)$$

It remains to multiply this inequality by $R^{n-2-2\delta}$, choose $R = \frac{1}{3}\,(\frac{2}{3})^i$, for $i \in \mathbb{N}$ and sum the result over i. We obtain

$$\|\nabla u\|^2_{L^2_{\delta-1}(B^*_{1/2})} + \|\nabla^2 u\|^2_{L^2_{\delta-2}(B^*_{1/2})} \leq c\,\left(\|u\|^2_{L^2_\delta(B^*_1)} + \|f\|^2_{L^2_\delta(B^*_1)} \right)$$

This completes the proof of the result.

1.3 The spectrum of the Laplacian on the unit sphere

We recall some well known facts about the spectrum of the Laplacian on the unit sphere.

Proposition 1.3.1 ([**?**], Theorem ??)**.** *The eigenvalues of $-\Delta_{S^{n-1}}$ are given by*

$$\lambda_j = j\,(n-2+j)$$

where $j \in \mathbb{N}$. The corresponding eigenspace will be denoted by E_j and the corresponding eigenfunctions are the restrictions to S^{n-1} of the homogeneous harmonic polynomials on $\mathbb{R}^n$.

One easy computation is the following : If P is a homogeneous harmonic polynomial of degree j, then $P(x) = |x|^j\,P(x/|x|)$ and hence

$$r\,\partial_r P = j\,P$$

Using the expression of the Laplacian in polar coordinates, we find that

$$r^2\,\Delta P = j\,(n-2+j)\,P + \Delta_{S^{n-1}} P$$

Since P is assumed to be harmonic, when restricted to the unit sphere this equality leads to

$$\Delta_{S^{n-1}} P = -j\,(n-2+j)\,P$$

This at least shows that the restrictions to S^{n-1} of the homogeneous harmonic polynomials of degree j on $\mathbb{R}^n$ belong to E_j.

Exercise 1.3.1. *What is the dimension of the j-th eigenspace E_j ?*

1.4 Indicial roots

We set

$$\delta_j := \frac{n-2}{2} + j$$

Definition 1.4.1. *The indicial roots of Δ at the origin are the real numbers given by*

$$\nu_j^{\pm} := \frac{2-n}{2} \pm \delta_j$$

for $j \in \mathbb{N}$.

The indicial roots are related to the asymptotic behavior of the solutions of the homogeneous problem $\Delta u = 0$ in $\mathbb{R}^n - \{0\}$. Indeed, a simple computation shows that

$$\Delta(|x|^{\nu_j^{\pm}}\,\phi) = 0$$

if $\phi \in E_j$.

1.5 A crucial *a priori* estimate

We now want to prove the key result which explains the importance of the parameters δ_j in the study of the operator $|x|^2\,\Delta$ when defined between weighted L^2-spaces. This is the purpose of the :

Proposition 1.5.1. *Assume that $\delta \neq \pm\delta_j$ for $j \in \mathbb{N}$. Then there exists a constant $c = c(n,\delta) > 0$ such that, for all $u, f \in L^2_\delta(B_1^*)$ satisfying*

$$|x|^2\,\Delta u = f$$

in B_1^*, we have

$$\|u\|_{L^2_\delta(B_1^*)} \leq c\,\left(\|f\|_{L^2_\delta(B_1^*)} + \|u\|_{L^2(B_1 - \bar{B}_{1/2})}\right)$$

Observe that this result states that we can control the weighted L^2-norm of u in terms of the weighted L^2-norm of f and some information about the function u away from the origin.

To prove the result let us perform the eigenfunction decomposition of both u and f. We write $x = r\,\theta$ where $r = |x|$ and $\theta = x/|x| \in S^{n-1}$ and we decompose

$$u(r,\theta) = \sum_{j\geq 0} u_j(r,\theta) \qquad \text{and} \qquad f(r,\theta) = \sum_{j} f_j(r,\theta)$$

where, for each $j \geq 0$, the functions $u_j(r,\,\cdot\,)$ and $f_j(r,\,\cdot\,)$ belong to E_j. In particular $\Delta_{S^{n-1}} u_j = -\lambda_j\, u_j$ and $\Delta_{S^{n-1}} f_j = -\lambda_j\, f_j$, wherever this makes sense.

Observe that

$$\int_{B_1} |u|^2\, |x|^{-2\delta-2}\, dx = \sum_{j\geq 0} \int_{B_1} |u_j|^2\, |x|^{-2\delta-2}\, dx = \sum_{j\geq 0} \int_0^1 \|u_j\|^2_{L^2(S^{n-1})}\, r^{n-3-2\delta}\, dr$$

and

$$\int_{B_1} |f|^2\, |x|^{-2\delta-2}\, dx = \sum_{j\geq 0} \int_{B_1} |f_j|^2\, |x|^{-2\delta-2}\, dx = \sum_{j\geq 0} \int_0^1 \|f_j\|^2_{L^2(S^{n-1})}\, r^{n-3-2\delta}\, dr$$

where $\|\cdot\|_{L^2(S^{n-1})}$ is the $L^2(S^{n-1})$ norm. In addition, the functions u_j and f_j satisfy

$$|x|^2\, \Delta u_j = f_j \tag{1.5}$$

in the sense of distributions in B_1^*. Indeed, making use of

$$\int_{B_1} u\, \Delta v\, dx = \int_{B_1} f\, v\, |x|^{-2}\, dx$$

with test functions of the form $v(r,\theta) = h(r)\,\phi(\theta)$ where $\phi \in E_j$ and h is a smooth function with compact support in $(0,1)$, we find that u_j is a E_j-valued function solution of (1.5). Using the decomposition of the Laplacian in polar coordinates, we also find that

$$r^2\, \partial_r^2 u_j + (n-1)\, r\, \partial_r u_j - \lambda_j\, u_j = f_j \tag{1.6}$$

in the sense of distribution. Moreover

$$\int_0^1 \|u_j\|^2_{L^2(S^{n-1})}\, r^{n-3-2\delta}\, dr < \infty \qquad \text{and} \qquad \int_0^1 \|f_j\|^2_{L^2(S^{n-1})}\, r^{n-3-2\delta}\, dr < \infty$$

The Sobolev Imbedding Theorem will help us justifying most of the forthcoming computation :

Proposition 1.5.2 ([?], Theorem 7.26)**.** *If $\alpha = 1 - \frac{n}{p}$ then*

$$W^{2,p}(\Omega) \subset \mathcal{C}^{1,\alpha}(\bar{\Omega})$$

provided Ω is a smooth bounded domain.

Observe that $u_j \in W^{2,2}((r,1))$ for all $r \in (0,1)$ and hence we find that $u_j \in \mathcal{C}^{1,1/2}_{loc}((0,1])$. Also, using the result of Proposition 1.2.1, we conclude that

$$\int_{B_1} |\partial_r u_j| \, |x|^{-2\delta} \, dx < \infty \qquad \text{and} \qquad \int_{B_1} |\partial_r^2 u_j| \, |x|^{2-2\delta} \, dx < \infty \tag{1.7}$$

Let j_0 denote the least index in $\mathbb{N}$ such that

$$|\delta| < \delta_{j_0} \tag{1.8}$$

The proof of Proposition 1.5.1 is now decomposed into two parts.

Part 1 : The case where $|\delta| < \delta_j$. Let χ be a cutoff function equal to 1 in $B_{1/2}$ and equal to 0 outside B_1, let us further assume that χ is radial. We multiply the equation (1.6) by $\chi^2 \, r^{-2\delta-2} \, u_j$ and integrate over B_1. We obtain using polar coordinates

$$\int_{B_1} \chi^2 \, r^{-2\delta} \, u_j \, \partial_r (r^{n-1} \, \partial_r u_j) \, dr \, d\theta - \lambda_j \int_{B_1} \chi^2 \, u_j^2 \, r^{n-3-2\delta} \, dr \, d\theta = \int_{B_1} \chi^2 \, u_j \, f_j \, r^{n-3-2\delta} \, dr \, d\theta$$

where $d\theta$ denotes the volume form on S^{n-1} and hence the Euclidean volume form is given by $dx = r^{n-1} \, dr \, d\theta$.

We integrate the first integral by parts to get

$$\begin{aligned} \int_{B_1} \chi^2 \, |\partial_r (\chi \, u_j)|^2 \, r^{-2\delta} \, dx + (\lambda_j + \delta \, (n-2-2\delta)) \int_{B_1} \chi^2 \, u_j^2 \, r^{-2-2\delta} \, dx \\ = \int_{B_1} (\delta \, \partial_r (\chi^2) - r \, |\partial_r \chi|^2) \, r^{-1-2\delta} \, u_j^2 \, dx - \int_{B_1} \chi^2 \, u_j \, f_j \, r^{-2-2\delta} \, dx \end{aligned} \tag{1.9}$$

Even is formally, this computation is correct, some care is needed to justify the integration by parts at 0. Let us explain how the integration by parts is performed : We write

$$\begin{aligned} -\chi^2 \, r^{-2\delta} \, u_j \, \partial_r (r^{n-1} \, \partial_r u_j) &= r^{n-1-2\delta} \, |\partial_r (\chi \, u_j)|^2 + \delta \, (n-2-2\delta) \, \chi^2 \, r^{n-3-2\delta} \, u_j^2 \\ &+ (\delta \, \partial_r (\chi^2) - r \, |\partial \chi|^2) \, r^{n-2-2\delta} \, u_j^2 \\ &- \partial_r \left(\chi^2 r^{n-1-2\delta} \, u_j \, \partial_r u_j + \delta \, \chi^2 \, r^{n-2-2\delta} \, u_j^2 \right) \end{aligned}$$

For all $R \in (0,1)$, we integrate this equality over $[R,1] \times S^{n-1}$ with respect to the measure $dr \, d\theta$ to get

$$\begin{aligned} -\int_{B_1 - \bar{B}_R} \chi^2 \, r^{-2\delta+1-n} \, u_j \, \partial_r (r^{n-1} \, \partial_r u_j) \, dx &= \int_{B_1 - \bar{B}_R} r^{-2\delta} \, |\partial_r (\chi \, u_j)|^2 \, dx \\ &+ \delta \, (n-2-2\delta) \int_{B_1 - \bar{B}_R} \chi^2 \, u_j^2 \, r^{-2-2\delta} \, dx \\ &+ \int_{B_1 - \bar{B}_R} (\delta \, \partial_r (\chi^2) - r \, |\partial \chi|^2) \, r^{-1-2\delta} \, u_j^2 \, dx \\ &+ \int_{\partial_{B_R}} \chi^2 \, r^{-2\delta} \, \left(r \, u_j \, \partial_r u_j + \delta \, r^{-1} \, u_j^2 \right) \, r^{n-1} \, d\theta \end{aligned} \tag{1.10}$$

Now, use the fact that, thanks to (1.7)

$$\int_0^1 \left(\int_{\partial B_r} r^{-2\delta} \, \left(|u_j \, \partial_r u_j| + r^{-1} \, u_j^2 \right) \, r^{n-1} \, d\theta \right) r^{-1} \, dr < \infty$$

to show that, for a sequence of R_i tending to 0 we have

$$\lim_{R_i \to 0} \int_{\partial_{B_{R_i}}} \chi^2 r^{-2\delta} \, \left(u_j \, \partial_r u_j + \delta \, r^{-1} \, u_j^2 \right) \, r^{n-1} \, d\theta = 0$$

We now use this sequence of radii and pass to the limit in (1.10) to get

$$\begin{aligned} - \int_{B_1} \chi^2 \, r^{-2\delta+1-n} \, u_j \, \partial_r (r^{n-1} \, \partial_r u_j) \, dx &= \int_{B_1} r^{-2\delta} \, |\partial_r (\chi \, u_j)|^2 \, dx \\ &+ \delta \, (n - 2 - 2\delta) \int_{B_1} \chi^2 \, u_j^2 \, r^{-2-2\delta} \, dx \\ &+ \int_{B_1} (\delta \, \partial_r (\chi^2) - r \, |\partial \chi|^2) \, r^{-1-2\delta} \, u_j^2 \, dx \end{aligned}$$

All subsequent integrations by parts can be justified using similar arguments, we shall leave the details to the reader.

We shall now make use of the following Hardy type inequality

Lemma 1.5.1. *The following inequality holds*

$$(n - 2 - 2\delta)^2 \int_{\mathbb{R}^n} r^{-2-2\delta} \, u^2 dx \leq \; 4 \int_{\mathbb{R}^n} r^{-2\delta} \, |\partial_r u|^2 \, dx$$

provided the integral on the left hand side is finite.

Using this Lemma together with (1.9) we conclude that

$$\left(\delta_j^2 - \delta^2 \right) \int_{B_1} \chi^2 \, u_j^2 \, |x|^{-2-2\delta} \, dx \leq \int_{B_1} \chi^2 \, |f_j| \, |u_j| \, |x|^{-2-2\delta} \, dx + c \int_{B_1 - B_{1/2}} u_j^2 \, dx$$

where the constant $c = c(n, \delta) > 0$ does not depend on j.

Now, we set

$$\eta := \delta_{j_0}^2 - \delta^2 > 0$$

This is where it is important that $\delta \neq \pm \delta_j$. Using Cauchy-Schwarz inequality together with the inequality

$$2 \, a \, b \leq \eta \, a^2 + \eta^{-1} \, b^2$$

we get

$$\left(2(\delta_j^2 - \delta^2) - \eta \right) \int_{B_1} \chi^2 \, u_j^2 \, |x|^{-2\delta-2} \, dx \leq \eta^{-1} \int_{B_1} \chi^2 \, f_j^2 \, |x|^{-2-2\delta} \, dx + 2 \, c \int_{B_1 - B_{1/2}} u_j^2 \, dx. \quad (1.11)$$

Observe that, for $j \geq j_0$,

$$2(\delta_j^2 - \delta^2) - \eta \geq 2(\delta_{j_0}^2 - \delta^2) - \eta = \eta.$$

We can sum all the inequalities (1.11) over $j \geq j_0$ to conclude that

$$\eta \int_{B_1} \chi^2 \, \tilde{u}^2 \, |x|^{-2\delta-2} \, dx \leq \eta^{-1} \int_{B_1} \chi^2 \, \tilde{f}^2 \, |x|^{-2-2\delta} \, dx + 2\, c \int_{B_1 - B_{\frac{1}{2}}} \tilde{u}^2 \, dx. \tag{1.12}$$

where we have set

$$\tilde{u} := \sum_{j \geq j_0} u_j \qquad \text{and} \qquad \tilde{f} := \sum_{j \geq j_0} f_j$$

Proof of Lemma 1.5.1: We now provide a proof of the Hardy type inequality we have used. Assume that $n - 2 \neq 2\,\delta$ and also that

$$\int_{\mathbb{R}^n} |\partial_r u|^2 \, |x|^{-2\delta} \, dx < \infty$$

since otherwise there is nothing to prove. Then, start with the identity

$$(n - 2 - 2\delta) \int_0^\infty v^2 \, r^{n-3-2\delta} \, dr = \int_0^\infty v^2 \, \partial_r (r^{n-2-2\delta}) dr = -2 \int_0^\infty v \, \partial_r v \, r^{n-2-2\delta} \, dr$$

where the last equality follows from an integration by parts. Use Cauchy-Schwarz inequality to conclude that

$$(n - 2 - 2\delta)^2 \int_0^\infty v^2 \, r^{n-3-2\delta} \, dr \leq 4 \int_0^\infty |\partial_r v|^2 \, r^{n-2-2\delta} \, dr$$

The inequality in Lemma 1.5.1 follows from the integration of this inequality over S^{n-1}. Observe that, in order to justify the integration by parts, it is enough to assume that $\int_0^\infty v^2 \, |x|^{-2\delta-2} \, dx$ converges.

Part 2 : The case where $|\delta| > \delta_j$ and $\delta_j \neq 0$. It remains to estimate u_j, for $j = 0, \ldots, j_0 - 1$. Here we simply use the fact that we have an explicit expression for u_j in terms of f_j. In order to simplify the discussion, we first assume that $\delta_j \neq 0$. Then, we define $\tilde{u}_j$ by

$$\tilde{u}_j(r, \cdot) = \frac{1}{2\delta_j} \left(r^{\frac{2-n}{2} + \delta_j} \int_*^r t^{\frac{n-4}{2} - \delta_j} \, f_j(t, \cdot) \, dt - r^{\frac{2-n}{2} - \delta_j} \int_*^r t^{\frac{n-4}{2} + \delta_j} \, f_j(t, \cdot) \, dt \right)$$

where $*$ has to be chosen according to the position of δ with respect to $\pm\delta_j$. In fact (see below) we will choose $* = 0$ when $\delta > \delta_j$ and $* = 1$ when $\delta < \delta_j$. It is easy to check that

$$|x|^2 \, \Delta \tilde{u}_j = \tilde{f}_j$$

in B_1^*.

Basic strategy : Consider a quantity of the form

$$u(r) = r^{\frac{2-n}{2} \pm \delta_j} \int_*^r t^{\frac{n-4}{2} \mp \delta_j} \, f(t) \, dt$$

where we assume that

$$\int_0^1 f^2(r) \, r^{n-3-2\delta} \, dr < \infty$$

It is a simple exercise to compute, using an integration by parts that

$$\int_R^1 u^2(r) \, r^{n-3-2\delta} \, dr \quad = \quad \frac{1}{2(\pm\delta_j - \delta)} \left(u(1)^2 - R^{n-2-2\delta} \, u(R)^2 - 2 \int_R^1 f(r) \, u(r) \, r^{n-3-2\delta} \, dr \right) \tag{1.13}$$

This is where, once again, it is important that $\delta \neq \pm\delta_j$.

A simple application of Cauchy-Schwarz inequality, yields

$$|u(r)|^2 \leq \frac{r^{2\delta+2-n}}{2|\delta \pm \delta_j|} \left(\int_0^1 f^2(t) \, t^{n-3-2\delta} \, dt \right),$$

provided we choose $* = 0$ when $\delta > \delta_j$ and $* = 1$ when $\delta < -\delta_j$. This is where the choice of $*$ is crucial.

Plugging this information in (1.13) and using Cauchy-Schwarz inequality, immediately implies that

$$\begin{aligned} \int_R^1 u^2 \, r^{n-3-2\delta} \, dr \quad &\leq \quad \frac{1}{2|\delta \pm \delta_j|^2} \left(\int_0^1 f^2(r) \, r^{n-3-2\delta} \, dr \right) \\ &+ \quad \frac{1}{|\delta \pm \delta_j|} \left(\int_0^1 u^2(r) \, r^{n-3-2\delta} \, dr \right)^{1/2} \left(\int_0^1 f^2(r) \, r^{n-3-2\delta} \, dr \right)^{1/2} \end{aligned}$$

It is a simple exercise to check that this implies that

$$\int_R^1 u^2 \, r^{n-3-2\delta} \, dr \quad \leq \quad c \int_0^1 f^2(r) \, r^{n-3-2\delta} \, dr$$

for some constant $c = c(\delta, n, j) > 0$.

Using this result, and passing to the limit as R tends to 0, we conclude that

$$\int_{B_1} \tilde{u}_j^2 \, |x|^{-2\delta-2} \, dx \quad \leq \quad c \int_{B_1} f_j^2 \, |x|^{-2\delta-2} \, dx \tag{1.14}$$

for some constant $c = c(\delta, n, j) > 0$.

