Moduli of polarized Hodge structures

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Abstract

Around 1970 Griffiths introduced the moduli of polarized Hodge structures/the period domain D and described a dream to enlarge D to a moduli space of degenerating polarized Hodge structures. Since in general D is not a Hermitian symmetric domain, he asked for the existence of a certain automorphic cohomology theory for D, generalizing the usual notion of automorphic forms on symmetric Hermitian domains. Since then there have been many efforts in the first part of Griffith's dream but the second part still lives in darkness. The objective of the present text is two-folded. First, we give an exposition of the subject. Second, we give another formulation of the Griffiths problem, based on the classical Weierstrass uniformization theorem.

1 Introduction

I got acquainted with the subject of present paper, when I was looking for works on abelian integrals in algebraic geometry. My initial aim was to collect some information and then try to apply them in the study of abelian integrals in differential equations/holomorphic foliations (see [38, 40]). In a canonical way, I found many articles of P. A. Griffiths around 1970 on periods of projective manifolds and in particular, his survey article [23] which contains almost all of his ideas on the subject. After a long period of investigation, I observed that I had gone far from my initial aim. But it was worthfull because, in my opinion, this is a very beautiful piece of mathematics which can attract many young mathematicians. Later, I got the idea of modular foliations (see [44, 43]) based on the generalized Griffiths domain and did not touch more the problems posed by Griffiths.

The conference "CIMPA-UNESCO-IPM School on Recent Topics in Geometric Analysis, 2006" was a nice occasion for me to start writing what I had collected so far on the subject. The present text is the main body of my talks at the mentioned conference and it is mainly expository. Its objective is to introduce the reader with the literature on the moduli of polarized Hodge structures and compactification problems after Satake, Baily and Borel. I have tried to introduce the subject from the point of view of the classical Weierstrass uniformization theorem. This seems to me the proper way of the realization of Griffiths dream on automorphic type functions for the moduli of polarized Hodge structures. Let us explain the contents of this text.

In §2 we sketch the objective of this text in the case of elliptic curves. This yields to the classical theory of Eisenstein series and modular forms. In §3 we recall the Hodge structures on the de Rham cohomologies of projective manifolds and the associated polarizations. In §4 we construct the classifying space of polarized Hodge structures D, called the Griffiths domain, and the action of an arithmetic group $\Gamma_{\mathbb{Z}}$ on D from the left. In §5 we recall Ehresmann's fibration theorem and then the fact that the period maps form coefficient spaces to $\Gamma_{\mathbb{Z}}\backslash D$ satisfy the so called Griffiths transversality. In §6 we state the Baily-Borel theorem on the unique algebraic structure of quotients of symmetric Hermitian domains by discrete arithmetic groups. Since except in few cases D is not a Hermitian symmetric domain, one cannot apply this theorem to D. We will mention such few cases which give origin to the notion of Shimura varieties in algebraic geometry. Finally, in §7 we give a new formulation of the automorphic type functions corresponding to families of hypersurfaces.

Currently, I am working on the text [44] in which an analytic variety P over the Griffiths domain is constructed and many notions of the present text, like the action of an arithmetic group, period map, Griffiths transversality and so on are extended to P.

2 Elliptic curves

In this section we sketch the objective of the present text in the case of polarized Hodge structures arising from elliptic curves. The reader who is not familiar with Hodge structures is recommended to read first this section and then the following sections.

2.1 Elliptic integrals and elliptic curves

Elliptic integrals

$$\int_I \frac{dx}{\sqrt{4x^3-t_2x-t_3}},\ t_1,t_2\in\mathbb{C}, \Delta := t_2^3-27t_3^2\neq 0,$$

where I is some interval in $\mathbb{R}$ with end points in the roots of $4x^3-t_2x-t_3=0$ or infinity, can be written, up to some algebraic constants, as $\int_\delta \frac{dx}{y}$, where $\delta\in H_1(E_t,\mathbb{Z})$ and E_t is an elliptic curve in the Weierstrass family of elliptic curves

$$(1)\qquad E_t: y^2-4x^3+t_2x+t_3=0,\ t=(t_1,t_2)\in\mathbb{U}_0:=\mathbb{C}^2.$$

A parameter t with $\Delta(t)=0$ corresponds to the singular E_t. In fact, after adding the point at infinity to E_t it turns to be a compact elliptic curve and by E_t we mean the compact one. The de-Rham cohomology (with complex valued differential forms) of E_t is a two dimensional $\mathbb{C}$-vector space generated by the classes ω and $\bar\omega$ of the differential forms $\frac{dx}{y}|_{E_t}$ and $\frac{d\bar x}{\bar y}|_{E_t}$, respectively, in $H^1_{\rm dR}(E_t)$.