It remains to evaluate the difference between the the functions u_j and $\tilde{u}_j$. Since

$$|x|^2 \, \Delta(u_j - \tilde{u}_j) = 0$$

we find that

$$u_j - \tilde{u}_j = r^{\frac{2-n}{2}+\delta_j}\,\phi + r^{\frac{2-n}{2}-\delta_j}\,\psi$$

where $\phi, \psi \in E_j$. Remembering that $u_j - \tilde{u}_j \in L^2_\delta(B_1^*)$ we find, in the case where $\delta > \delta_j$, that the only possibility is $\phi = \psi = 0$. Therefore, in this case the proof is already complete since (1.14) provides the desired estimate. When $\delta < -\delta_j$, it is very likely that ϕ and ψ are not equal to 0. In this case, we evaluate

$$\|\phi\|_{L^2(S^{n-1})} + \|\psi\|_{L^2(S^{n-1})} \leq c\,\|\tilde{u}_j - u_j\|_{L^2(B_1 - \bar{B}_{1/2})}$$

for some constant $c = c(\delta, j, n) > 0$. To obtain this estimate without much work observe that the space of functions

$$\{r^{\frac{2-n}{2}+\delta_j}\,\phi + r^{\frac{2-n}{2}-\delta_j}\,\psi \qquad : \qquad \phi, \psi \in E_j\}$$

is finite dimensional and that we have two (equivalent) norms on it. Namely

$$N_1(r^{\frac{2-n}{2}+\delta_j}\,\phi + r^{\frac{2-n}{2}-\delta_j}\,\psi) := \|\phi\|_{L^2(S^{n-1})} + \|\psi\|_{L^2(S^{n-1})}$$

and

$$N_2(r^{\frac{2-n}{2}+\delta_j}\,\phi + r^{\frac{2-n}{2}-\delta_j}\,\psi) := \|r^{\frac{2-n}{2}+\delta_j}\,\phi + r^{\frac{2-n}{2}-\delta_j}\,\psi\|_{L^2(B_1 - \bar{B}_{1/2})}.$$

Observe that we have implicitly used the fact that $\delta_j \neq 0$ and hence the functions $r \longrightarrow r^{\frac{2-n}{2}+\delta_j}$ and $r \longrightarrow r^{\frac{2-n}{2}-\delta_j}$ are linearly independent.

Granted this estimate, we conclude that

$$\|u_j\|_{L^2_\delta(B_1^*)} \leq c\,\left(\|f_j\|_{L^2_\delta(B_1^*)} + \|u_j\|_{L^2(B_1 - B_{1/2})}\right)$$

This completes the proof of the result when all $\delta_j \neq 0$. Collecting this estimates together with (1.12) this completes the proof of the Proposition 1.5.1 when $\delta_j \neq 0$, for all $j \in \mathbb{N}$.

Part 3 : The case where $|\delta| > \delta_j = 0$. We now turn to the case where $\delta_j = 0$. This case happens when $n = 2$ and $j = 0$. The equation satisfied by u_0 reads

$$r^2\,\partial_r^2\,u_0 + r\,\partial_r u_0 = f_0$$

This time, the explicit formula we will use is

$$\tilde{u}_0(r) := \int_*^r s^{-1}\left(\int_*^s t^{-1}\,f_0(t)\,dt\right)\,ds$$

where $*$ will be chosen appropriately, namely $* = 0$ when $\delta > 0$ and $* = 1$ when $\delta < 0$. Again, one can check directly that $|x|^2\,\Delta\tilde{u}_0 = f_0$.

To start with use the strategy developed above to prove that

$$\|\partial_r \tilde{u}_0\|_{L^2_{\delta-1}(B_1^*)} \leq c\,\|f_0\|_{L^2_\delta(B_1^*)}$$

We leave the details to the reader. Once this is done, use again the above strategy to show that

$$\begin{aligned}\int_R^1 \tilde{u}_0^2\, r^{-2\delta-1}\, dr \quad &\leq \quad \frac{1}{2|\delta|^2}\left(\int_0^1 f_0^2(r)\, r^{-2\delta-1}\, dr\right)\\ &+ \quad \frac{1}{|\delta|}\left(\int_0^1 \tilde{u}_0^2(r)\, r^{-2\delta-1}\, dr\right)^{1/2}\left(\int_0^1 |\partial\tilde{u}_0|^2(r)\, r^{-2\delta+1}\, dr\right)^{1/2}\end{aligned}$$

Collecting these two estimates, we conclude that

$$\|\tilde{u}_0\|^2_{L^2_\delta(B_1^*)} \leq c\left(\|f_0\|^2_{L^2_\delta(B_1^*)} + \|f_0\|_{L^2_\delta(B_1^*)}\|\tilde{u}_0\|_{L^2_\delta(B_1^*)}\right)$$

from which it follows that

$$\|\tilde{u}_0\|_{L^2_\delta(B_1^*)} \leq c\, \|f_0\|_{L^2_\delta(B_1^*)}$$

Once this estimate has been obtained, we observe that

$$u_0 - \tilde{u}_0 = \alpha + \beta \log r$$

When $\delta > 0$, $\alpha = \beta = 0$ since $u_0 - \tilde{u}_0 \in L^2_\delta(B_1^*)$ and when $\delta < 0$ we can argue as what has been already done when $\delta_j \neq 0$ to obtain

$$|\alpha| + |\beta| \leq c\, \|\tilde{u}_0 - u_0\|_{L^2(B_1 - \bar{B}_{1/2})}$$

for some constant $c = c(n, \delta) > 0$. Collecting all the estimate, we conclude that

$$\|u_0\|_{L^2_\delta(B_1^*)} \leq c\left(\|f_0\|_{L^2_\delta(B_1^*)} + \|u_0\|_{L^2(B_1 - B_{1/2})}\right) \tag{1.15}$$

This completes the proof in all cases.

Exercise 1.5.1. *Observe that there is another formula we could have used for $\tilde{u}_0$, namely*

$$\tilde{u}_0(r) = \log r \int_*^r t^{-1}\, f_0(t)\, dt - \int_*^r t^{-1}\, \log t\, f_0(t)\, dt.$$

Prove the estimate (1.15) starting from this formula.

Exercise 1.5.2. *Show that, in the main estimate in the statement of Proposition 1.5.1, one can replace $\|u\|_{L^2(B_1 - \bar{B}_{1/2})}$ by $\|u\|_{L^1(B_1 - \bar{B}_{1/2})}$.*

Exercise 1.5.3. *Let $a : B_1^* \longrightarrow \mathbb{R}$ be a function which satisfies the bound*

$$|a(x)| \leq\ c\, |x|^{-2+\alpha}$$

in B_1^*, for some $\alpha > 0$. Show that the result of Proposition 1.5.1 remains true if the operator $|x|^2\, \Delta$ is replaced by the operator $|x|^2\, (\Delta + a)$.

Exercise 1.5.4. † *Show that the result of Proposition 1.5.1 remains true if the operator* $|x|^2\,\Delta$ *is replaced by the operator* $|x|^2\,\Delta + d$*, where* $d \in \mathbb{R}$ *is fixed, provided we define*

$$\delta_j = \Re\left(\left(\frac{n-2}{2} + j\right)^2 + d\right)^{1/2}$$

Chapter 2

Weighted L^2 analysis on a punctured manifold

2.1 The Laplace-Beltrami operator in normal geodesic coordinates

Given a Riemannian manifold (M, g), the Laplace-Beltrami operator is defined in local coordinates $x^1, \ldots, x^n$

$$\Delta_g = \sum_{i,j} \frac{1}{\sqrt{\det g}} \, \partial_{x^i} \left(\sqrt{\det(g)} \, g^{ij} \, \partial_{x^j} \right)$$

where g^{ij} are the coefficients of the inverse of the matrix $(g_{ij})_{i,j}$.

Recall that, in local coordinates, the volume form on M is given by

$$dvol_g = \sqrt{\det g} \, dx^1 \ldots dx^n$$

In particular, if u is a smooth function of M, we have

$$\int_M u \, \Delta_g u \, dvol_g = - \int_M g^{ij} \partial_{x^i} u \, \partial_{x^j} u \, dvol_g = - \int_M g^{ij} |\nabla u|_g^2 \, dvol_g$$

Using the exponential mapping, we can define normal geodesic coordinates in a neighborhood of a point $p \in M$ as follows : first choose an orthonormal basis $e_1, \ldots, e_m$ of T_pM. Then define the mapping

$$F(x^1, \ldots, x^m) := \mathrm{Exp}_p \left(\sum_i x^i \, e_i \right)$$

One can prove that F is a local diffeomorphism from a neighborhood of 0 in (M, g) into a neighborhood of p in M.

Proposition 2.1.1. *([?], Theorem ??) In normal geodesic coordinates, the coefficients of the metric g can be expanded as*

$$g_{ij} = \delta_{ij} + \mathcal{O}(|x|^2)$$

The functions $\mathcal{O}(|x|^2)$ are smooth function which vanish quadratically at the origin. As a simple consequence of this result, we have the expansion of the Laplace-Beltrami operator in normal geodesic coordinates :

$$\Delta_g = \Delta_{eucl} + \mathcal{O}(|x|^2)\, \partial_{x^i}\, \partial_{x^j} + \mathcal{O}(|x|)\, \partial_{x^k} \tag{2.1}$$

The operator $\mathcal{O}(|x|^2)\, \partial_{x^i}\, \partial_{x^j}$ is a second order differential operator whose coefficients are smooth and vanish quadratically at the origin and the operator $\mathcal{O}(|x|^2)\, \partial_{x^k}$ is a first order differential operator whose coefficients are smooth and vanish at the origin. This last expansion follows from a direct computation using the formula of Δ_g in local coordinates and the result of Proposition 2.1.1.

2.2 Two global results

Using the normal geodesic coordinates, we extend the results of Proposition 1.1.1 and Proposition 1.5.1 in a global setting.

As in the previous section (M, g) is a compact n-dimensional Riemannian manifold without boundary. We choose points $p_1, \ldots, p_k \in M$ and denote by

$$M^* := M - \{p_1, \ldots, p_k\}$$

Given R small enough, we define $B_R(p) \subset M$ (resp. $\bar{B}_R(p) \subset M$) to be the open (resp. closed) geodesic ball of radius R centered at p. The corresponding punctured balls are denoted by $B_R^*(p)$ and $\bar{B}_R^*(p)$. Finally, we set

$$M_R := M - \cup_j \bar{B}_R(p_j)$$

We fix a smooth function

$$\gamma : M^* \longrightarrow (0, \infty)$$

such that, for all $j = 1, \ldots, k$

$$\gamma(p) = \mathrm{dist}(p, p_j)$$

in some neighborhood of p_j.

Given $\delta \in \mathbb{R}$ we define the space

$$L^2_\delta(M^*) := \gamma^{\delta+1}\, L^2(M)$$

This space is endowed with the norm

$$\|u\|_{L^2_\delta(M^*)} := \left(\int_M |u|^2\, \gamma^{-2\delta-2}\, dvol_g \right)^{1/2}$$

Again, we have

Lemma 2.2.1. *The space* $(L^2_\delta(M^*), \| \cdot \|_{L^2_\delta(M^*)})$ *is a Banach space.*

We define the unbounded operator A_δ by

$$\begin{array}{rccc} A_\delta : & L^2_\delta(M^*) & \longrightarrow & L^2_\delta(M^*) \\ & u & \longmapsto & \gamma^2\,(\Delta_g u + a\,u) \end{array}$$

where a is a smooth function on M. The domain $D(A_\delta)$ of this operator is the set of functions $u \in L^2_\delta(M^*)$ such that $A_\delta u = f \in L^2_\delta(M^*)$ in the sense of distributions : This means that $u \in W^{2,2}(M_R)$, for all $R > 0$ small enough and

$$\int_M u\,(\Delta_g v + a\,v)\,dvol_g = \int_M f\,v\,\gamma^{-2}\,dvol_g$$

for all $\mathcal{C}^\infty$ functions v with compact support in M^*. It is easy to check that

Lemma 2.2.2. *The domain of the operator* A_δ *is dense in* $L^2_\delta(M^*)$ *and the graph of* A_δ *is closed.*

Exercise 2.2.1. *Give a proof of Lemma 2.2.2.*

The result we have obtain in Proposition 1.2.1 translates immediately into :

Proposition 2.2.1. *Assume* $\delta \in \mathbb{R}$ *is fixed. There exists a constant* $c = c(n, \delta) > 0$ *such that for all* $u, f \in L^2_\delta(M^*)$ *satisfying* $\gamma^2\,(\Delta_g u + a\,u) = f$ *in* M^* *we have*

$$\|\nabla u\|_{L^2_{\delta-1}(M^*)} + \|\nabla^2 u\|_{L^2_{\delta-2}(M^*)} \leq c\,(\|f\|_{L^2_\delta(M^*)} + \|u\|_{L^2_\delta(M^*)})$$

The proof of the result goes as follows : First observe that the result of Proposition 1.1.1 remains true if one changes B_1^* with B_R^*. In which case the estimate of Proposition 1.1.1 has to be replaced by

$$\|\nabla u\|_{L^2_{\delta-1}(B_R^*)} + \|\nabla^2 u\|_{L^2_{\delta-2}(B_R^*)} \leq c\,\left(\|f\|_{L^2_\delta(B_R^*)} + \|u\|_{L^2_\delta(B_R^*)}\right) \tag{2.2}$$

if $u, f \in L^2_\delta(B_R^*)$ satisfy $|x|^2\,\Delta u = f$ in B_R^*. This can be seen easily by performing a simple change $v(x) = u(R\,x)$ and $g(x) = f(R\,x)$ so that v and g satisfy $|x|^2\,\Delta\,v = g$ in B_1^*, then the estimate follows from the corresponding estimate in Proposition 1.1.1.

Close to the puncture p_j we use normal geodesic coordinates so that $\gamma = |x|$ and write the equation $\gamma^2\,(\Delta_g\,u + a\,u) = f$ as

$$|x|^2\,\Delta_{eucl} u = f + |x|^2\,(\Delta_{eucl} - \Delta_g)\,u - |x|^2\,a\,u$$

Using the result of Proposition 1.1.1 together with the result of Proposition 2.1.1, we evaluate

$$\||x|^2 (\Delta_{eucl} - \Delta_g) u - |x|^2 a u\|_{L^2_\delta(B^*_R)} \leq c R^2 \left(\|u\|_{L^2_\delta(M^*)} + \|\nabla u\|_{L^2_{\delta-1}(M^*)} + \|\nabla^2 u\|_{L^2_{\delta-2}(M^*)} \right)$$

for some constant $c = c(n, \delta) > 0$ which does not depend on $R > 0$, small enough. Next we apply (2.2) to conclude that

$$\begin{aligned} \|\nabla u\|_{L^2_{\delta-1}(B^*_R)} + \|\nabla^2 u\|_{L^2_{\delta-2}(B^*_R)} \quad \leq \quad & c \Big(\|f\|_{L^2_\delta(B^*_R)} + \|u\|_{L^2_\delta(M^*)} \\ + \quad & R^2 \left(\|\nabla u\|_{L^2_{\delta-1}(M^*)} + \|\nabla^2 u\|_{L^2_{\delta-2}(M^*)} \right) \Big) \end{aligned}$$

for some constant $c = c(n, \delta) > 0$ independent of $R > 0$ small enough. This can also be written as

$$(1 - c R^2) \left(\|\nabla u\|_{L^2_{\delta-1}(B^*_R)} + \|\nabla^2 u\|_{L^2_{\delta-2}(B^*_R)} \right) \leq c \left(\|f\|_{L^2_\delta(B^*_R)} + \|u\|_{L^2_\delta(M^*)} \right)$$

If $R > 0$ is chosen so that $c R^2 \leq 1/2$ we conclude that

$$\|\nabla u\|_{L^2_{\delta-1}(B^*_R)} + \|\nabla^2 u\|_{L^2_{\delta-2}(B^*_R)} \leq 2 c \left(\|f\|_{L^2_\delta(B^*_R)} + \|u\|_{L^2_\delta(M^*)} \right)$$

We now use the elliptic estimates provided by

Proposition 2.2.2. *([?], Theorem ??) Assume we are given $\Omega \subset M$, $\Omega' \subset\subset \Omega$ and $p \in (1, \infty)$. Then there exists $c = c(M, g, \Omega, \Omega') > 0$ such that, if $v \in W^{2,p}$ and $g \in L^2(\Omega)$ satisfy $\Delta_g v = g$ in Ω, then*

$$\|\nabla v\|_{L^p(\Omega')} + \|\nabla^2 v\|_{L^p(\Omega')} \leq c \left(\|g\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right)$$

with $\Omega = M_{R/2}$ and $\Omega' = M_R$ to show that

$$\|\nabla u\|_{L^2_{\delta-1}(M_R)} + \|\nabla^2 u\|_{L^2_{\delta-2}(M_R)} \leq c \left(\|f\|_{L^2_\delta(M_{R/2})} + \|u\|_{L^2_\delta(M_{R/2})} \right)$$

for some constant $c = c(n, R) > 0$. The estimate then follows from the sum of the two estimates we have obtained.

The following result is a consequence of Proposition 1.5.1.

Proposition 2.2.3. *Assume that $\delta \neq \pm\delta_j$ for $j \in \mathbb{N}$. Then there exists a constant $c = c(n, \delta)$ and a compact K in M^* such that, for all $u, f \in L^2_\delta(M^*)$ satisfying*

$$\gamma^2 (\Delta_g u + a u) = f$$

in M^*, we have

$$\|u\|_{L^2_\delta(M^*)} \leq c \left(\|f\|_{L^2_\delta(M^*)} + \|u\|_{L^2(K)} \right)$$

Again this result states that we can control the weighted L^2- norm of u in terms of the weighted L^2-norm of f and some information about the function u away from the punctures.

The proof of this second Proposition, also follows from a perturbation argument. First observe that the result of Proposition 1.5.1 remains true if one changes B_1^* with B_R^*. In which case the estimate of Proposition 1.5.1 has to be replaced by

$$\|u\|_{L^2_\delta(B_R^*)} \leq c\left(\|f\|_{L^2_\delta(B_R^*)} + R^{-\delta-1}\,\|u\|_{L^2(B_R-\bar{B}_{R/2})}\right) \tag{2.3}$$

if $u, f \in L^2_\delta(B_R^*)$ satisfy $|x|^2\,\Delta u = f$ in B_R^*. This can be seen easily by performing a simple change $v(x) = u(R\,x)$ and $g(x) = f(R\,x)$ so that v and g satisfy $|x|^2\,\Delta\,v = g$ in B_1^*, then the estimate follows from the corresponding estimate in Proposition 1.5.1.

Close to the puncture p_j we use normal geodesic coordinates so that $\gamma = |x|$ and write the equation $\gamma^2\,(\Delta_g\,u + a\,u) = f$ as

$$|x|^2\,\Delta_{eucl} u = f + |x|^2\,(\Delta_{eucl} - \Delta_g)\,u - |x|^2\,a\,u$$

Using the result of Proposition 2.2.1 together with the result of Proposition 2.1.1, we evaluate

$$\||x|^2\,(\Delta_{eucl} - \Delta_g)\,u - |x|^2\,a\,u\|_{L^2_\delta(B_R^*)} \leq c\,R^2\,\|u\|_{L^2_\delta(M^*)}$$

for some constant $c = c(n,\delta) > 0$ which does not depend on $R > 0$, small enough. Next we apply (2.3) to conclude that

$$\|u\|_{L^2_\delta(B_R^*)} \leq c\left(\|f\|_{L^2_\delta(B_R^*)} + R^2\,\|u\|_{L^2_\delta(M^*)} + R^{-\delta-1}\,\|u\|_{L^2(B_R-\bar{B}_{R/2})}\right)$$

for some constant $c = c(n,\delta) > 0$ independent of $R > 0$ small enough. Adding on both sides $\|u\|_{L^2(M_R)}$ we conclude that

$$\|u\|_{L^2_\delta(M^*)} \leq c\left(\|f\|_{L^2_\delta(B_R^*)} + R^2\,\|u\|_{L^2_\delta(M^*)} + R^{-\delta-1}\,\|u\|_{L^2(M_{R/2})}\right)$$

where $c = c(n,\delta) > 0$ does not depend on $R > 0$ small enough. In other words

$$(1 - c\,R^2)\,\|u\|_{L^2_\delta(M^*)} \leq c\left(\|f\|_{L^2_\delta(B_R^*)} + R^{-\delta-1}\,\|u\|_{L^2(M_{R/2})}\right)$$

It remains to fix $R > 0$ such that $c\,R^2 \leq 1/2$ and let $K = M_{R/2}$. this completes the proof of the result.

Exercise 2.2.2. *Show that the result of Proposition 2.2.3 remains true if the function $a : M^* \longrightarrow \mathbb{R}$ only belongs to $L^\infty_{loc}(M^*)$ and satisfies the bound*

$$|a| \leq\; c\,\gamma^{-2+\alpha}$$

in M^*, for some $\alpha > 0$.

Exercise 2.2.3. † *Show that the result of Proposition 2.2.3 remains true if, near any of the p_i, the function a can be decomposed as $a = d\,\gamma^{-2} + \tilde{a}_i$ where $d \in \mathbb{R}$ is a constant and the function $\tilde{a}_i$ satisfies the bound*

$$|\tilde{a}_i| \leq \, c\,\gamma^{-2+\alpha_i}$$

in $B_{R_i}(p_i)$, for some $\alpha_i > 0$ and provided we define

$$\delta_j = \Re\left(\left(\frac{n-2}{2} + j\right)^2 + c\right)^{1/2}$$

Exercise 2.2.4. † *Show that the result of Proposition 2.2.3 remains true if, near any of the p_i there exists local coordinates $x^1, \dots, x^n$ in which the coefficients of the metric can be expanded as*

$$g_{ij} = \delta_{ij} + \mathcal{O}(|x|^\beta)$$

and if in addition

$$\nabla g_{ij} = \mathcal{O}(|x|^{\beta-1})$$

for some $\beta > 0$.

Exercise 2.2.5. ‡ *Extend the result of Proposition 2.2.3 to handle the case where, near any of the p_i, the function a can be decomposed as*

$$a = d_i\,\gamma^{-2} + \tilde{a}_i$$

where $d_i \in \mathbb{R}$ are constants and the function $\tilde{a}_i$ satisfies the bound

$$|\tilde{a}_i| \leq \, c\,\gamma^{-2+\alpha_i}$$

in $B_{R_i}(p_i)$, for some $\alpha_i > 0$.

2.3 The kernel of the operator A_δ

The results of the previous sections will now be used to derive the functional analytic properties of the operator A_δ. We start with the :

Theorem 2.3.1. *The kernel of A_δ is finite dimensional.*

For the time being, let us assume that $\delta \neq \pm\delta_j$. We argue by contradiction and assume that the result is not true. Then, there would exist a sequence $(u^m)_m$ of elements of $L^2_\delta(M^*)$ which satisfy $A_\delta\, u^m = 0$.

Without loss of generality we can assume that the sequence is normalized so that

$$\int_M |u^m|^2\,\gamma^{-2\delta-2}\,dvol_g = 1 \tag{2.4}$$

and also that

$$\int_M u^m \, u^{m'} \, \gamma^{-2\delta-2} \, dvol_g = 0. \tag{2.5}$$

for all $m \neq m'$. Using the result of Proposition 2.2.3 we obtain

$$\|u^m - u^{m'}\|_{L^2_\delta(M^*)} \leq c \, \|u^m - u^{m'}\|_{L^2(K)} \tag{2.6}$$

where $c = c(n, \delta) > 0$ does not depend on m.

Using (2.4) together with the result of Proposition 2.2.1 we conclude that u_m is bounded in $W^{1,2}(K)$. Now, we apply Rellich's compactness result :

Proposition 2.3.1. *([**?**], Theorem ??) Given a smooth bounded domain $\Omega \subset M$, the imbedding*

$$W^{1,2}(\Omega) \longrightarrow L^2(\Omega)$$

is compact.

This result allows us to extract some subsequence (which we will still denote by $(u^m)_m$) which converges in $L^2(K)$. In particular, the sequence $(u_m)_m$ is a Cauchy sequence in $L^2(K)$. In view of (2.6) we see that the sequence $(u_m)_m$ is a Cauchy sequence in $L^2_\delta(M^*)$. This space being a Banach space, we conclude that this sequence converges in $L^2_\delta(M^*)$ to some function u.

Clearly, passing to the limit in (2.4) we see that

$$\int_\Omega |u|^2 \, \gamma^{-2\delta-2} \, dvol_g = 1$$

While, passing to the limit $m' \longrightarrow \infty$ in (2.5), we get

$$\int_\Omega u^m \, u \, r^{-2\delta-2} \, dx = 0$$

and then passing to the limit as m tends to ∞, we conclude that

$$\int_\Omega u^2 \, r^{-2\delta-2} \, dx = 0$$

Clearly a contradiction. This completes the proof when $\delta \neq \pm\delta_j$, for all $j \in \mathbb{N}$. In order to complete the proof is all cases it is enough to observe that if $u \in \mathrm{Ker} A_\delta$ then $u \in \mathrm{Ker} A_{\delta'}$ for all $\delta' \leq \delta$. Therefore, one can always reduce to the case where $\delta' \neq \pm\delta_j$ for all $j \in \mathbb{N}$.

2.4 The range of the operator A_δ

We pursue our quest of the mapping properties of the operators A_δ by studying the range of this operator. Thanks to the results of the previous sections, we are in a position to prove the :

Theorem 2.4.1. *Assume that $\delta \neq \pm\delta_j$ for $j \in \mathbb{N}$. Then the range of A_δ is closed.*

Let $u^m, f^m \in L^2_\delta(M^*)$ be sequences such that $f^m := A_\delta\, u^m$ converges to f in $L^2_\delta(M^*)$. Since we now know that $\operatorname{Ker} A_\delta$ is finite dimensional, it is closed and we can project every each u^m onto

$$\left\{ u \in L^2_\delta(M^*) \quad : \quad \int_M u\, v\, r^{-2\delta-2}\, dx = 0 \quad \forall v \in \operatorname{Ker} A_\delta \right\}$$

the orthogonal complement of $\operatorname{Ker} A_\delta$ in $L^2_\delta(M^*)$, with respect to the scalar product associated to the weighted norm. Therefore, without loss of generality, we can assume that u^m is L^2_δ-orthogonal to $\operatorname{Ker} A_\delta$.

Since f^m converges in $L^2_\delta(M^*)$, there exists $c > 0$ such that

$$\|f^m\|_{L^2_\delta(M^*)} \leq c. \tag{2.7}$$

Now, we claim that the sequence $(u^m)_m$ is bounded in $L^2_\delta(M^*)$. To prove this claim, we argue by contradiction and assume that (at least for a subsequence still denoted $(u^m)_m$)

$$\lim_{m \to +\infty} \|u^m\|_{L^2_\delta(M^*)} = \infty$$

We set

$$v^m := \frac{u^m}{\|u^m\|_{L^2_\delta(M^*)}} \qquad \text{and} \qquad g^m := \frac{f^m}{\|u^m\|_{L^2_\delta(M^*)}}$$

so that $A_\delta v^m = g^m$. Applying the result of Proposition 2.2.1, we conclude that the sequence $(v^m)_m$ is bounded in $W^{1,2}(K)$ and hence, using Rellich's Theorem, we conclude that a subsequence (still denoted $(v^m)_m$) converges in $L^2(K)$. Now the result of Proposition 2.2.3 yields

$$\|v^m - v^{m'}\|_{L^2_\delta(M^*)} \leq c \left(\|g^m - g^{m'}\|_{L^2_\delta(M^*)} + \|v^m - v^{m'}\|_{L^2(K)} \right). \tag{2.8}$$

On the right hand side, the sequence $(g^m)_m$ tends to 0 in $L^2_\delta(M^*)$ and the sequence $(v^m)_m$ converges in $L^2(K)$. Therefore, we conclude that $(v^m)_m$ is a Cauchy sequence in $L^2_\delta(M^*)$ and hence converges to $v \in L^2_\delta(M^*)$.

To reach a contradiction, we first pass to the limit in the identity $A_\delta\, v^m = g^m$ to get that the function v is a solution of $A_\delta v = 0$ and hence $v \in \operatorname{Ker} A_\delta$. But by construction $\|v\|_{L^2_\delta(M^*)} = 1$ and also

$$\int_M v^m\, v\, \gamma^{-2\delta-2}\, dvol_g = 0$$

(since $v \in \operatorname{Ker} A_\delta$) and, passing to the limit in this last identity we find that $\|v\|_{L^2_\delta(M^*)} = 0$. A contradiction.

Now that the claim is proved, we use the result of Proposition 2.2.1 together with Rellich's Theorem to extract, form the sequence $(u^m)_m$ some subsequence which converges to u in $L^2_\delta(M^*)$. Once more, Proposition 2.2.3 implies that

$$\|u^m - u^{m'}\|_{L^2_\delta(M^*)} \leq c \left(\|f^m - f^{m'}\|_{L^2_\delta(M^*)} + \|u^m - u^{m'}\|_{L^2(K)} \right). \tag{2.9}$$

This time, on the right hand side, the sequence $(f^m)_m$ converges in $L^2_\delta(M^*)$ and the sequence $(u^m)_m$ converges in $L^2(K)$. Therefore, we conclude that $(u^m)_m$ is a Cauchy sequence in $L^2_\delta(M^*)$ and hence converges to $u \in L^2_\delta(M^*)$. Passing to the limit in the identity $A_\delta\, u^m = f^m$ we conclude that $A_\delta u = f$ and hence f belongs to the range of A_δ. This completes the proof of the result.

2.5 Fredholm properties for A_δ

It will be convenient to identify the dual of $L^2_\delta(M^*)$ with $L^2_{-\delta}(M^*)$. This is done using the scalar product

$$\langle u, v\rangle := \int_M u\, v\, \gamma^{-2}\, dvol_g \tag{2.10}$$

Clearly, given $v \in L^2_{-\delta}(M^*)$, we can define $T_v \in \left(L^2_\delta(M^*)\right)'$ by

$$T_v(u) = \langle u, v\rangle$$

Moreover, we have

$$\|T_v\|_{\left(L^2_\delta(M^*)\right)'} = \|v\|_{L^2_{-\delta}(M^*)}$$

Conversely, given $T \in \left(L^2_\delta(M^*)\right)'$ there exists a unique $v \in L^2_{-\delta}(M^*)$ such that $\langle u, v\rangle = T(u)$ for all $u \in L^2_\delta(M^*)$.

We define A^*_δ, the adjoint of A_δ

$$A^*_\delta : \left(L^2_\delta(M^*)\right)' \longrightarrow \left(L^2_\delta(M^*)\right)'$$

is defined to be an unbounded operator. An element $T \in \left(L^2_\delta(M^*)\right)'$ belongs to $D(A^*_\delta)$, the domain of A^*_δ, if and only if there exists $S \in \left(L^2_\delta(M^*)\right)'$ such that

$$T(A_\delta\, v) = S(v)$$

for all $v \in D(A_\delta)$. We will write $A^*_\delta(T) = S$.

Granted the above identification of $\left(L^2_\delta(M^*)\right)'$ with $L^2_\delta(M^*)$ it is easy to check that we can identify A^*_δ with $A_{-\delta}$. Indeed, if we write $T = T_u$ and $A^*_\delta(T) = T_f$, for $u, f \in L^2_{-\delta}(M^*)$, then, by definition

$$T_u(A_\delta v) := \langle u, A_\delta v\rangle$$

and

$$A^*_\delta(T)(v) := \langle f, v\rangle$$

for all $v \in D(A_\delta)$. Hence, we have

$$\int_M u(\Delta_g + a)v\, dvol_g = \int_M f\, v\, \gamma^{-2} dvol_g$$

for all $v \in D(A_\delta)$. This in particular implies that $\gamma^2 (\Delta_g + a) u = f$ in the sense of distributions. Since $u, f \in L^2_{-\delta}(M^*)$, we conclude that $u \in D(A_{-\delta})$ and $f = A_{-\delta} u$.

Conversely, if $u \in D(A_{-\delta})$, we can write for all $v \in D(A_\delta)$

$$\begin{aligned} \langle u, A_\delta v \rangle &= \int_M u\,(\Delta_g + a) v \, dvol_g \\ &= \int_M v\,(\Delta_g + a) u \, dvol_g \\ &= \langle A_{-\delta}\, u, v \rangle \end{aligned}$$

The integrations by parts can be justified since, according to the result of Proposition 2.2.1, we have $\nabla v \in L^2_{\delta-1}(M^*)$, $\nabla u \in L^2_{-\delta-1}(M^*)$, $\nabla^2 v \in L^2_{\delta-2}(M^*)$ and $\nabla^2 u \in L^2_{-\delta-2}(M^*)$. Therefore $T_u \in D(A^*_\delta)$ and $A^*_\delta(T_u) = T_{A_{-\delta} u}$.

With these identifications in mind, we can state the

Theorem 2.5.1. *Assume that $\delta \neq \pm\delta_j$ for all $j \in \mathbb{N}$. Then*

$$Ker\, A_\delta = (Im\, A_{-\delta})^\perp$$

and

$$Im\, A_\delta = (Ker\, A_{-\delta})^\perp$$

The first part is a classical property for unbounded operators with closed graph and dense domain (see Corollary II.17 in [**?**]). The second result follows from classical results for unbounded operators with dense domains, closed graph and closed range (see Theorem II.18 in [**?**]).

Observe that, because of our identifications, $F^\perp$ is obtained from F using the scalar product defined in (2.10).

Very useful for us will be the :

Corollary 2.5.1. *Assume that $\delta \neq \pm\delta_j$ for all $j \in \mathbb{N}$. Then A_δ is injective if and only if $A_{-\delta}$ is surjective.*

2.6 The deficiency space

Even though the previous results seem already a great achievement, since it will provide right inverses for some operators, we will need a more refined result. As usual, this result for operators defined on the punctured manifold M^* are obtained by perturbing the corresponding results in Euclidean space.

To start with, let us prove the :

Lemma 2.6.1. *Assume that $\delta \neq \pm\delta_j$, for $j \in \mathbb{N}$. There exist an operator*

$$G_\delta : L^2_\delta(B_1^*) \longrightarrow L^2_\delta(B_1^*)$$

and $c = c(n, \delta) > 0$ such that for all $f \in L^2_\delta(B_1^*)$, the function $u := G_\delta(f)$ is **a** *solution of*

$$|x|^2 \,\Delta\, u = f$$

in B_1^* and

$$\|u\|_{L^2_\delta(B_1^*)} + \|\nabla u\|_{L^2_{\delta-1}(B_1^*)} + \|\nabla^2 u\|_{L^2_{\delta-2}(B_1^*)} \leq c\, \|f\|_{L^2_\delta(B_1^*)}$$

At first glance this result looks rather strange wince we are not imposing any boundary data. Nevertheless, some boundary data are hidden in the construction of the operator G_δ. Observe that we state the existence of G_δ and do not state any uniqueness of this operator !

The proof of the existence of G_δ relies on the eigenfunction decomposition of the function f. We decompose as usual

$$f = \sum_{j \geq 0} f_j$$

where $f(r, \cdot) \in E_j$ for all $j \in \mathbb{N}$. Let $j_0 \in \mathbb{N}$ be the least index for which

$$|\delta| < \delta_{j_0}$$

We set

$$\tilde{f} = \sum_{j \geq j_0} f_j$$

Clearly $\tilde{f} \in L^2_\delta(B_1^*)$ and, for all $R \in (0, 1/2)$ one can solve

$$\left\{ \begin{array}{rclcl} |x|^2\, \Delta \tilde{u}_R & = & \tilde{f} & \text{in} & B_1 - \bar{B}_R \\ \tilde{u}_R & = & 0 & \text{on} & \partial B_1 \cup \partial B_R \end{array} \right.$$

The existence of $\tilde{u}_R$ follows from Proposition 1.1.2 and we have the estimate

$$\|\tilde{u}_R\|_{L^2(B_1 - \bar{B}_R)} \leq c\, \|\tilde{f}\|_{L^2(B_1 - \bar{B}_R)}$$

for some constant $c = c(n, R) > 0$. We claim that there exists a constant $c = c(n, \delta) > 0$ such that

$$\|\tilde{u}_R\|_{L^2_\delta(B_1 - \bar{B}_R)} \leq c\, \|\tilde{f}\|_{L^2_\delta(B_1 - \bar{B}_R)} \tag{2.11}$$

Here the norm in $L^2_\delta(B_1 - \bar{B}_R)$ is nothing but the restriction of the restriction of the norm in $L^2_\delta(B_1^*)$ to functions which are defined in $B_1 - \bar{B}_R$. The proof of the claim follows the Part 1 of the proof of Proposition 1.5.1. We omit the details.

Arguing as in the proof of Proposition 1.2.1 we conclude that there exists a constant $c = c(n,\delta) > 0$ such that

$$\|\nabla \tilde{u}_R\|_{L^2_{\delta-1}(B_1-\bar{B}_R)} \leq c\,\|\tilde{f}\|_{L^2_\delta(B_1-\bar{B}_R)}$$

In particular, given $R' \in (0,1/2)$, there exists $c = c(n,\delta,R') > 0$ such that

$$\|\tilde{u}_R\|_{L^2(B_1-\bar{B}'_R)} + \|\nabla \tilde{u}_R\|_{L^2(B_1-\bar{B}'_R)} \leq c\,\|\tilde{f}\|_{L^2_\delta(B_1^*)}$$

for all $R \in (0,R')$. Then using Rellich's Theorem together with a simple diagonal argument, we conclude that there exists a sequence of radii R_i tending to 0 such that the sequence $(\tilde{u}_{R_i})_i$ converges in $L^2(B_1 - \bar{B}_R)$, for all $R \in (0,1/2)$. Passing to the limit in the equation we obtain a solution $\tilde{u}$ of

$$\left\{\begin{array}{rcccc} |x|^2\,\Delta\tilde{u} & = & \tilde{f} & \text{in} & B_1^* \\ \tilde{u} & = & 0 & \text{on} & \partial B_1 \end{array}\right. \tag{2.12}$$

Moreover, passing to the limit in (2.11), we have the estimate

$$\|\tilde{u}\|_{L^2_\delta(B_1^*)} \leq c\,\|\tilde{f}\|_{L^2_\delta(B_1^*)}$$

To finish this study observe that the solution of (2.12) which belongs to $L^2_\delta(B_1^*)$ is unique. To see this, argue by contradiction. If the claim were not true there would exists two solutions and taking the difference we would obtain a function $\tilde{w} \in L^2_\delta(B_1^*)$ satisfying

$$\left\{\begin{array}{rcccc} |x|^2\,\Delta\tilde{w} & = & 0 & \text{in} & B_1^* \\ \tilde{u} & = & 0 & \text{on} & \partial B_1 \end{array}\right.$$

Performing the eigenfunction decomposition of $\tilde{w}$ as

$$\tilde{w} = \sum_{j \geq j_0} \tilde{w}_j$$

we find that

$$\tilde{w}_j = r^{\frac{2-n}{2}+\delta_j}\,\phi_j + r^{\frac{2-n}{2}-\delta_j}\,\psi_j$$

where $\phi_j, \psi_j \in E_j$. Using the fact that $\tilde{w}_j \in L^2_\delta(B_1^*)$ we conclude that $\psi_j = 0$. Next, using the fact that $\tilde{w}_j = 0$ on ∂B_1, we get $\phi_j = 0$ and hence $\tilde{w} = 0$.

Therefore, we can define

$$G_\delta(\tilde{f}) = \tilde{u}.$$

It remains to understand the definition of G_δ acting on f_j, for $j \leq j_0 - 1$. For the sake of simplicity, we assume that $\delta_j = 0$ (When $\delta_j = 0$, the formula has to be changed according to what we have already done in Part 3 of the proof of Proposition 1.5.1) and we use an explicit formula

$$G_\delta(f_j) = \frac{1}{2\delta_j}\left(r^{\frac{2-n}{2}+\delta_j}\int_*^r t^{\frac{n-4}{2}-\delta_j}\,f_j(t)\,dt - r^{\frac{2-n}{2}-\delta_j}\int_*^r t^{\frac{n-4}{2}+\delta_j}\,f_j(t)\,dt\right)$$

where $* = 0$ if $\delta > \delta_j$ and $* = 1$ if $\delta < -\delta_j$. The estimate follows at once from the arguments developed in Part 2 of the proof of Proposition 1.5.1. We omit the details.

Let us now provide a few applications of this result.

Application # 1 : The first application is concerned with the extension of the previous result to the operator defined on the manifold in a neighborhood of one puncture.

Lemma 2.6.2. *Assume that* $\delta \neq \pm\delta_j$, *for* $j \in \mathbb{N}$. *Given* $p_i \in M$ *one of the punctures, there exists* $R_i = R(p_i, n, \delta) > 0$, *an operator*

$$G_\delta^{(i)} : L^2_\delta(B^*_{R_i}(p_i)) \longrightarrow L^2_\delta(B^*_{R_i}(p_i))$$

and $c = c(n, \delta, p_i) > 0$ *such that for all* $f \in L^2_\delta(B^*_{R_i}(p_i))$, *the function* $u := G_\delta^{(i)}(f)$ *is* **a** *solution of*

$$\gamma^2\,(\Delta_g + a)\,u = f$$

in $B^*_{R_i}(p_i)$ *and*

$$\|u\|_{L^2_\delta(B^*_{R_i}(p_i))} + \|\nabla u\|_{L^2_{\delta-1}(B^*_{R_i}(p_i))} + \|\nabla^2 u\|_{L^2_{\delta-2}(B^*_{R_i}(p_i))} \leq c\,\|f\|_{L^2_\delta(B^*_{R_i}(p_i))}$$

This result follows from a simple perturbation argument. First observe that, a scaling argument shows that the result of Lemma 2.6.1 holds when the radius of the ball, which was chosen to be 1, is replaced by R. The corresponding operator will be denoted by $G_{\delta,R}$ and the estimate holds with a constant which does not depend on $R > 0$. We leave this as an exercise.

Thanks to the result of Proposition 2.1.1 we can write

$$\begin{aligned}\|\gamma^2\,(\Delta_g - \Delta_{eucl} + a)\,u\|_{L^2_\delta(B^*_{R_i}(p_i))} \quad \leq \quad & c\,R^2\,\Big(\|u\|_{L^2_\delta(B^*_{R_i}(p_i))} + \|\nabla u\|_{L^2_{\delta-1}(B^*_{R_i}(p_i))} \\ & + \|\nabla^2 u\|_{L^2_{\delta-2}(B^*_{R_i}(p_i))}\Big)\end{aligned}$$

provided $R > 0$ is small enough. This implies that

$$\|f - A_\delta \circ G_{\delta,R}\,f\|_{L^2_\delta(B^*_{R_i}(p_i))} \leq c\,R^2\,\|f\|_{L^2_\delta(B^*_{R_i}(p_i))}$$

for some constant $c = c(n, \delta) > 0$ which does not depend on R. This clearly implies that the operator $A_\delta \circ G_{\delta,R}$ is invertible provided R is fixed small enough, say $R = R_i$. To obtain the result, it is enough to define

$$G_\delta^{(i)} := G_{\delta,R_i} \circ (A_\delta \circ G_{\delta,R_i}).$$

The relevant estimate then follows at once.

Application # 2 : Recall that the functions

$$|x|^{\frac{2-n}{2} \pm \delta_j}\,\phi$$

are harmonic in B_1^* provided $\phi \in E_j$. Building on the result of the previous application, we now prove that one can perturb these functions to get, near any puncture p_i a solution of the homogeneous problem associated with the operator $\gamma^2 (\Delta_g + a)$. This is the content of the following :

Lemma 2.6.3. *For all puncture $p_i \in M$, given $j \in \mathbb{N}$ and $\phi \in E_j$, there exists $W^{\pm\,(i)}_{j,\phi}$ which is defined in $B^*_{R_i}(p_i)$ and which satisfies*

$$\gamma^2 (\Delta_g + a) \, W^{\pm\,(i)}_{j,\phi} = 0$$

in $B^*_{R_i}(p_i)$. In addition,

$$W^{\pm\,(i)}_{j,\phi} - |x|^{\frac{2-n}{2} \pm \delta_j} \, \phi \in L^2_\delta(B^*_{R_i}(p_i))$$

for all $\delta < \pm\delta_j + 2$. Finally the mapping

$$\phi \in E_j \longrightarrow W^{\pm\,(i)}_{j,\phi}$$

is linear.

In this result, R_i is the radius given in Lemma 2.6.2 and x are normal geodesic coordinates near p_i.

The proof of this Lemma uses the following computation which follows at once from Proposition 2.1.1

$$\gamma^2 (\Delta_g + a) \, |x|^{\frac{2-n}{2} \pm \delta_j} \, \phi = \gamma^2 (\Delta_g - \Delta_{eucl} + a) \, |x|^{\frac{2-n}{2} \pm \delta_j} \, \phi \in L^2_\delta(B^*_{R_i}(p_i))$$

for all $\delta < \pm\delta_j + 2$. The result then follows from Lemma 2.6.2.

For each $i = 1, \ldots, k$, we define $\chi^{(i)}$ to be a cutoff function which is identically equal to 1 in $B_{R_i/2}(p_i)$ and identically equal to 0 in $M - B_{3R_i/4}(p_i)$.

The main result of this section is :

Proposition 2.6.1. *Given $\delta < \delta'$, $\delta, \delta' \neq \pm\delta_j$, for all $j \in \mathbb{N}$. Assume that $u \in L^2_\delta(M^*)$ and $f \in L^2_{\delta'}(M^*)$ satisfy*

$$\gamma^2 (\Delta_g + a) \, u = f$$

in M^*. Then, there exists $v \in L^2_{\delta'}(M^*)$ such that

$$u - v \in D_{\delta,\delta'} := Span \{ \chi^{(i)} \, W^{\pm(i)}_{j,\phi}, \quad : \quad \phi \in E_j, \qquad \delta < \pm\delta_j < \delta' \}$$

In addition

$$\|v\|_{L^2_{\delta'}(M^*)} + \|u - v\|_{D_{\delta,\delta'}} \leq c \, (\|f\|_{L^2_{\delta'}(M^*)} + \|u\|_{L^2_\delta(M^*)})$$

for some constant $c = c(n, \delta, \delta') > 0$.

The proof of this result relies on the corresponding result for the Laplacian in the punctured unit ball.

Lemma 2.6.4. *Given $\delta < \delta'$, $\delta, \delta' \neq \pm\delta_j$, for all $j \in \mathbb{N}$. Assume that $u \in L^2_\delta(B_1^*)$ and $f \in L^2_{\delta'}(B_1^*)$ satisfy*

$$|x|^2 \, \Delta \, u = f$$

in B_1^*. Then, there exists $v \in L^2_{\delta'}(B_1^*)$ such that

$$u - v \in D_{\delta,\delta'} := Span\{|x|^{\frac{2-n}{2} \pm j} \, \phi, \quad : \quad \phi \in E_j, \qquad \delta < \pm\delta_j < \delta'\}$$

In addition

$$\|v\|_{L^2_{\delta'}(B_1^*)} + \|u - v\|_{D_{\delta,\delta'}} \leq c\,(\|f\|_{L^2_{\delta'}(B_1^*)} + \|u\|_{L^2_\delta(B_1^*)})$$

for some constant $c = c(n, \delta, \delta') > 0$.

To prove the Lemma, we use the result of Lemma 2.6.1 and set $\bar{v} = G_{\delta'} \, f \in L^2_{\delta'}(B_1^*)$. Therefore

$$|x|^2 \, \Delta \, (u - \bar{v}) = 0$$

in B_1^*. We have

$$\|\bar{v}\|_{L^2_{\delta'}(B_1^*)} \leq c \, \|f\|_{L^2_{\delta'}(B_1^*)}$$

for some constant $c = c(n, \delta) > 0$. We set $w = u - \bar{v}$ which we decompose as usual

$$w = \sum_j w_j$$

where $w_j(r, \cdot) \in E_j$. We fix j_0 to be the least index for which

$$|\delta| < \delta_{j_0} \qquad \text{and} \qquad |\delta'| < \delta_{j_0}$$

We define

$$\tilde{w} = \sum_{j \geq j_0} w_j$$

We claim that $\tilde{w} \in L^2_{\delta'}(B_1^*)$ and also that

$$\|v\|_{L^2_{\delta'}(B_1^*)} \leq c \, \|w\|_{L^2(B_1 - \bar{B}_{1/2})}$$

for some constant $c = c(n, \delta) > 0$. The proof of the claim follows the arguments of Part 1 in the proof of Proposition 1.5.1. We omit the details.

Next, observe that, for $j = 0, \ldots, j_0 - 1$ the function w_j is given by

$$w_j = |x|^{\frac{2-n}{2} + \delta_j} \, \phi_j + |x|^{\frac{2-n}{2} - \delta_j} \, \psi_j$$

for some $\phi_j, \psi_j \in E_j$. Observe that $\phi_j = 0$ if $\delta_j < \delta$ and $\psi_j = 0$ if $-\delta_j < \delta$ since $w_j \in L^2_\delta(B_1^*)$. It is easy to see that

$$\|\phi_j\|_{L^2(S^{n-1})} + \|\phi_j\|_{L^2(S^{n-1})} \leq c\, \|w_j\|_{L^2(B_1 - \bar{B}_{1/2})}$$

for some constant $c = c(n, j) > 0$.

We set

$$v = \bar{v} + w + \sum_{j=0,\dots,j_0-1, \quad \delta_j > \delta'} |x|^{\frac{2-n}{2} + \delta_j}\, \phi_j + \sum_{j=0,\dots,j_0-1, \quad -\delta_j > \delta'} |x|^{\frac{2-n}{2} - \delta_j}\, \psi_j$$

so that

$$u - v = \sum_{j=0,\dots,j_0-1, \quad \delta < \delta_j < \delta'} |x|^{\frac{2-n}{2} + \delta_j}\, \phi_j + \sum_{j=0,\dots,j_0-1, \quad \delta < -\delta_j < \delta'} |x|^{\frac{2-n}{2} - \delta_j}\, \psi_j$$

The estimate follows from collecting the above estimates. This completes the proof of Lemma 2.6.4.

We proceed with the proof of Proposition 2.6.1. Choose

$$\tilde{\delta} \geq \inf(\delta', \delta + 1)$$

such that $\tilde{\delta} \neq \pm\delta_j$, for all $j \in \mathbb{N}$. Using the result of Proposition 2.2.1 we have

$$\|\nabla u\|_{L^2_{\tilde{\delta}-1}(M^*)} + \|\nabla^2 u\|_{L^2_{\tilde{\delta}-2}(M^*)} \leq c\, (\|f\|_{L^2_{\delta'}(M^*)} + \|u\|_{L^2_{\delta}(M^*)})$$

Using the decomposition given in Proposition 2.1.1, we conclude that, near any puncture p_i, we have

$$|x|^2\, \Delta\, u = f - |x|^2\, (\delta_g - \delta_{eucl} + a) \in L^2_{\tilde{\delta}}(B^*_{R_i})$$

We apply the previous result which yields the decomposition

$$u = v + \sum_{\delta < \pm\delta_j < \delta'} |x|^{\frac{2-n}{2} \pm \delta_j}\, \phi$$

where $\phi \in E_j$. Next use the result of Lemma 2.6.3 and replace all $|x|^{\frac{2-n}{2} \pm \delta_j}\, \phi$ by $\chi^{(i)}\, W^{\pm(i)}_{j,\phi}$ to get the decomposition

$$u = \left(v + \sum_{\delta < \pm\delta_j < \delta'} (|x|^{\frac{2-n}{2} \pm \delta_j}\, \phi - \chi^{(i)}\, W^{\pm(i)}_{j,\phi}) \right) + \sum_{\delta < \pm\delta_j < \delta'} \chi^{(i)}\, W^{\pm(i)}_{j,\phi}$$

Observe that the function

$$\tilde{u} = v + \sum_{\delta < \pm\delta_j < \delta'} (|x|^{\frac{2-n}{2} \pm \delta_j}\, \phi - \chi^{(i)}\, W^{\pm(i)}_{j,\phi}) \in L^2_{\tilde{\delta}}(M^*)$$

and also that $\gamma^2\, (\Delta_g + a)\, \tilde{u} = \tilde{f} \in L^2_{\delta'}(M^*)$. If $\tilde{\delta} = \delta'$ then the roof is complete. If not, apply the same argument with u replaced by $\tilde{u}$, f replaced by $\tilde{f}$ and δ replaced by $\tilde{\delta}$ and proceed until the gap between δ and δ' is covered.

We now give some important consequences of this result :

2.6.1 The kernel of A_δ revisited :

Thanks to the result of Proposition 2.6.1 we can state the :

Lemma 2.6.5. *Fix $\delta < \delta'$ such that $\delta, \delta' \neq \pm\delta_j$, for $j \in \mathbb{N}$. Assume that $u \in L^2_\delta(M^*)$ satisfied*

$$\gamma^2 \left(\Delta_g u + a\, u\right) = 0$$

in M^*. Then $u \in L^2_{\delta'}(M^*)$ provided the interval (δ, δ') does not contain any $\pm\delta_j$, for some $j \in \mathbb{N}$.

This Lemma is a direct consequence of the result of Proposition 2.6.1. It essentially states that the kernel of the operator A_δ does not change as δ remains in some interval which does not contain any $\pm\delta_j$, for $j \in \mathbb{N}$.

2.6.2 The deficiency space :

We now define

Definition 2.6.1. *Given $\delta > 0$, $\delta \neq \delta_j$, for all $j \in \mathbb{N}$, the deficiency space D_δ is defined by*

$$D_\delta := Span\{\chi^{(i)}\, W^{\pm(i)}_{j,\phi}, \qquad : \qquad \phi \in E_j, \qquad -\delta < \pm\delta_j < \delta\}$$

Observe that the dimension of D_δ can be computed as follows

$$\mathrm{dim} D_\delta = 2 \sum_{j, \delta_j < |\delta|} \dim E_j$$

As a first by product, we obtain

Proposition 2.6.2. *Given $\delta > 0$, $\delta \neq \delta_j$, for all $j \in \mathbb{N}$. Assume that A_δ is injective. Then the operator*

$$\begin{array}{rccc} \tilde{A}_\delta : & L^2_\delta(M^*) \oplus D_\delta & \longrightarrow & L^2_\delta(M^*) \\ & u & \longmapsto & \gamma^2 \left(\Delta_g u + a\, u\right) \end{array}$$

is surjective and

$$Ker\, A_{-\delta} = Ker\, \tilde{A}_\delta$$

As a consequence of the previous Proposition, we have the

Corollary 2.6.1. *Given $\delta > 0$, $\delta \neq \delta_j$, for all $j \in \mathbb{N}$. Assume that A_δ is injective. Then*

$$dim\, Ker\, A_{-\delta} = codim\, Im\, A_\delta = \frac{1}{2}\, dim\, D_\delta$$

Under the assumptions of the Corollary, we have

$$\dim \operatorname{Ker} A_{-\delta} = \dim \operatorname{Ker} \tilde{A}_{\delta}$$

and

$$\dim D_{\delta} = \dim \dim \operatorname{Ker} A_{-\delta} + \operatorname{codim} \operatorname{Im} A_{\delta}$$

But, by duality, we have $\dim \operatorname{Ker} A_{-\delta} = \operatorname{codim} \operatorname{Im} A_{\delta}$. The result then follows at once.

Exercise 2.6.1. ‡ *Extend the results of Corollary 2.6.1 to the case there A_δ is not injective.*

Chapter 3

Weighted $C^{2,\alpha}$ analysis on a punctured manifold

3.1 From weighted Lebesgue spaces to weighted Hölder spaces

As far as linear analysis is concerned the results of the previous sections are sufficient. However, we would like to apply them to nonlinear problems for which is will be more convenient to work in the framework of Hölder spaces. The purpose of this section is to explain how the analysis of the previous section can be extended to weighted Hölder spaces.

We begin with the definition of weighted Hölder spaces.

Definition 3.1.1. *Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, we define $\mathcal{C}^{\ell,\alpha}_{\delta}(M^*)$ to be the space of functions $u \in \mathcal{C}^{\ell,\alpha}_{loc}(M^*)$ for which the following norm*

$$\|u\|_{\mathcal{C}^{\ell,\alpha}_{\delta}(M^*)} := \|u\|_{\mathcal{C}^{\ell,\alpha}_{\delta}(M_R)} + \sum_{i=1}^{k} \sup_{\rho \in (0,R)} \rho^{\frac{n-2}{2}-\delta} \, \|u(Exp_{p_i}(\rho\, \cdot))\|_{\mathcal{C}^{\ell,\alpha}(\bar{B}_2 - B_1 \subset T_{p_i}M)}$$

is finite.

For example, the function $\gamma^{\frac{2-n}{2}+\delta} \in \mathcal{C}^{\ell,\alpha}_{\delta'}(M^*)$ if and only if $\delta \geq \delta'$. It also follows directly from this definition that

$$\mathcal{C}^{\ell,\alpha}_{\delta}(M^*) \subset L^2_{\delta'}(M^*)$$

for all $\delta > \delta'$.

Lemma 3.1.1. *The space $(\mathcal{C}^{\ell,\alpha}_{\delta}(M^*), \|\cdot\|_{\mathcal{C}^{\ell,\alpha}_{\delta}(M^*)})$ is a Banach space.*

Exercise 3.1.1. *Show that the embedding*

$$\mathcal{C}^{\ell,\alpha}_{\delta}(M^*) \longrightarrow \mathcal{C}^{\ell',\alpha'}_{\delta'}(M^*)$$

is compact provided $\ell' + \alpha' < \ell + \alpha$ *and* $\delta < \delta'$.

The last easy observation is that the operator

$$\begin{array}{rccc} \mathcal{A}_\delta : & \mathcal{C}^{\ell,\alpha}_{\delta}(M^*) & \longrightarrow & \mathcal{C}^{\ell,\alpha}_{\delta}(M^*) \\ & u & \longrightarrow & \gamma^2 \left(\Delta_g u + a u\right) \end{array}$$

is well defined and bounded.

The extension of our results to weighted Hölder spaces rely on the following regularity result.

Proposition 3.1.1. *Assume that* $\delta, \delta' \in \mathbb{R}$ *are fixed with* $\delta < \delta'$. *Further assume that the interval* $[\delta, \delta']$ *does not contain any* $\pm\delta_j$ *for* $j \in \mathbb{N}$. *Then, there exists* $c = c(n, \delta, \delta') > 0$ *such that for all* $u, f \in L^2_\delta(B^*_R(p_i))$ *satisfying*

$$\gamma^2 \left(\Delta_g + a\right) u = f$$

in M^*, *if* $f \in \mathcal{C}^{0,\alpha}_{\delta'}(M^*)$ *then* $u \in \mathcal{C}^{2,\alpha}_{\delta'}(M^*)$ *and*

$$\|u\|_{\mathcal{C}^{2,\alpha}_{\delta'}(M^*)} \leq c \left(\|f\|_{\mathcal{C}^{0,\alpha}_{\delta'}(M^*)} + \|u\|_{L^2_\delta(M^*)} \right)$$

Before we proceed to the proof of this result, let us explain how it can be used.

Application # 1: The first application of the result of Proposition 3.1.1 is concerned with the kernel of the operator A_δ.

Lemma 3.1.2. *Assume that* $\delta \in \mathbb{R}$ *is fixed with* $\delta \neq \delta_j$, *for* $j \in \mathbb{N}$. *Further assume that* $u \in L^2_\delta(M^*)$ *is a solution of*

$$\gamma^2 \left(\Delta_g + a\, u\right) = 0$$

in M^*. *Then* $u \in \mathcal{C}^{2,\alpha}_{\delta}(M^*)$.

In other words, in order to check the injectivity of A_δ, it is enough to check the injectivity of $\mathcal{A}_\delta$, which in practical situation is easier to perform.

Application # 2 : Observe that, if $u \in L^2_\delta(M^*)$ is in the kernel of A_δ then u is also in the kernel of $A_{\delta'}$ for all $\delta' \leq \delta$ since $L^2_\delta(M^*) \subset L^2_\delta(M^*)$. However, it follows from Proposition 3.1.1 the the following is also true :

Lemma 3.1.3. *Assume that* $\delta \in \mathbb{R}$ *is fixed with* $\delta \neq \delta_j$, *for* $j \in \mathbb{N}$. *Further assume that* $u \in L^2_\delta(M^*)$ *is in the kernel of* A_δ. *Then* u *is also in the kernel of* $A_{\delta'}$ *for all* $\delta' > \delta$ *for which* $[\delta, \delta']$ *does not contain any* $\pm\delta_j$, *for* $j \in \mathbb{N}$

Application # 3 : The third application of the result of Proposition 3.1.1 is concerned with the extension of the result of Proposition 2.6.1 to weighted Hölder spaces and this will be useful when dealing with nonlinear differential operators. We have the :

Proposition 3.1.2. *Given $\delta < \delta'$, $\delta, \delta' \neq \pm\delta_j$, for all $j \in \mathbb{N}$. Assume that $u \in L^2_\delta(M^*)$ and $f \in \mathcal{C}^{0,\alpha}_{\delta'}(M^*)$ satisfy*

$$\gamma^2\,(\Delta_g + a)\,u = f$$

in M^*. Then, there exists $v \in \mathcal{C}^{2,\alpha}_{\delta'}(M^*)$ such that

$$u - v \in D_{\delta,\delta'} := Span\,\{\chi^{(i)}\,W^{\pm(i)}_{j,\phi}, \quad : \quad \phi \in E_j, \qquad \delta < \pm\delta_j < \delta'\}$$

In addition

$$\|v\|_{\mathcal{C}^{2,\alpha}_{\delta'}(M^*)} + \|u - v\|_{D_{\delta,\delta'}} \leq c\,(\|f\|_{\mathcal{C}^{0,\alpha}_{\delta'}(M^*)} + \|u\|_{L^2_\delta(M^*)})$$

for some constant $c = c(n, \delta, \delta') > 0$.

There are important by products of this result :

Given δ, $\delta > \delta_0$, $\delta \neq \delta_j$ for all $j \in \mathbb{N}$. Assume that A_δ is injective, then, according to the result of Corollary 2.5.1, the operator $A_{-\delta}$ is surjective and hence there exists

$$G_{-\delta} : L^2_{-\delta}(M^*) \longrightarrow L^2_{-\delta}(M^*).$$

a right inverse for $A_{-\delta}$ (i.e. $A_{-\delta} \circ G_{-\delta} = I$). In particular, given

$$f \in \mathcal{C}^{0,\alpha}_\delta(M^*) \subset L^2_{-\delta}(M^*),$$

the function $u := G_{-\delta}\, f \in L^2_{-\delta}(M^*)$ solves

$$A_{-\delta}\, u = f$$

in M^*. Applying the result of Proposition 3.1.2, we see that there exists $v \in \mathcal{C}^{2,\alpha}_\delta(M^*)$ such that

$$u - v \in D_\delta := \mathrm{Span}\,\{\chi^{(i)}\,W^{\pm(i)}_{j,\phi}, \quad : \quad \phi \in E_j, \qquad -\delta < \pm\delta_j < \delta\}$$

and in addition

$$\|v\|_{\mathcal{C}^{2,\alpha}_\delta(M^*)} + \|u - v\|_{D_\delta} \leq c\,(\|f\|_{\mathcal{C}^{0,\alpha}_\delta(M^*)} + \|u\|_{L^2_\delta(M^*)})$$

for some constant $c = c(n, \delta) > 0$.

If $\delta \in (-\delta_0, \delta_0)$ and if A_δ is injective. Then, according to the result of Lemma 3.1.3 the operator $A_{-\delta'}$ is also injective for all $\delta' \in (-\delta_0, \delta_0)$. Therefore, according to the result of Corollary 2.5.1 the operator $A_{\delta'}$ is surjective. This implies that there exists

$$G_{\delta'} : L^2_{\delta'}(M^*) \longrightarrow L^2_{\delta'}(M^*).$$

a right inverse for $A_{\delta'}$. In particular, if $-\delta_0 < \delta' < \delta < \delta_0$, given

$$f \in \mathcal{C}^{0,\alpha}_{\delta}(M^*) \subset L^2_{\delta'}(M^*),$$

the function $u := G_{\delta'}\, f \in L^2_{\delta'}(M^*)$ solves

$$A_{\delta'}\, u = f$$

in M^*. Applying the result of Proposition 3.1.2, we see that $u \in \mathcal{C}^{2,\alpha}_{\delta}(M^*)$ and in addition

$$\|u\|_{\mathcal{C}^{2,\alpha}_{\delta}(M^*)} + \|u - v\|_{D_\delta} \leq c\,(\|f\|_{\mathcal{C}^{0,\alpha}_{\delta}(M^*)} + \|u\|_{L^2_{\delta}(M^*)})$$

for some constant $c = c(n, \delta) > 0$. Collecting these result, we have proven the :

Proposition 3.1.3. *Given $\delta > -\delta_0$, $\delta \neq \delta_j$, for all $j \in \mathbb{N}$, let us assume that A_δ is injective, then the operator*

$$\begin{array}{rccc} \tilde{\mathcal{A}}_\delta : & \mathcal{C}^{\ell,\alpha}_{\delta}(M^*) \oplus D_\delta & \longrightarrow & \mathcal{C}^{\ell,\alpha}_{\delta}(M^*) \\ & u & \longrightarrow & \gamma^2\,(\Delta_g u + a u) \end{array}$$

is well defined, bounded and surjective. In addition $dim\, Ker(\tilde{\mathcal{A}}_\delta) = \frac{1}{2}\, dim D_\delta$.

In particular, under the assumptions of the Proposition, there exists an operator

$$\mathcal{G}_\delta : \mathcal{C}^{0,\alpha}_{\delta}(M^*) \longrightarrow \mathcal{C}^{2,\alpha}_{\delta}(M^*) \oplus D_\delta.$$

which is a right inverse for the operator $\gamma^2\,(\Delta_g + a)$.

As a special case, when $\delta \in (\delta_0, \delta_0)$, then D_δ is empty and the above statement simplifies into the :

Proposition 3.1.4. *Given $\delta \in (-\delta_0, \delta_0)$. Let us assume that A_δ is injective, then the operator $\mathcal{A}_{\delta'}$ is an isomorphism for all $\delta' \in (-\delta_0, \delta_0)$.*

We now proceed with the proof of Proposition 3.1.1. We start with the :

Lemma 3.1.4. *Assume that $\delta, \delta' \in \mathbb{R}$ are fixed with $\delta < \delta'$. There exists $c = c(n, \delta, \delta') > 0$ such that for all $u, f \in L^2_\delta(M^*)$ satisfying*

$$\gamma^2\,(\Delta_g + a)\, u = f$$

in M^*, if $f \in \mathcal{C}^{0,\alpha}_{\delta'}(M^*)$ then $u \in \mathcal{C}^{2,\alpha}_{\delta}(M^*)$ and

$$\|u\|_{\mathcal{C}^{2,\alpha}_{\delta}(M^*)} \leq c\,\left(\|f\|_{\mathcal{C}^{0,\alpha}_{\delta'}(M^*)} + \|u\|_{L^2_{\delta}(M^*)}\right)$$

Away from the punctures p_j, the regularity of u follows from classical elliptic regularity :

Proposition 3.1.5. *([?], Theorem ??) Assume that $\bar{\Omega}' \subset\subset \Omega$ is fixed. There exists $c = c(n, \Omega', \Omega) > 0$ such that for all $u, f \in L^2(\Omega)$ satisfying*

$$(\Delta_g + a)\, u = f$$

in Ω, if $f \in \mathcal{C}^{0,\alpha}(\bar{\Omega})$ then $u \in \mathcal{C}^{2,\alpha}_\delta(\bar{\Omega}')$ and

$$\|u\|_{\mathcal{C}^{2,\alpha}(\bar{\Omega}')} \leq c\, \left(\|f\|_{\mathcal{C}^{0,\alpha}(\bar{\Omega})} + \|u\|_{L^2(\Omega)} \right)$$

Close to the punctures, we use normal geodesic coordinates together with (2.1) and write the equation satisfied by u as

$$|x|^2\, (\Delta_{eucl}\, u + \mathcal{O}(|x|^2)\, \partial_{x^i}\, \partial_{x^j}\, u + \mathcal{O}(|x|)\, \partial_{x^i} u + \mathcal{O}(1) u) = f$$

For all $r \in (0, R)$ we defined the rescaled functions

$$\hat{u}(x) = u(R\, x) \qquad \text{and} \qquad \hat{f}(x) = f(R\, x)$$

so that

$$|x|^2\, (\Delta_{eucl}\, \hat{u} + \mathcal{O}(R^2)\, \partial_{x^i}\, \partial_{x^j}\, \hat{u} + \mathcal{O}(R^2)\, \partial_{x^i} \hat{u} + \mathcal{O}(R^2)\, \hat{u}) = \hat{f}$$

in $B_2 - \bar{B}_{1/2}$. Applying the result of Proposition 3.1.5 with $\Omega = B_2 - \bar{B}_1$ and $\Omega' = B_{3/2} - \bar{B}_{3/4}$ we conclude that

$$\|\hat{u}\|_{\mathcal{C}^{2,\alpha}(\bar{B}_{3/2} - B_{3/4})} \leq c\, \left(\|\hat{f}\|_{\mathcal{C}^{0,\alpha}(\bar{B}_2 - B_1)} + \|\hat{u}\|_{L^2(B2 - \bar{B}_1)} \right)$$

But we have

$$\|\hat{f}\|_{\mathcal{C}^{0,\alpha}(\bar{B}_2 - B_1)} \leq c\, R^{\frac{2-n}{2} + \delta'}\, \|f\|_{\mathcal{C}^{0,\alpha}_\delta(M^*)} \leq c\, R^{\frac{2-n}{2} + \delta}\, \|f\|_{\mathcal{C}^{0,\alpha}_\delta(M^*)}$$

and

$$\|\hat{u}\|_{L^2(B2 - \bar{B}_1)} \leq c\, R^{\frac{2-n}{2} + \delta}\, \|u\|_{L^2_\delta(M^*)}$$

for some constant $c = c(n, \delta, \delta') > 0$. Therefore, we conclude that

$$\|\hat{u}\|_{\mathcal{C}^{2,\alpha}(\bar{B}_{3/2} - B_{3/4})} \leq c\, R^{\frac{2-n}{2} + \delta}\, \left(\|f\|_{\mathcal{C}^{0,\alpha}_{\delta'}(M^*)} + \|u\|_{L^2_\delta(M^*)} \right)$$

which by definition of the weighted Hölder norm, implies that

$$\|u\|_{\mathcal{C}^{2,\alpha}_\delta(\bar{B}^*_{2R})} \leq c\, \left(\|f\|_{\mathcal{C}^{0,\alpha}_{\delta'}(M^*)} + \|u\|_{L^2_\delta(M^*)} \right)$$

This completes the proof of the Lemma.

The next result we will need reads :

Lemma 3.1.5. *Assume that $\delta < \delta' \in \mathbb{R}$ and further assume that $[\delta, \delta']$ does not contain any $\pm\delta_j$ for all $j \in \mathbb{R}$. Let $u \in \mathcal{C}^{2,\alpha}_{\delta}(\bar{B}^*_1)$ and $f \in \mathcal{C}^{0,\alpha}_{\delta'}(\bar{B}^*_1)$ satisfy*

$$|x|^2 \, \Delta \, u = f$$

in B^*_1. Then $u \in \mathcal{C}^{0,\alpha}_{\delta'}(\bar{B}^*_1)$ and

$$\|u\|_{\mathcal{C}^{2,\alpha}_{\delta'}(\bar{B}^*_1)} \leq c \left(\|f\|_{\mathcal{C}^{0,\alpha}_{\delta'}(\bar{B}^*_1)} + \|u\|_{\mathcal{C}^{2,\alpha}_{\delta}(\bar{B}^*_1)} \right)$$

The proof of this last Lemma goes as follows. As usual, we perform the eigenfunction decomposition of both u and f in B^*_1

$$u = \sum_j u_j \qquad \text{and} \qquad f = \sum_j f_j$$

We define $j_0 \in \mathbb{N}$ to be the least index for which $|\delta| < \delta_{j_0}$. For $j = 0, \ldots, , j_0 - 1$ one can use the explicit formula we have provided in the proof of Proposition 1.5.1 to show directly that $u_j \in \mathcal{C}^{2,\alpha}_{\delta'}(\bar{B}^*_1)$ and that

$$\|u_j\|_{\mathcal{C}^{2,\alpha}_{\delta'}(\bar{B}^*_1)} \leq c \left(\|f_j\|_{\mathcal{C}^{0,\alpha}_{\delta'}(\bar{B}^*_1)} + \|u_j\|_{\mathcal{C}^{2,\alpha}_{\delta}(\bar{B}^*_1)} \right)$$

We denote

$$\tilde{u} = \sum_{j \geq j_0} u_j \qquad \text{and} \qquad \tilde{f} = \sum_{j \geq j_0} f_j$$

The strategy is now to construct $\tilde{v} \in \mathcal{C}^{2,\alpha}_{\delta'}(B^*_1)$ solution of

$$|x|^2 \, \Delta \, \tilde{v} = \tilde{f}$$

in B^*_1 with $\tilde{v} = \tilde{u}$ on ∂B_1 and also to prove that

$$\|\tilde{v}\|_{\mathcal{C}^{2,\alpha}_{\delta'}(\bar{B}^*_1)} \leq c \left(\|\tilde{f}\|_{\mathcal{C}^{0,\alpha}_{\delta'}(\bar{B}^*_1)} + \|\tilde{u}\|_{\mathcal{C}^{2,\alpha}_{\delta}(\bar{B}^*_1)} \right) \tag{3.1}$$

Assuming we have already done so, the difference $\tilde{u} - \tilde{v}$ is harmonic in B^*_1 and equal to 0 on ∂B_1. The eigenfunction decomposition of the function $\tilde{u} - \tilde{v}$ shows that

$$\tilde{u} - \tilde{v} = \sum_{j \geq j_0} \left(|x|^{\frac{2-n}{2} + \delta_j} \, \phi_j + |x|^{\frac{2-n}{2} - \delta_j} \, \psi_j \right)$$

where $\phi_j, \psi_j \in E_j$. But $\tilde{u} - \tilde{v} \in \mathcal{C}^{2,\alpha}_{\delta}(B^*_1)$ and hence ψ_j all have to be equal to 0. Using the fact that $\tilde{u} - \tilde{v} = 0$ on ∂B_1, we also get that $\phi_j = 0$. Hence $\tilde{u} = \tilde{v}$. This will complete the proof of the Lemma.

Therefore, the only missing part is the existence of $\tilde{v}$ and the *a priori* estimate (3.1). To simplify the argument, let us first reduce to the case where $\tilde{u} = 0$ on ∂B_1. To this aim, we choose a cutoff function χ which is radial, identically equal to 1 on $B_1 - B_{3/4}$ and identically equal to 0 in $B_{1/2}$. Then, define

$$\tilde{v}_0(x) = \chi(x)\, \tilde{u}(x/|x|)$$

and set

$$\tilde{w} = \tilde{v} - \tilde{v}_0 \qquad \text{and} \qquad \tilde{g} = \tilde{f} - |x|^2\, \Delta \tilde{v}_0$$

so that the equation we have to solve now reads

$$|x|^2\, \Delta \tilde{w} = \tilde{g}$$

in B_1^* with $\tilde{w} = 0$ on ∂B_1. Obviously the existence of $\tilde{v}$ is equivalent to the existence of $\tilde{w}$ and (3.1) will follow at once from

$$\|\tilde{w}\|_{\mathcal{C}^{2,\alpha}_{\delta'}(\bar{B}_1^*)} \leq c\, \|\tilde{g}\|_{\mathcal{C}^{0,\alpha}_{\delta'}(\bar{B}_1^*)}$$

since

$$\|\tilde{f} - |x|^2\, \Delta \tilde{u}_0\|_{\mathcal{C}^{0,\alpha}_{\delta'}(\bar{B}_1^*)} \leq c\, \left(\|\tilde{f}\|_{\mathcal{C}^{0,\alpha}_{\delta'}(\bar{B}_1^*)} + \|\tilde{u}\|_{\mathcal{C}^{2,\alpha}_{\delta}(\bar{B}_1^*)} \right)$$

The existence of $\tilde{w}$ follows from the arguments already developed to prove Proposition 1.1.1. However the derivation of the estimate is more involved an requires new technics since it is not possible to construct barrier solutions anymore. In any case, for all $R \in (0, 1/2)$, we solve

$$|x|^2\, \Delta \tilde{w}_R = \tilde{g}$$

in $B_1 - \bar{B}_R$ with $\tilde{w}_R = 0$ on $\partial B_1 \cup \partial B_R$.

We claim that there exists a constant $c = c(n, \delta') > 0$ such that

$$\sup_{B_1 - \bar{B}_R} |x|^{\frac{n-2}{2} - \delta'}\, |\tilde{w}_R| \leq c\, \sup_{B_1^*} |x|^{\frac{n-2}{2} - \delta'}\, |\tilde{g}|$$

When R remains bounded away from 0, the claim is certainly true and follows from standard elliptic estimate (use Proposition 1.1.2 and Proposition 1.1.3). In order to prove the claim, we argue by contradiction and assume that, for a sequence R_i tending to 0, for a sequence of functions $\tilde{g}_i \in \mathcal{C}^{0,\alpha}_{\delta'}(\bar{B}_1^*)$, we have

$$\sup_{B_1^*} |x|^{\frac{n-2}{2} - \delta'}\, |\tilde{g}_i| = 1$$

while, for the corresponding sequence of solutions $\tilde{w}_{R_i}$

$$A_i := \sup_{B_1^*} |x|^{\frac{n-2}{2} - \delta'}\, |\tilde{w}_{R_i}|$$

tends to ∞. One should keep in mind that the eigenfunction decomposition of both $\tilde{g}_i$ and $\tilde{w}_{R_i}$ have no component over E_j for $j < j_0$.

Observe that the function $\tilde{w}_R$ is continuous and we can choose a point $x_i \in B_1 - \bar{B}_R$ where A_i is achieved. We define the rescaled functions

$$\hat{w}_i(x) := A_i^{-1}\, \tilde{w}_{R_i}(|x_i|\, x) \qquad \text{and} \qquad \hat{g}_i(x) := A_i^{-1}\, \tilde{g}_{R_i}(|x_i|\, x)$$

Obviously

$$|x|^2\, \Delta \hat{w}_i = \hat{g}_i$$

in their common domain of definition.

Using the result of Proposition 1.1.5, we get the estimate

$$\|\nabla \tilde{w}_{R_i}\|_{L^\infty(B_1 - \bar{B}_{3/4})} \leq c \left(\|\tilde{w}_{R_i}\|_{L^\infty(B_1 - \bar{B}_{1/2})} + \|\tilde{g}_i\|_{L^\infty(B_1 - \bar{B}_{1/2})} \right)$$

for some constant $c = c(n) > 0$ And hence

$$\|\nabla \tilde{w}_{R_i}\|_{L^\infty(B_1 - \bar{B}_{3/4})} \leq c\,(1 + A_i)$$

This implies that

$$|x|^{\frac{n-2}{2} - \delta}\, |\tilde{w}_{R_i}| \leq c\,(1 + R^{\frac{n-2}{2} - \delta})\,(1 - R)\,(1 + A_i)$$

for all $x \in B_1 - \bar{B}_{3/4}$. Therefore, if $\rho \in (3/4, 1)$ is fixed so that

$$c\,(1 + \rho^{\frac{n-2}{2} - \delta})\,(1 - \rho) \leq 1/2$$

we conclude that, for i large enough, $x_i \notin B_1 - \bar{B}_\rho$.

Working near ∂B_R and using similar arguments one can show that there exists $\bar{\rho} \in (1, 3/2)$ such that $x_i \notin B_{\bar{\rho}\, R} - \bar{B}_R$. Therefore we conclude that

$$R < \bar{\rho}\, R \leq |x_i| \leq \rho < 1 \tag{3.2}$$

As in the proof of Proposition 1.1.1 we pass to the limit for a subsequence of i tending to ∞ to obtain $\hat{w}$ a solution of

$$|x|^2\, \Delta\, \hat{w} = 0$$

in one of the following domains

(i) $\mathbb{R}^{n\,*}$ (which occurs when $r_2 := \lim 1/|x_i| = \infty$ and $r_1 := \lim R_i/|x_i| = 0$).

(ii) $\mathbb{R}^n - \bar{B}_{r_1}$ (which occurs when $r_2 := \lim 1/|x_i| = \infty$ and $r_1 := \lim R_i/|x_i| < 1$).

(iii) $B_{r_2} - \bar{B}_{r_1}$ (which occurs when $r_2 := \lim 1/|x_i| > 1$ and $r_1 \lim R_i/|x_i| < 1$).

(iv) $B^*_{r_2}$ (which occurs when $r_2 := \lim 1/|x_i| > 1$ and $r_1 := \lim R_i/|x_i| = 0$).

Observe that, using (3.2), we always have $r_1 < 1 < r_2$

In addition,

$$\sup |x|^{\frac{n-2}{2}-\delta} |\hat{w}| = 1 \tag{3.3}$$

where the supremum is taken over the domain of definition of $\hat{w}$ and finally $\hat{w} = 0$ on either ∂B_{r_j} if either r_1 or r_2 is finite. As usual, we perform the eigenfunction decomposition of $\hat{w}$ as

$$\hat{w} = \sum_{j \geq j_0} \hat{w}_j$$

only depends on n and δ. It is easy to rule out case (iii) since $\hat{w}$ is harmonic in the annulus and has zero boundary data. In order to rule out case (iv), it is enough to look at the behavior of the function $\hat{w}_j$ near 0. Using (3.3) together with $|\delta| < \delta_j$ we conclude that

$$\hat{w}_j = |x|^{\frac{2-n}{2}+\delta_j} \phi_j$$

for some $\phi_j \in E_j$. But $\hat{w}_j = 0$ on ∂B_{r_2} and hence $\hat{w}_j \equiv 0$. The other cases can be ruled out using similar arguments and we leave the details to the reader.

Hence $\hat{w} \equiv 0$ and this clearly contradicts (3.3). This completes the proof of the claim.

Now that we have proven the claim, we use elliptic estimates and Ascoli's Theorem to pass to the limit as R tends to 0 in the sequence $\tilde{w}_R$ and obtain a solution of

$$|x|^2 \Delta \tilde{w} = \tilde{g}$$

in B_1^* with $\tilde{w} = 0$ on ∂B_1 and

$$\sup_{B_1^*} |x|^{\frac{n-2}{2}-\delta'} |\tilde{w}| \leq c \sup_{B_1^*} |x|^{\frac{n-2}{2}-\delta'} |\tilde{g}|$$

To obtain the relevant estimates for the derivative, we use again the result of Proposition 3.1.5. This completes the proof of Lemma 3.1.5.

To complete the proof of Proposition 3.1.1, we argue as follows : We start by applying the result of Lemma 3.1.4 which implies that $u \in \mathcal{C}^{2,\alpha}_{\delta}(M^*)$ and hence, thanks to (2.1) we can write, near any of the punctures

$$|x|^2 \Delta \in \mathcal{C}^{0,\alpha}_{\tilde{\delta}}(B_R^*(p_j))$$

for $\tilde{\delta} = \min(\delta', \delta - 2)$. Next we apply the result of Lemma 3.1.5 which guaranties that $u \in \mathcal{C}^{2,\alpha}_{\tilde{\delta}}(B_R^*(p_j))$. If $\tilde{\delta} = \delta'$ then the proof is complete. If not, we iterate the argument starting from $\tilde{\delta}$ and proceed in this way until the interval $[\delta, \delta']$ has been entirely covered. The proof of the estimate follows from the estimates given in Lemma 3.1.4 and Lemma 3.1.5.

3.2 An example

As a typical example, we consider the study of the operator $\Delta_{S^n} + \lambda$ where $\lambda \in \mathbb{R}$.

Given $p_1, \ldots, p_k \in S^n$, we define γ to be a smooth positive function on $S^{n\,*}$ which coincides with the distance to p_i in some neighborhood of p_i. Then, we define the operators

$$\begin{array}{rccc} A_\delta : & L^2_\delta(S^{n\,*}) & \longrightarrow & L^2_\delta(S^{n\,*}) \\ & u & \longmapsto & \gamma^2(\Delta_{S^n}\, u + \lambda\, u) \end{array}$$

and

$$\begin{array}{rccc} \mathcal{A}_\delta : & \mathcal{C}^{2,\alpha}_\delta(S^{n\,*}) & \longrightarrow & \mathcal{C}^{0,\alpha}_\delta(S^{n\,*}) \\ & u & \longmapsto & \gamma^2(\Delta_{S^n}\, u + \lambda\, u) \end{array}$$

Let us assume that λ is not an eigenvalue of $-\Delta_{S^n}$ (namely $\lambda \neq j\,(n-1+j)$, for all $j \in \mathbb{N}$). Then the theory we have developed leads to the following :

Lemma 3.2.1. *Assume that λ is not an eigenvalue of $-\Delta_{S^n}$, then the operators A_δ and $\mathcal{A}_\delta$ are injective for all $\delta > \frac{2-n}{2}$.*

The proof of this result goes as follows. First observe that if $u \in \operatorname{Ker} A_\delta$, then, according to the result of Proposition 3.1.1, $u \in \operatorname{Ker} \mathcal{A}_\delta$ and hence $|u| \leq c\,\gamma^{\frac{2-n}{2}+\delta}$ and $|\nabla u| \leq c\,\gamma^{-\frac{n}{2}+\delta}$. Using these, once can show that the function i is a solution in the sense of distributions of the equation

$$\Delta_{S^n}\, u + \lambda\, u = 0$$

Regularity theory then implies that $u \in \mathcal{C}^\infty(S^n)$, and hence $u \equiv 0$.

Exercise 3.2.1. *Prove that if the function u solution of $\Delta_{S^n} u + \lambda\, u = 0$ in $S^n *$ satisfies $|u| \leq c\,\gamma^{\frac{2-n}{2}+\delta}$ and $|\nabla u| \leq c\,\gamma^{-\frac{n}{2}+\delta}$, then u is a solution of $\Delta_{S^n} u + \lambda\, u = 0$ in S^n, in the sense of distributions.*

The consequences of this result are :

Corollary 3.2.1. *Assume that λ is not an eigenvalue of $-\Delta_{S^n}$, then the operator $\mathcal{A}_\delta$ is an isomorphism for all $|\delta| < \frac{n-2}{2}$.*

Exercise 3.2.2. *If λ is not an eigenvalue of $-\Delta_{S^n}$, then the operator $\mathcal{A}_\delta$ is surjective for $\delta \in (-\frac{n}{2} - j, \frac{2-n}{2} - j)$, for all $j \in \mathbb{N}$. Characterize the kernel of this operator according to the value of j.*

Let us now assume that $\lambda = 0$. The result of Lemma 3.2.1 is now changed into :

Lemma 3.2.2. *The operators A_δ and $\mathcal{A}_\delta$ are injective for all $\delta > \frac{n-2}{2}$.*

Exercise 3.2.3. *The operator $\mathcal{A}_\delta$ is also surjective for $\delta \in (-\frac{n}{2} - j, \frac{2-n}{2} - j)$, for all $j \in \mathbb{N}$. Characterize the kernel of this operator according to the value of j.*

Let us now assume that $\lambda = n$, the second eigenvalue of $-\Delta_{S^n}$. Then, following similar arguments, we have the :

Lemma 3.2.3. *The operators A_δ and $\mathcal{A}_\delta$ are injective for all $\delta > \frac{n+2}{2}$.*

The proof of this result proceeds as before, fist we show that any element in the

Lemma 3.2.4. *Assume that Span$\{p_1, \ldots, p_k\} = \mathbb{R}^{n+1}$. Then the operators A_δ and $\mathcal{A}_\delta$ are injective for all $\delta > \frac{n-2}{2}$.*

Exercise 3.2.4. *The operator $\mathcal{A}_\delta$ is also surjective for $\delta \in (-\frac{n}{2} - j, \frac{2-n}{2} - j)$, for all $j \in \mathbb{N}$. Characterize the kernel of this operator according to the value of j.*

Chapter 4

Analysis on ALE spaces

4.1 Asymptotically Locally Euclidean spaces

We will say that a complete, noncompact n-dimensional manifold (M, g) is an ALE space (Asymptotically Locally Euclidean space) if it can be decomposed into the union of a compact piece $K \subset\subset M$ and finitely many ends $\mathcal{E}_1, \ldots, \mathcal{E}_k$ which are diffeomorphic to the complement of a ball in $\mathbb{R}^n$ and on which there exists coordinates $(x^1, \ldots, x^n)$ in which the coefficients of the metric g satisfy

$$g_{ij} = \delta_{ij} + \mathcal{O}(|x|^{-\alpha})$$

and

$$\nabla^\ell g_{ij} = \mathcal{O}(|x|^{-\alpha-\ell})$$

for some $\alpha > 0$.

We would like to study operators of the form

$$\Delta_g + a$$

where the function $a = M \longrightarrow \mathbb{R}$ satisfies

$$\nabla^\ell a = \mathcal{O}(|x|^{-2-\beta-\ell})$$

for some $\beta > 0$, on each end $\mathcal{E}_j$.

We define a smooth positive function $\gamma : M \longrightarrow (0, \infty)$ which coincides with $|x|$ on each end $\mathcal{E}_j$ of M. As in the case of a punctured manifold, we defined weighted L^2-spaces and weighted Hölder spaces by defining the norms in these spaces as follows

$$\|u\|_{L^2_\delta(M)} := \left(\int_M |u|^2 \, \gamma^{-2\delta-2} \, dvol_g\right)^{1/2}$$

and

$$\|u\|_{\mathcal{C}^{\ell,\alpha}_\delta(M)} := \|u\|_{\mathcal{C}^{\ell,\alpha}(K)} + \sum_j \sup_{r\geq R} r^{\frac{n-2}{2}-\delta} \, \|u(\rho\,\cdot)\|_{\mathcal{C}^{\ell,\alpha}(\bar{B}_1 - B_{1/2})}$$

As before we define the unbounded operator

$$\begin{array}{rccc} A_\delta : & L^2_\delta(M^*) & \longrightarrow & L^2_\delta(M^*) \\ & u & \longmapsto & \gamma^2\,(\Delta_g\, u + a\, u) \end{array}$$

as well as the bounded operator

$$\begin{array}{rccc} \mathcal{A}_\delta : & C^{2,\alpha}_\delta(M^*) & \longrightarrow & \mathcal{C}^{0,\alpha}_\delta(M^*) \\ & u & \longmapsto & \gamma^2\,(\Delta_g\, u + a\, u) \end{array}$$

The key remark is that, if u and f solve

$$|x|^2\,\Delta u = f$$

in B_1^* then, setting

$$v(x) = |x|^{2-n}\, u(x/|x|^2) \qquad \text{and} \qquad g(x) = |x|^{2-n}\, f(x/|x|^2)$$

one can check that

$$|x|^2\,\Delta v = f$$

in $\mathbb{R}^n - \bar{B}_1^*$.

Further observe that

$$\int_{B_1} u^2(x)\,|x|^{-2\delta-2}\,dx = \int_{\mathbb{R}^n - \bar{B}_1} v^2(y)\,|y|^{2\delta-2}\,dy$$

and

$$\|u\|_{\mathcal{C}^{\ell,\alpha}_\delta(\bar{B}_1^*)} = \|v\|_{\mathcal{C}^{\ell,\alpha}_{-\delta}(\mathbb{R}^n - B_1^*)}$$

These remarks allow one to extend all the previous results on a punctured manifold to this noncompact complete setting. We leave the details to the reader.

In particular, we have the :

Proposition 4.1.1. *Given $\delta < \delta_0$, $\delta \neq -\delta_j$, for all $j \in \mathbb{N}$, let us assume that A_δ is injective, then the operator*

$$\begin{array}{rccc} \tilde{\mathcal{A}}_\delta : & \mathcal{C}^{\ell,\alpha}_\delta(M^*) \oplus D_\delta & \longrightarrow & \mathcal{C}^{\ell,\alpha}_\delta(M^*) \\ & u & \longrightarrow & \gamma^2\,(\Delta_g u + au) \end{array}$$

is well defined, bounded and surjective. In addition dim Ker$(\tilde{\mathcal{A}}_\delta) = \frac{1}{2}$ dimD_δ.

As usual the deficiency space is defined by

$$D_\delta := \mathrm{Span}\,\{\chi^{(i)}\, W^{\pm(i)}_{j,\phi}, \qquad : \qquad \phi \in E_j, \qquad -\delta < \pm\delta_j < \delta\}$$

where the functions $W^{\pm(i)}_{j,\psi}$ are solutions of $\gamma^2\,(\Delta_g + a)\, W^{\pm(i)}_{j,\phi} = 0$ on $\mathcal{E}_i$ and satisfy

$$W^{\pm(i)}_{j,\phi} = |x|^{\frac{2-n}{2}\pm\delta_j}\,\phi + \mathcal{O}(|x|^{\frac{2-n}{2}\pm\delta_j-\eta})$$

at infinity, for some $\eta > 0$.

Observe that when $\delta \in (-\delta_0, \delta_0)$, then D_δ is empty and hence, under the assumption of Proposition 4.1.1, the operator $\mathcal{A}_{\delta'}$ is an isomorphism, for any '$\delta' \in (-\delta_0, \delta_0)$.

4.2 An example from conformal geometry

Consider, in dimension $n \geq 3$ the semilinear elliptic equation

$$\Delta u + \frac{n(n-2)}{4}\, u^{\frac{n+2}{n-2}} = 0 \tag{4.1}$$

where $u > 0$ in $\mathbb{R}^n$. We set

$$u_0(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}}$$

The reader should check that u_0 is a solution of (4.1). The operator we would like to study is the linearised operator about the solution u_0. Namely

$$\mathcal{L} := \Delta + \frac{n(n+2)}{(1+|x|^2)^2}$$

Observe that we are precisely in the setting described in the previous section. We define a function γ which is positive and coincides with $|x|$ on the complement of the unit ball and the unbounded operator

$$\begin{array}{rccl} A_\delta : & L^2_\delta(M^*) & \longrightarrow & L^2_\delta(M^*) \\ & u & \longmapsto & \gamma^2\,(\Delta\, u + \frac{n(n+2)}{(1+|x|^2)^2}\, u) \end{array}$$

We prove the :

Lemma 4.2.1. *The operator A_δ is injective provided $\delta < -\frac{n}{2}$*

The proof goes as follows. Any solution $w \in L^2_\delta(\mathbb{R}^n)$ of $A_\delta\, w = 0$ belongs to $\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^n)$. We then proceed to the eigenfunction decomposition of a solution w of $A_\delta\, w = 0$ as

$$w = \sum_j w_j$$

First let us show that $w_0 \equiv 0$. The idea is that the function w_0 satisfies a second order ordinary differential equation

$$\left(\partial_r^2 + \frac{n-1}{r}\,\partial_r + \frac{n(n+2)}{4}\,\frac{4}{(1+|x|^2)^2}\right) w_0 = 0 \tag{4.2}$$

but, we know an explicit solution of this homogeneous equation namely

$$\tilde{w}_0 = \frac{n-2}{2}\,u_0 + r\,\partial_r u_0$$

The fact that $\tilde{w}_0$ is also a solution of this equation either follows from direct computation or from the observation that (4.1) is invariant under scaling in the sense that, whenever u is a solution of (4.1) then so does

$$u_\alpha(x) = \alpha^{\frac{n-2}{2}}\,u(\alpha x)$$

for all $\alpha > 0$. Taking the derivative with respect to α when $\alpha = 1$ in

$$\Delta u_\alpha + \frac{n(n-2)}{4}\,u_\alpha^{\frac{n+2}{n-2}} = 0$$

we conclude that

$$\Delta(\partial_\alpha u_\alpha) + \frac{n(n+2)}{4}\,u_\alpha^{\frac{4}{n-2}}\,(\partial_\alpha u_\alpha) = 0$$

Just observe that $\tilde{w}_0 = \partial_\alpha u_\alpha|_{\alpha=1}$. It turns out that the other independent solution of (4.2) but this solution blows up at the origin like a constant times r^{2-n} and tends at infinity to some constant. This shows that w_0 has to be a multiple of $\tilde{w}_0$. But the function $\tilde{w}_0$ is asymptotic to a constant times r^{2-n} at infinity and is certainly not bounded by a constant times $r^{\frac{2-n}{2}-\delta}$ when $\delta < \frac{2-n}{2}$. Therefore, we conclude that $w_0 = 0$.

The fact that $w_1 \equiv 0$ follows from a similar argument. This time we use the invariance of the problem under translations. The E_1 valued function w_1 satisfies a second order ordinary differential equation

$$\left(\partial_r^2 + \frac{n-1}{r}\,\partial_r - \frac{n-1}{r} + \frac{n(n+2)}{4}\,\frac{4}{(1+|x|^2)^2}\right) w_1 = 0 \tag{4.3}$$

but, we know n explicit solutions of this homogeneous equation namely

$$\tilde{w}_1^{(j)} = \partial_{x^j} u_0$$

for $j = 1, \ldots, n$. The fact that $\tilde{w}_1^{(j)}$ are solutions of this equation either follows from direct computation or from the observation that (4.1) is invariant under translation in the sense that, whenever u is a solution of (4.1) then so does $u(\cdot + \alpha\, e_j)$. Further observe from the definition of u_0 that $w_1^{(j)}$ decay like r^{1-n} at infinity and hence are not bounded by a constant times $r^{\frac{2-n}{2}-\delta}$ when $\delta < -\frac{n}{2}$. As above we conclude easily that, on the one hand, w_1 being smooth at the origin it has to be a linear combination of the $\tilde{w}_1^{(j)}$ and on the other hand does not have the right decay at infinity unless it is identically equal to 0. This shows that $w_1 = 0$

It remains to show that $w_j = 0$ for all $j \geq 2$. this time an explicit solution of the corresponding ordinary differential equation is not available. However, using Hardy's inequality, we can show that indeed $w_j = 0$. The proof is very similar to what we have already done in Part 1 of the proof of Proposition 1.5.1. We set

$$w_j(r, \theta) = \bar{w}_j(r)\, \phi(\theta)$$

for some $\phi \in E_j$. The scalar function w_j satisfies

$$\left(\partial_r^2 + \frac{n-1}{r}\, \partial_r - \frac{j(n-2+j)}{r} + \frac{n(n+2)}{4}\, \frac{4}{(1+|x|^2)^2} \right) \bar{w}_j = 0$$

Multiply this equation by $r^{n-1}\, \bar{w}_j$ and integrate by parts over $(0, \infty)$ to conclude that

$$\int_0^\infty |\partial_r \bar{w}_j|^2\, r^{n-1}\, dr + j(n-2+j) \int_0^\infty |\bar{w}_j|^2\, r^{n-3}\, dr = \frac{n(n+2}{4} \int_0^\infty \frac{4r^2}{(1+r^2)^2}\, \bar{|}w_j|^2\, r^{n-1}\, dr$$

But,

$$\frac{4r^2}{(1+r^2)^2} \leq 1$$

and hence we get the inequality

$$\int_0^\infty |\partial_r \bar{w}_j|^2\, r^{n-1}\, dr + j(n-2+j) \int_0^\infty |\bar{w}_j|^2\, r^{n-3}\, dr \leq \frac{n(n+2}{4} \int_0^\infty |\bar{w}_j|^2\, r^{n-1}\, dr$$

Hardy's inequality reads

$$(n-2)^2 \int_0^\infty |\bar{w}_j|^2\, r^{n-3}\, dr \leq 4 \int_0^\infty |\partial_r \bar{w}_j|^2\, r^{n-1}\, dr$$

and hence we conclude that

$$\left(\frac{n-2}{2} + j \right)^2 \int_0^\infty |\bar{w}_j|^2\, r^{n-3}\, dr \leq \frac{n(n+2)}{4} \int_0^\infty |\bar{w}_j|^2\, r^{n-1}\, dr$$

which precisely implies that $\bar{w}_j = 0$ provided $j \geq 2$.

Exercise 4.2.1. *Justify the integrations by parts in the last proof, using the fact that $w \in L^2_\delta(\mathbb{R}^n)$ and hence $w \in \mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^n)$.*

Exercise 4.2.2. *Extend the previous analysis to the semilinear elliptic equation*

$$\Delta u + 8\, e^{u} = 0$$

where this time the explicit solution is given by

$$u_0(x) = -2 \log(1 + |x|^2)$$

It will be useful to observe that the equation is invariant under the following transformation

$$u_\alpha(x) = 2 \log \alpha + u(\alpha\, x)$$

for $\alpha > 0$.

Chapter 5

Mean curvature of hypersurfaces

5.1 The mean curvature

Assume that $\Sigma \subset \mathbb{R}^{n+1}$ is an oriented hypersurface. We denote by N the unit normal vector field on Σ which is compatible with the orientation. Given a small (smooth) function w with compact support defined on Σ, we define Σ_w to be the image of Σ by

$$p \longmapsto p + w(p)\, N(p)$$

We can describe Σ locally using a local chart. Let $(x^1, \ldots, x^n)$ be local coordinates and X parameterizes locally Σ. The first fundamental form (also referred to as the induced metric) on Σ is defined by

$$g(T_1, T_2) := T_1 \cdot T_2$$

for any tangent vectors $T_1, T_2 \in T_p\Sigma$. In the above parametrization, a basis of the tangent space at p is given by

$$\partial_{x^1} X, \ldots, \partial_{x^n} X$$

and, in this basis, the coefficients g_{ij} of the induced metric on Σ read

$$g_{ij} := \partial_{x^i} X \cdot \partial_{x^j} X$$

We denote by

$$N : \Sigma \longrightarrow S^{n-1}$$

the Gauss map ($N(p)$ is nothing but the normal vector field to Σ) and by

$$DN_p : T_p\Sigma \longrightarrow T_{N(p)} S^{n-1}$$

its differential at p.

Exercise 5.1.1. *Check that, after having identified $T_p\Sigma$ with $T_{N(p)}S^{n-1}$, the mapping D_pN is a symmetric endomorphism (i.e. $D_pN(T_1) \cdot T_2 = T_1 \cdot D_pN(T_2)$, for all $T_1, T_2 \in T_p\Sigma$).*

The second fundamental form h_Σ is defined by

$$h_\Sigma(T_1, T_2) = -T_1 \cdot D_pN(T_2)$$

for any tangent vectors $T_1, T_2 \in T_p\Sigma$. In the above parametrization, the coefficients of the second fundamental form are given by

$$h_{ij} := -\,\partial_{x^i} X \cdot \partial_{x^j} N$$

where we have identified $N \circ X$ with N so that, with slight abuse of notations, $\partial_{x^i} N$ is in fact equal to $D_X N(\partial_{x^i} X)$. It will be convenient to observe that

$$\partial_{x_i} N = \sum_j h_{ij}\, g^{jk}\, X_k$$

where $(g^{ij})_{ij}$ denotes the inverse of $(g_{ij})_{ij}$. We leave this as an exercise in linear algebra (simply observe that $N \cdot N \equiv 1$ and hence $\partial_{x^i} N \cdot N \equiv 0$.

The tangent vector fields to Σ_w are then given by

$$\partial_j Y = \partial_j X_j + \partial_j w\, N + w\, \partial_j N$$

Hence the coefficients $\tilde{g}_{ij}$ of the induced metric on Σ_w are given by

$$\tilde{g}_{ij} := \partial_{x^i} Y \cdot \partial_{x^j} Y == g_{ij} - 2\, h_{ij}\, w + \partial_{x^i} w\, \partial_{x^j} w + a_{ij}\, w^2$$

where we have defined

$$a_{ij} := h_{ik}, g^{k\ell}\, h_{\ell j}$$

Using the classical expansion

$$\det(I + B) = 1 + \mathrm{Tr}(B) + \tfrac{1}{2}\, \left((\mathrm{Tr} B)^2 - \mathrm{Tr}\, B^2\right) + \mathcal{O}(\|B\|^3)$$

for any (small) square matrix B, we find the expansion

$$\sqrt{\det \tilde{g}} = \Big(1 - \mathrm{Tr} A + \tfrac{1}{2}\, (|\nabla w|_g^2 - \mathrm{Tr}(A^2)\, w^2 + (\mathrm{Tr} A)^2\, w^2) + \mathcal{O}(\|w\|^3_{\mathcal{C}^1})\Big)\, \sqrt{\det g}$$

where we have defined $A := -DN$.

We now assume that w is small and has compact support in Σ which is bounded (but not necessarily closed) so that we can compute the n-dimensional volume of Σ_w. The expansion of $\mathrm{Vol}_n(\Sigma_w)$ in powers of w is given by

$$\mathrm{Vol}_n(\Sigma_w) = \int_\Sigma \Big(1 - \mathrm{Tr} A + \tfrac{1}{2}\, (|\nabla w|_g^2 - \mathrm{Tr}(A^2)\, w^2 + (\mathrm{Tr} A)^2\, w^2) + \mathcal{O}(\|w\|^3_{\mathcal{C}^1})\Big)\, dvol_\Sigma$$

In particular, we conclude that the first and second variations of the n dimensional volume form are given by

$$D_w \mathrm{Vol}_n(\Sigma_w)_{|w=0}(v) = -\int_\Sigma \mathrm{Tr} A \, v \, dvol_\Sigma$$

and

$$D_w^2 \mathrm{Vol}_n(\Sigma_w)_{|w=0}(v,v) = \int_\Sigma \left(|\nabla v|_g^2 - \mathrm{Tr}(A^2)\, v^2 + (\mathrm{Tr} A)^2 \, v^2 \right) \, dvol_\Sigma \tag{5.1}$$

The trace of the operator A, which appears in the first variation of the n-dimensional volume, is called the mean curvature of the hypersurface Σ and is denoted by $H(\Sigma)$. In local coordinates

$$H(\Sigma) = \sum_{ij} h_{ij} g^{ji}$$

Exercise 5.1.2. *Show that the mean curvature of $S^n(r)$ the sphere of radius $r > 0$ in $\mathbb{R}^{n+1}$ is equal to $\frac{n}{r}$ when the normal vector is inward pointing. It will be useful to observe that the Gauss map is given by $N(p) = -\frac{1}{r}\, p$, for all $p \in S^n(r)$.*

Exercise 5.1.3. *Show that the mean curvature of $S^{n_1}(r) \times \mathbb{R}^{n_2}$ in $\mathbb{R}^{n_1+n_2+1}$ is equal to $\frac{n_1}{r}$ when the normal vector is inward pointing.*

5.2 Jacobi operator and Jacobi fields

The second order partial differential operator which appears in the second variation of the n-dimensional volume, is called the Jacobi operator

$$J_\Sigma = \Delta_g + \mathrm{Tr}(A^2)$$

This corresponds to the linearized mean curvature operator, that is the linearization of the mapping

$$\tilde{H}(w) := H(\Sigma_w)$$

Indeed, by definition of the mean curvature, the first variation of the n-dimensional volume is given by

$$D_w \mathrm{Vol}_n(\Sigma)(v) = -\int_\Sigma H(\Sigma) \, v \, dvol_\Sigma$$

And hence, the second variation of the n-dimensional volume is given by

$$D_w^2 \mathrm{Vol}_n(\Sigma)(v_1, v_2) = -\int_\Sigma D_w \tilde{H}_\Sigma(v_1) \, v_2 \, dvol_\Sigma + \int_\Sigma H(\Sigma)^2 \, v_1 \, v_2 \, dvol_\Sigma$$

Comparing this formula with (5.1) we conclude that

$$\tilde{H}_\Sigma(v) = J_\Sigma \, v$$

Exercise 5.2.1. *Show that the Jacobi operator about $S^n(r)$ in $\mathbb{R}^{n+1}$ is equal to $J = r^2\,(\Delta_{S^n} + n)$.*

The last object we need to introduce are called *Jacobi fields* they are nothing but so solutions of the homogeneous problem $J_\Sigma\, w = 0$. We describe some general recipe which allows one to find explicit Jacobi fields. Assume we are given a vector field Ξ in $\mathbb{R}^{n+1}$. We denote by $\Phi_\xi(s;\,\cdot\,)$ the associated flow, namely for all $x \in \mathbb{R}^{n+1}$ $\Phi_\Xi(s;x)$ is the solution at time s of the dynamical system

$$\frac{dy}{ds} = \Xi(y)$$

with $y(0) = x$.

For s small enough and $p \in \Sigma$, we can write

$$\Phi_\Xi(s;p) = q(s;p) + w(s;p)\,N(s;p)$$

Using the fact that

$$(q,t) \longrightarrow q + t\,N(q)$$

is a local diffeomorphism from a neighborhood of $(p,0)$ in $\Sigma \times \mathbb{R}$ to a neighborhood of $p \in \mathbb{R}^{n+1}$, it is easy to see that

$$\partial_s q(0,p) = \Pi_{T_p\Sigma} N(p)$$

where $\Pi_{T_p\Sigma}$ denotes the orthogonal projection over $T_p\Sigma$ and

$$\partial_s w(0;p) = \Xi(p) \cdot N(p)$$

We define $\Sigma(s)$ to be the image of Σ by $\Phi_\Xi(s,\,\cdot\,)$. Then, we have the general formula

$$\partial_s H(\Sigma(s))|_{s=0} = D_w \tilde{H}_\Sigma(\Xi \cdot N) + \nabla H_\Sigma \cdot \Pi_{T_p\Sigma} N \tag{5.2}$$

This formula is very useful when the hypersurface Σ has constant mean curvature in which case (5.2) reduces to

$$\partial_s H(\Sigma(s))|_{s=0} = D_w \tilde{H}_\Sigma(\Xi \cdot N)$$

There are 3 families of vector fields which will be useful for us : The vector fields associated to translations which are simply constant vector fields

$$\Xi_{t,e}(x) = e$$

The vector fields associated to rotations which are given by

$$\Xi_{r,A}(x) = A\,x$$

where $A \in M_{n+1}(\mathbb{R})$ is skew symmetric ${}^tA = -A$. Finally the vector field associated to the dilation centered at the origin is given by

$$\Xi_d(x) = x$$

We have proved the

Proposition 5.2.1. *Assume that Σ is a constant mean curvature hypersurface. Then $N \cdot X_t$, $N \cdot X_r$ are Jacobi fields. If in addition the mean curvature of Σ is equal to 0 then $N \cdot X_d$ is also a Jacobi field on Σ.*

Indeed, when $\Xi = \Xi_{t,e}$ or $\Xi = \Xi_{r,A}$, then $\Phi_\Xi(s;\,\cdot\,)$ is an isometry. Hence $\partial_s H(\Sigma(s))|_{s=0} = 0$ in this case, which immediately implies that $D_w \tilde{H}_\Sigma(\Xi \cdot N) = 0$ if the mean curvature of Σ is constant. When $\Xi = \Xi_d$ then $\Phi_\Xi(s;\,\cdot\,) = e^s\, Id$ and hence $\partial_s H(\Sigma(s))|_{s=0} = \partial_s e^{-s}|_{s=0} H(\Sigma) = -H(\sigma)$ in this case. We conclude that

$$D_w \tilde{H}_\Sigma(\Xi_d \cdot N) = -H(\Sigma)$$

This completes the proof of the Lemma.

Definition 5.2.1. *We will say that Σ is a constant mean curvature hypersurface is $H(\Sigma)$ is a constant function on Σ. We will say that Σ is a minimal hypersurface if $H(\Sigma) = 0$ on Σ.*

There is a nice variational characterization of both minimal and constant mean curvature hypersurfaces. Indeed, it follows from the above considerations that minimal hypersurfaces are critical points of the n-dimensional volume functional while constant mean curvature hypersurfaces are critical points of the n-dimensional volume functional with some $(n+1)$-dimensional volume constraint (this amounts to consider perturbation which involve functions whose mean over Σ is zero.

Chapter 6

Minimal hypersurfaces with catenoidal ends

6.1 The n-catenoid

The n-catenoid C is a minimal hypersurface of revolution about the x_{n+1}-axis. It will be convenient to consider a parametrization

$$X : \mathbb{R} \times S^{n-1} \longrightarrow \mathbb{R}^{n+1}$$

of C for which the induced metric is conformal to the product metric on $\mathbb{R} \times S^{n-1}$. This parametrization is given by

$$X(t, z) := \big(\varphi(t)\, z\,, \psi(t)\big), \tag{6.1}$$

where $t \in \mathbb{R}$, $z \in S^{n-1}$ and where the functions φ and ψ are explicitly given by

$$\varphi(t) := (\cosh((n-1)t))^{\frac{1}{n-1}} \qquad \text{and} \qquad \psi(t) := \int_0^t \varphi^{2-n}\, ds.$$

It is easy to check that the induced metric on C is given by

$$g := \varphi^2\, (dt^2 + g_{S^{n-1}})$$

and, if the orientation of C is chosen so that the unit normal vector field is given by

$$N := \big(-\varphi^{1-n}\, z, \partial_t\, \ln \varphi\big), \tag{6.2}$$

then, the second fundamental form h about C is given by

$$h := \varphi^{2-n}\, \big((1-n)\, dt^2 + g_{S^{n-1}}\big)\,.$$

From these expressions, it is easy to check that the hypersurface parameterized by X is indeed minimal.

The Jacobi operator about the n-catenoid is given, in the above defined parametrization, by

$$J_C = \frac{1}{\varphi^n}\, \partial_t\, \left(\varphi^{n-2}\, \partial_t\, \cdot\right) + \frac{1}{\varphi^2}\, \Delta_{S^{n-1}} + n\,(n-1)\, \frac{1}{\varphi^{2n}}$$

We apply the above recipe to obtain globally defined Jacobi fields

(i) The function $\Phi_0^- := \varphi^{-1}\, \partial_t \varphi$, which is associated to the translation of C along its axis.

(ii) The function $\Phi_0^+ := \varphi^{-1}\left(\varphi\, \partial_t \psi - \psi\, \partial_t \varphi\right)$, which is associated to the dilation of C,

(iii) The functions $\Phi_{1,e}^- := \varphi^{1-n}\,(z\,\cdot\, e)$ for $e \in \mathbb{R}^n \times \{0\}$, which is associated to the translation of C along the direction e orthogonal to its axis.

(iv) The functions $\Phi_{1,e}^+ := \varphi^{-1}\,(\psi\, \partial_t \psi + \varphi\, \partial_t \varphi)\,(z\,\cdot\, e)$ for $e \in \mathbb{R}^n \times \{0\}$, which is associated to the rotation of the axis of C in a direction e orthogonal to its axis.

The interested reader should check that these constitute $2(n+1)$ linearly independent Jacobi fields.

We claim that the n-catenoid is an ALE space as defined in Chapter 4. Indeed, we change variables and write

$$x = \varphi(t)\, z$$

for $t \geq 0$ and $z \in S^{n-1}$. So that the upper end of the n-catenoid can now be parameterized as a vertical graph over the horizontal hyperplane $x^{n+1} = 0$ for a function u_C. It is a simple exercise to check that, at infinity, the function u_C can be expanded as

$$u_C(x) = \log|x| + \log 2 + \mathcal{O}(|x|^{-2})$$

when $n = 2$ and

$$u_C(x) = u_C(\infty) - \tfrac{1}{n-2}\, |x|^{2-n} + \mathcal{O}(|x|^{4-2n})$$

In addition, in these coordinates, the metric g can be expanded as

$$g = g_{eucl} + \mathcal{O}(|x|^{2-2n})$$

and the potential in the Jacobi operator satisfies

$$\mathrm{Tr}(A^2) = \mathcal{O}(|x|^{-2n})$$

We are just in the situation described in Chapter 4.

6.2 Unmarked space of minimal hypersurfaces with catenoidal ends

We now defined the set of hypersurfaces we are interested in. The *unmarked space* M_u is the set of minimal hypersurfaces which have finitely many ends asymptotic to a properly rescaled, translated and rotated n-catenoid. More precisely, this means that each element of M_u can be decomposed into the union of a compact piece and finitely many ends which (up to a translation, a rotation and a dilation) can be written as a normal graph over one end of the n-catenoid for a function which decays like a constant times $|x|^{\frac{2-n}{2}+\delta}$ for some $\delta \in (-\frac{n+2}{2}, -\frac{n}{2})$.

We have the important :

Definition 6.2.1. *An element $\Sigma \in M_u$ is said to be unmarked-nondegenerate if the operator*

$$A_\delta := \gamma^2 \left(\Delta_\Sigma + Tr(A^2)\right)$$

is injective on $L^2_\delta(\Sigma)$ for all $\delta < -\frac{n}{2}$.

As usual, the function γ is a

For the time being we do not have many example of minimal hypersurface with catenoidal ends, except the n-catenoid itself. We here prove that this hypersurface is unmarked-nondegenerate.

Lemma 6.2.1. *Assume that $\delta < -\frac{n}{2}$. Let $w \in L^2_\delta(\Sigma)$ be a solution of $A_\delta \, w = 0$ then $w \equiv 0$.*

A simple proof of this result can be obtained as follows. Proceed with the eigenfunction decomposition of w,

$$w(t,z) = \sum_j w_j(t)$$

where $w_j(t,\cdot) \in E_j$. Observe that w_j is a solution of

$$\frac{1}{\varphi^n} \, \partial_t \, \left(\varphi^{n-2} \, \partial_t \, w_j\right) - \frac{\lambda_j}{\varphi^2} \, w_j + n\,(n-1) \, \frac{1}{\varphi^{2n}} \, w_j = 0$$

which is bounded by a constant times $(\cosh t)^\delta$. When $j = 0$ (resp. when $j = 1$) then all solutions are explicitly known and are described above, therefore w_0 (resp. w_j) is a linear combination of the Jacobi fields $\Phi_0^\pm$ (resp. $\Phi_{1,e}^\pm$, for $e \in \mathbb{R}^n$). It is easy to check that no such solution is bounded by a constant times $(\cosh t)^\delta$ unless it is identically equal to 0 since we have chosen $\delta < -\frac{n}{2}$. Therefore, $w_0 = w_1 = 0$.

Now, when $j \geq 2$ we write $w_j(t,z) = v_j(t) \, \phi_j(z)$ where $\phi_j \in E_j$. Observe that the function v_j being bounded by a constant times $(\cosh t)^\delta$ for $\delta < -\frac{n}{2}$ has to decay at infinity like $(\cosh t)^{-\delta_j}$. Then, we can write

$$\frac{1}{\varphi^n} \, \partial_t \, \left(\varphi^{n-2} \, \partial_t \, v_j\right) - \frac{\lambda_j}{\varphi^2} \, v_j + n\,(n-1) \, \frac{1}{\varphi^{2n}} \, v_j = 0$$

Define $v_1 = \varphi^{1-n}$. Using the fact that $\Phi^+_{1,e}$ is a Jacobi field, we conclude that

$$\frac{1}{\varphi^n}\,\partial_t\,\left(\varphi^{n-2}\,\partial_t\,v_1\right) - \frac{\lambda_1}{\varphi^2}\,v_j + n\,(n-1)\,\frac{1}{\varphi^{2n}}\,v_1 = 0$$

For all $s \in \mathbb{R}$, we set $v^{(s)} := v_1 - s\,v_j$. Using the above equations, we have

$$\frac{1}{\varphi^n}\,\partial_t\,\left(\varphi^{n-2}\,\partial_t\,v^{(s)}\right) - \frac{\lambda_j}{\varphi^2}\,v^{(s)} + n\,(n-1)\,\frac{1}{\varphi^{2n}}\,v^{(s)} = -(\lambda_1 - \lambda_j)\,\varphi^{-2}\,v^{(s)} \tag{6.3}$$

For all $s \in \mathbb{R}$, v_s is positive near $\pm\infty$ (because the function v_j tends to 0 at $\pm\infty$ much faster than the function v_1). We choose s to be the sup of the reals for which $v_s \geq 0$. Then v_s vanishes in $\mathbb{R}$ and at this point, which is a minimum point for v_s, (6.3) yields $\partial_t^2\,v_s < 0$. A contradiction. This completes the proof of the result.

The main result of this Chapter states that, if Σ is unmarked nondegenerate, then there exists an open manifold of dimension $k(n+1)$ which contains Σ and is included in M_u, where k is the number of catenoidal ends of Σ.

The proof of the result is an almost simple consequence of the implicit function theorem. To prove this result, we apply the result of Proposition 3.1.3 with $\delta \in (-\frac{n+2}{2}, -\frac{n}{2})$. Since Σ is unmarked-nondegenerate, this yields the :

Proposition 6.2.1. *The operator*

$$\begin{array}{rlcl} \tilde{\mathcal{A}}_\delta : & \mathcal{C}^{2,\alpha}_\delta(M^*) \oplus D_\delta & \longrightarrow & \mathcal{C}^{0,\alpha}_\delta(M^*) \\ & u & \longrightarrow & \gamma^2\,(\Delta_\Sigma u + Tr(A)^2 u) \end{array}$$

is well defined, bounded and surjective. In addition $\dim Ker(\tilde{\mathcal{A}}_\delta) = k\,(n+1)$, *where* k *is the number of ends of* Σ.

where we recall that the deficiency space D_δ is given by

$$D_\delta := \mathrm{Span}\,\{\chi^{(i)}\,W^{\pm(i)}_{j,\phi}, \qquad : \qquad \phi \in E_j, \qquad j = 0,1\}$$

recall that the functions $W^{\pm(i)}_{j,\phi}$ are constructed in such a way that $\gamma^2\,(\delta_\Sigma + \mathrm{Tr}(A)^2) W^{\pm(i)}_{j,\phi} = 0$ near the i-th end and also that $W^{\pm(i)}_{j,\phi}$ is asymptotic to $|x|^{\frac{2-n}{2}\pm\delta_j}\,\phi$. Observe that, in the statement of Proposition 6.2.1 one can replace D_δ by

$$\tilde{D}_\delta := \mathrm{Span}\,\{\chi^{(i)}\,\Phi^{\pm(i)}_0, \chi^{(i)}\,\Phi^{\pm(i)}_{1,e} \qquad : \qquad e \in \mathbb{R}^n\}$$

where $\Phi^{\pm(i)}_0$ and $\Phi^{\pm(i)}_{1,e}$ are the Jacobi fields associated to dilations, translations and rotations of the i-th end of the hypersurface Σ.

We would like to apply the implicit function theorem to some nonlinear mapping $\mathcal{N}$ defined on a neighborhood of 0 in $\mathcal{C}^{2,\alpha}_\delta(M^*) \oplus D_\delta$ and whose differential at 0 is the operator $\tilde{\mathcal{A}}_\delta$. Assuming this nonlinear operator is already obtained, we have immediately that the dimension of the zero set of this operator is equal to the dimension of the kernel of $\tilde{\mathcal{A}}_\delta$. Hence it is equal to $k\,(n+1)$, where k is the number of ends of Σ. This yields the existence of a smooth $k\,(n+1)$-dimensional family of minimal hypersurfaces with catenoidal ends, which contains Σ and is embedded in M_u when Σ is unmarked nondegenerate.

In order to define the nonlinear mapping $\mathcal{N}$, we first observe that any element $w \in D_\delta$ is associated to some hypersurface which can be constructed by moving slightly ends ends of Σ. To make things precise let us consider the elements of D_δ which are supported on $\mathcal{E}_i$. They are of the form

$$a_0^+\,\Phi_0^{+(i)} + a_0^-\,\Phi_0^{-(i)} + a_{1,e}^+\,\Phi_{1,e}^{+(i)} + a_{1,\tilde{e}}^-\,\Phi_{1,\tilde{e}}^{-(i)}$$

where $e, \tilde{e} \in \mathbb{R}^n$ and are normalized to have unit norm. We consider the end $\mathbb{E}_i$ which we translate along its axis by a_0^+, which we dilate by $(1+a_0^-)$, which we translate in the direction e orthogonal to is axis by $a_{1,e}$ and whose axis we rotate in the direction $\tilde{e}$ b an angle $a_{1,\tilde{e}}$. This gives an end of a minimal hypersurface which is asymptotic to a catenoidal end and which can be smoothly connected to Σ. A similar construction can be performed for all other ends. We obtain some embedding

$$I_w : \Sigma \longrightarrow \Sigma_w$$

for all w in some neighborhood of 0 in D_δ. Observe that, by construction,the mean curvature of Σ_w is equal to 0 except on some compact pieces where the perturbed end is connected to the initial end.

Next, to any element $u = v + w_i n\mathcal{C}_\delta^{2,\alpha}(\Sigma) \times D_\delta$ we first construct the hypersurface Σ_w corresponding to the component w of u which belongs to D_δ and then take the normal graph over it for the function $v \circ I_w^{-1}$ corresponding to the component of u belonging to $\mathcal{C}_\delta^{2,\alpha}(\Sigma)$. Once this hypersurface is defined, we compute its mean curvature and pull back the result on Σ using I_w. This defined a nonlinear mapping

$$\mathcal{N} : \mathcal{C}_\delta^{2,\alpha}(\Sigma) \times D_\delta \longrightarrow \mathcal{C}_\delta^{0,\alpha}(\Sigma)$$

It is a simple exercise to check that this mapping is smooth and that the differential of this mapping coincides with $\tilde{\mathcal{A}}_\delta$ (maybe up to a change in the definition of the cutoff function used to define D_δ).

6.3 The marked space of minimal hypersurfaces with catenoidal ends

Here we introduce another space of minimal hypersurfaces with catenoidal ends : the marked space of minimal hypersurfaces with catenoidal ends. The idea is that, in the unmarked space, one is allowed to translate, rotate, and dilate the end. In the marked moduli space, the only modifications allowed are the translation of the end along its axis and the dilation of the end.

Paralleling what we have already done for the unmarked space, we define the *marked space* M_m as the set of minimal hypersurfaces which have finitely many ends asymptotic to the end or a catenoid whose axis is fixed but which is properly rescaled and translated along its axis. More precisely, this means that each element of M_m can be decomposed into the union of a compact piece and finitely many ends which (up to a translation, a rotation and a dilation) can be written as a normal graph over one end of the n-catenoid, whose axis is fixed, for a function which decays like a constant times $|x|^{\frac{2-n}{2}+\delta}$ for some $\delta \in (-\frac{n}{2}, \frac{2-n}{2})$.

We have the corresponding notion of nondegeneracy :

Definition 6.3.1. *An element $\Sigma \in M_m$ is said to be marked-nondegenerate if the operator*

$$A_\delta := \gamma^2 \left(\Delta_\Sigma + Tr(A^2)\right)$$

is injective on $L^2_\delta(\Sigma)$ for all $\delta < \frac{2-n}{2}$.

Applying the implicit function theorem as above we find that, if Σ is marked nondegenerate, then there exists an open manifold of dimension k which contains Σ and is included in M_m, where k is the number of catenoidal ends of Σ.

Chapter 7

Analysis on manifolds with cylindrical ends

7.1 Manifolds with cylindrical ends

We will say that a complete, noncompact n-dimensional manifold (M, g) is a manifold with cylindrical ends if it can be decomposed into the union of a compact piece $K \subset\subset M$ and finitely many ends $\mathcal{E}_1, \ldots, \mathcal{E}_k$ which are diffeomorphic to $(0, \infty) \times \Sigma_i$ where Σ_i is a $(n-1)$-dimensional manifold and if the metric g is asymptotic to the product metric

$$g_{cyl} = dt^2 + d_{\Sigma_i}$$

in the sense that the coefficients of $g - g_{cyl}$ satisfy

$$\nabla^\ell (g - g_{cyl})_{ij} = \mathcal{O}(e^{-\alpha\, t})$$

for some $\alpha > 0$.

On (M, g) we would like to study operators of the form

$$\Delta_g + a$$

where this time the function $a = M \longrightarrow \mathbb{R}$ satisfies

$$\nabla^\ell (a - a_j) = \mathcal{O}(e^{-\beta\, t})$$

for some $\beta > 0$, on each end $\mathcal{E}_j$ where $a_j \in \mathbb{R}$.

We define a smooth positive function $\gamma : M \longrightarrow (0, \infty)$ which coincides with e^t on each end $\mathcal{E}_j$ of M. We define weighted L^2-spaces and weighted Hölder spaces by defining the norms in these spaces as follows

$$\|u\|_{L^2_\delta(M)} := \left(\int_M |u|^2\, \gamma^{-2\delta}\, dvol_g \right)^{1/2}$$

and

$$\|u\|_{\mathcal{C}^{\ell,\alpha}_\delta(M)} := \|u\|_{\mathcal{C}^{\ell,\alpha}(K)} + \sum_j \sup_{t\geq 0} e^{-\delta\, t}\, \|u(t+\cdot,\cdot)\|_{\mathcal{C}^{\ell,\alpha}([t,t+1]\times\Sigma_j)}$$

As before we define the unbounded operator

$$\begin{array}{rccc} A_\delta : & L^2_\delta(M^*) & \longrightarrow & L^2_\delta(M^*) \\ & u & \longmapsto & \Delta_g\, u + a\, u \end{array}$$

as well as the bounded operator

$$\begin{array}{rccc} \mathcal{A}_\delta : & C^{2,\alpha}_\delta(M^*) & \longrightarrow & \mathcal{C}^{0,\alpha}_\delta(M^*) \\ & u & \longmapsto & \Delta_g\, u + a\, u \end{array}$$

In order to extend the previous analysis to this framework, the key remark is that, if

$$|x|^2\, \Delta u = f$$

in B_1^* then, setting

$$v(t,z) = e^{\frac{n-2}{2}t}\, u(e^{-t}\, z) \qquad \text{and} \qquad g(x) = e^{\frac{n-2}{2}t}\, f(e^{-t}\, z)$$

one can check that

$$\Delta_{g_{cyl}} v - \left(\frac{n-2}{2}\right)^2 v = f$$

in $(0,\infty)\times S^{n-1}$.

Further observe that

$$\int_{B_1} u^2(x)\, |x|^{-2\delta-2}\, dx = \int_{(0,\infty)\times S^{n-1}} v^2(t,z)\, e^{2\delta t}\, dt\, dvol_{S^{n-1}}$$

and

$$\|u\|_{\mathcal{C}^{\ell,\alpha}_\delta(\bar{B}_1^*)} = \|v\|_{\mathcal{C}^{\ell,\alpha}_{-\delta}((0,\infty)\times S^{n-1})}$$

These remarks allow one to extend all the previous results on a punctured manifold to this noncompact complete setting when all Σ_j are equal to S^{n-1} and $a_j = -\left(\frac{n-2}{2}\right)^2$. We leave the details to the reader.

In order to further extend the result to all manifolds with cylindrical ends, observe that the parameters δ_j are given by

$$\delta_j = \left(\left(\frac{n-2}{2}\right)^2 + \lambda_j\right)^{1/2}$$

are related to the asymptotic behavior of the solutions of the homogeneous equation

$$\left(\partial_t^2 - \lambda_j - \left(\frac{n-2}{2}\right)^2\right) w = 0$$

In the general case, this equation has to be replaced by

$$\left(\partial_t^2 - \mu_j^{(i)} + a_i\right) w = 0$$

where $\mu_j^{(i)}$ are the eigenvalues of $-\delta_{\Sigma_i}$. This gives the general definition of the parameters $\delta_j^{(i)}$ as

$$\delta_j^{(i)} = \Re\left(\mu_j^{(i)} - a_i\right)^{1/2}$$

In particular, we have the :

Proposition 7.1.1. *Given $\delta < \min_i \delta_0^{(i)}$, $\delta \neq -\delta_j^{(i)}$, for all $j \in \mathbb{N}$ and all $i = 1, \ldots, k$, let us assume that A_δ is injective, then the operator*

$$\begin{array}{rlcl} \tilde{\mathcal{A}}_\delta : & \mathcal{C}_\delta^{\ell,\alpha}(M^*) \oplus D_\delta & \longrightarrow & \mathcal{C}_\delta^{\ell,\alpha}(M^*) \\ & u & \longrightarrow & \Delta_g u + a\, u \end{array}$$

is well defined, bounded and surjective. In addition dim Ker$(\tilde{\mathcal{A}}_\delta) = \frac{1}{2}$ dimD_δ.

As usual the deficiency space is defined by

$$D_\delta := \oplus_{i=1}^k \operatorname{Span}\{\chi^{(i)}\, W_{j,\phi}^{\pm(i)}, \qquad : \qquad \phi \in E_j, \qquad -\delta < \pm\delta_j^{(i)} < \delta\}$$

where the functions $W_{j,\psi}^{\pm(i)}$ are solutions of $(\Delta_g + a)\, W_{j,\phi}^{\pm(i)} = 0$ on $\mathcal{E}_i$ and satisfy

$$W_{j,\phi}^{\pm(i)} = e^{\pm\delta_j\, t}\, \phi + \mathcal{O}(e^{(\pm\delta_j - \eta)\, t})$$

at infinity, for some $\eta > 0$.

Observe that when $\delta \in (-\min_i \delta_0^{(i)}, \min_i \delta_0^{(i)})$, then D_δ is empty and hence, under the assumption of Proposition 7.1.1, the operator $\mathcal{A}_{\delta'}$ is an isomorphism, for any $\delta' \in (-\min_i \delta_0^{(i)}, \min_i \delta_0^{(i)})$.

7.2 Manifolds with periodic-cylindrical ends

There is a last class of manifolds we will have to consider : manifolds with periodic-cylindrical ends. We will say that a complete, noncompact n-dimensional manifold (M, g) is a manifold with periodic-cylindrical ends if it can be decomposed into the union of a compact piece $K \subset\subset M$

and finitely many ends $\mathcal{E}_1, \ldots, \mathcal{E}_k$ which are diffeomorphic to $(0, \infty) \times \Sigma_i$ where Σ_i is a $(n-1)$-dimensional manifold and if the metric g is asymptotic to the metric

$$g_{per} = \varphi^2 \left(dt^2 + d_{\Sigma_i} \right)$$

where φ is a function defined on $(0, \infty)$ which is T_i-periodic, in the sense that the coefficients of $g - g_{per}$ satisfy

$$\nabla^\ell (g - g_{per})_{ij} = \mathcal{O}(e^{-\alpha\, t})$$

for some $\alpha > 0$.

On (M, g) we would like to study operators of the form

$$\Delta_g + a$$

where this time the function $a = M \longrightarrow \mathbb{R}$ satisfies

$$\nabla^\ell (a - a_j) = \mathcal{O}(e^{-\beta\, t})$$

for some $\beta > 0$, on each end $\mathcal{E}_j$ where a_j is a T_i-periodic function defined on $(0, \infty)$ (and in particular a_j does not depend on $z \in \Sigma_i$).

The definitions of the weighted L^2-spaces as well as the weighted Hölder spaces are the same as in the above sections as well as the definition of the operators A_δ and $\mathcal{A}_\delta$.

Observe that the Laplace-Beltrami operator associated to the metric g_{per} is explicitly given by

$$\Delta_{g_{per}} = \frac{1}{\varphi^n} \, \partial_t \left(\varphi^{2-n} \, \partial_t \, \cdot \right) + \frac{1}{\varphi^2} \, \Delta_{\Sigma_i}$$

All our analysis in Chapters 1 to 3 is based on the fact that we can study the model operator

$$\Delta_{g_{per}} + a_i = \frac{1}{\varphi^n} \, \partial_t \left(\varphi^{2-n} \, \partial_t \, \cdot \right) + \frac{1}{\varphi^2} \, \Delta_{\Sigma_i} + a_i$$

on $(0, \infty) \times \Sigma_i$. Then the corresponding properties for $\Delta_g + a$ are obtained using perturbation arguments.

Now, in order to study the model operator $\Delta_{g_{per}} + a_i$, it is easier to perform the eigenfunction decomposition of a function defined on $(0, \infty) \times \Sigma_i$ as

$$w(t, z) = \sum_j w_j$$

where for all $t \in (0, \infty)$, $w_j(t, \cdot) \in E_j$ the j-th eigenspace of $-\delta_{\Sigma_i}$. Using this the solvability of the equation

$$(\Delta_{g_{per}} + a_j) w = f$$

reduces to the solvability of the second order ordinary differential equations

$$\frac{1}{\varphi^n}\,\partial_t\left(\varphi^{2-n}\,\partial_t\,w_j\right) - \frac{1}{\varphi^2}\,\mu_j^{(i)}\,w_j + a_i\,w_j = f_j$$

where $\mu_j^{(i)}$ are the eigenvalues of $-\delta_{\Sigma_i}$.

It turns out that it is easier to study the conjugate operator

$$\varphi^{\frac{n+2}{2}}\,(\Delta_{g_{per}} + a_j)\,\varphi^{\frac{2-n}{2}}\,\bar{w} = f$$

Since $\varphi > 0$ is bounded there is no loss of generality in doing so. Therefore, the ordinary differential equations we need to solve now read

$$\partial_t^2\,\bar{w}_j + b_j^{(i)}\,\bar{w}_j = f_j \tag{7.1}$$

where $b_j^{(i)}$ can be expressed in terms of the function a_i, the eigenvalue $\mu_j^{(i)}$, the function φ ant its derivative with respect to t. The exact expression of the function $b_j^{(i)}$ is not really important for the time being. the only important fact is that $b_j^{(i)}$ is T periodic.

These equations in turn can be solved using the "variation of the constant formula" namely, if $\bar{w}_j^{+(i)}$ and $\bar{w}_j^{-(i)}$ are two linearly independent solutions of the homogeneous equation

$$\partial_t^2\,\bar{w}_j^{\pm(i)} + b_j^{(i)}\,\bar{w}_j^{\pm(i)} = 0$$

then all the solutions of (7.1) are given by

$$\bar{w}_j = \frac{1}{W_j^{(i)}}\left(\bar{w}_j^{+(i)}\int \bar{w}_j^{-(i)}\,f_j\,ds - w_j^{-(i)}\int \bar{w}_j^{+(i)}\,f_j\,ds\right) + a_j^+\,\bar{w}_j^{+(i)} + a_j^-\,\bar{w}_j^{-(i)}$$

Therefore, all properties will follow from the corresponding properties of the functions $\bar{w}_j^{\pm(i)}$.

To make the notation as simple as possible, we drop all indices $^{(i)}$ and $_j$. Given initial data (a_0, a_1) we consider w the unique solution of

$$\partial_t^2 w + b\,w = 0 \tag{7.2}$$

with $w(0) = a_0$ and $\partial_t w(0) = a_1$. We define the mapping

$$B(a_0, a_1) = (w(T), \partial_t w(T))$$

Clearly B is linear and $\mathrm{Tr} B \in \mathbb{R}$. We claim that

$$\det B = 1$$

To see this, consider the solution w_0 associated to the initial data $w_0(0) = 1$ and $\partial_t w_0(0) = 0$ and the solution w_1 associated to the initial data $w_1(0) = 0$ and $\partial_t w_1(0) = 1$. The Wronskian

$$W = w_1 \, \partial_t w_0 - w_0 \, \partial_t w_1$$

does not depend on t and $W(0) = 1$. Now, given the definition of B we have

$$W(T) = \det B \, W(0)$$

and hence $\det B = 1$ as claimed. We now distinguish a few cases according to the spectrum of the operator B.

Assume that B can be diagonalized (in $\mathbb{C}^2$). Since the determinant of B is equal to 1 then the eigenvalues are given by λ and $1/\lambda$ where $\lambda \in \mathbb{C}$ and $|\lambda| \geq 1$. If $|\lambda| > 1$ then necessarily $\lambda \in \mathbb{R}$ since the trace of B is a real number. The eigenvector of B associated to λ corresponds to $\bar{w}^+$ a solution of (7.2) which blows up at infinity exponentially and the eigenvector of B associated to $1/\lambda$ corresponds to $\bar{w}^-$ a solution of (7.2) which tends to 0 exponentially at infinity. In this case we define

$$\delta = \frac{1}{T} \, \log |\lambda|$$

so that δ is precisely the exponential rate at which the solutions $w^\pm$ tend to 0 or infinity at infinity.

When $|\lambda| = 1$ then $\lambda = e^{i\mu}$ and $1/\lambda = e^{-i\mu}$. The eigenvectors of B are associated to $w^\pm$ The eigenvector of B associated to λ corresponds to $\bar{w}^+$ solutions of (7.2) which are bounded in $\mathbb{R}$. In this case we define

$$\delta = 0$$

To end this discussion, we consider the case where B can't be diagonalized. In this case the eigenvalue λ necessarily satisfies $\lambda^2 = 1$ since the determinant of B is equal to 1. The eigenvector e_1 of B associated to the eigenvalue λ corresponds to w^+ a periodic solution of (7.2) (this is clear when $\lambda = 1$ since the solution is then T periodic and when $\lambda = -1$ then the solution is then $2T$ periodic). Since the operator B is not diagonalized then there exists a vector e_2 such that

$$B(e_2) = \lambda \, e_2 + \mu \, e_1$$

In other words e_1, e_2 is a Jordan basis associated to B. We denote by w^- the solution of (7.2) associated to e_2. By definition we have

$$e_1 = (w^+(0), \partial_t w^+(0)) \qquad \text{and} \qquad e_1 = (w^-(0), \partial_t w^-(0))$$

Now, on the one hand

$$B \, e_2 = (w^-(T), \partial_t w^+(T))$$

and on the other hand

$$B \, e_2 = \lambda \, (w^-(0), \partial_t w^-(0)) + \mu \, (w^+(0), \partial_t w^+(0))$$

Therefore, we have the identity

$$(w^-(T), \partial_t w^+(T)) = \lambda\,(w^-(0), \partial_t w^-(0)) + \mu\,(w^+(0), \partial_t w^+(0))$$

which implies that

$$w^-(t+T) = \lambda\, w^-(t) + \mu\, w^+(t)$$

for all $t \in \mathbb{R}$ (simply use the uniqueness of the solutions of (7.2) with given initial data). This shows that

$$v(t) = w^-(t) - \lambda\,\mu\,\frac{t}{T}\,w^+(t)$$

satisfies

$$v(t+T) = \lambda\, v(t)$$

Therefore, v is T periodic when $\lambda = 1$ and $2T$ periodic when $\lambda = -1$. In any case,

$$w^-(t) = v(t) + \lambda\,\mu\,\frac{t}{T}\,w^+(t)$$

where both v and w^+ are periodic. When $\mu \neq 0$, we will say that w^- is "linearly growing" and we set

$$\delta = 0$$

The result of Proposition 7.1.1 holds in this framework when the values of δ_j are obtained as above. Let us emphasize that for operators with periodic (nonconstant) coefficients the explicit determination of the values of $\delta_j^{(i)}$ is in general not possible.