2.2 Polarized Hodge structures

We have the Hodge decomposition

$$H^1_{\rm dR}(E_t):=H^{10}\oplus H^{01}, H^{01}=\overline{H^{10}},$$

where H^{10} is the one dimensional $\mathbb{C}$-vector space generated by ω. From another side we have

$$H^1(E_t,\mathbb{Z})\otimes\mathbb{C}=H^1(E_t,\mathbb{C})\cong H^1_{\rm dR}(E_t),$$

where $H^1(E_t,\mathbb{Z})$ is the dual of the $\mathbb{Z}$-module $H_1(E_t,\mathbb{Z})$. In $H^1(E_t,\mathbb{Z})$ we have the non-degenerated bilinear map

$$H^1(E_t,\mathbb{Z})\times H^1(E_t,\mathbb{Z})\to\mathbb{Z}, \psi(a,b)=\int_{E_t} a\wedge b \ \text{ up to a constant}$$

which is dual (by Poincaré duality, see [21] p. 59 and §3.2) to the intersection mapping in $H_1(E_t,\mathbb{Z})$. It is also called a polarization. It satisfies $-\sqrt{-1}\psi(\omega,\bar{\omega})>0$ which is equivalent to:

$$\mathrm{Im}(z)>0,\ z:=\frac{\int_{\delta_1}\omega}{\int_{\delta_2}\omega}, \tag{2}$$

where $\delta_i,\ i=1,2$ is a basis of the $\mathbb{Z}$-module $H_1(E_t,\mathbb{Z})$. Usually one takes the symplectic basis of $H_1(E_t,\mathbb{Z})$, i.e. $\langle\delta_1,\delta_2\rangle=1$, and so in this basis the intersection matrix in $H_1(E_t,\mathbb{Z})$ is

$$\Psi:=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Let $\check{\delta}_i\in H^1(E_t,\mathbb{Z}),\ i=1,2$ be the dual of δ_i. We have

$$\omega=(\int_{\delta_1}\omega)\check{\delta}_1+(\int_{\delta_2}\omega)\check{\delta}_2$$

and so elliptic integrals are encoded in an abstract structure consisting of: A $\mathbb{Z}$-module $V_{\mathbb{Z}}$ of rank two and a polarization $\psi_{\mathbb{Z}}:V_{\mathbb{Z}}\times V_{\mathbb{Z}}\to\mathbb{Z}$, which is a non-degenerated bilinear map, and a Hodge structure

$$V_{\mathbb{C}}:=V_{\mathbb{Z}}\otimes\mathbb{C}=H^{10}\oplus H^{01},\ H^{01}=\overline{H^{10}},\ \dim H^{01}=\dim H^{10}=1,$$

$$-\sqrt{-1}\psi(v,\bar{v})>0,\ \forall v\in H^{10}.$$

One call this a polarized Hodge structure of type $\Phi:=(m,h^{10},h^{01},\Psi)$, where $m=1$ and $h^{10}=h^{01}=1$.

2.3 Moduli of polarized Hodge structures

Next, we want to construct the moduli of polarized Hodge structures of type Φ. Fix a $\mathbb{Z}$-module of rank 2 and a polarization $\psi_{\mathbb{Z}}$ on $V_{\mathbb{Z}}$. Let D be the space of all polarized Hodge structures of type Φ. It is in fact isomorphic to the Poincaré upper half plane

$$\mathbb{H}:=\{x+iy\in\mathbb{C}\mid y>0\}.$$

The isomorphism $\mathbb{H}\to D$ is constructed in the following way: To $z\in\mathbb{H}$ we associate the Hodge structure in which H^{10} is generated by $z\check{\delta}_1+\check{\delta}_2$, where $\{\check{\delta}_1,\check{\delta}_2\}$ is a basis of $V_{\mathbb{Z}}$ with $\psi_{\mathbb{Z}}(\check{\delta}_1,\check{\delta}_2)=1$. The group

$$\mathrm{SL}(2,\mathbb{Z}):=\{A\in\mathrm{Mat}(2,\mathbb{Z})\mid\det(A)=1\}$$

acts on $\mathbb{H}$ by $\begin{pmatrix}a&b\\c&d\end{pmatrix}z:=\frac{az+b}{cz+d}$ and the Hodge structures associated to z and $Az,\ A\in\mathrm{SL}(2,\mathbb{Z})$ are isomorphic. It is not difficult to see that $\Gamma_{\mathbb{Z}}\backslash\mathbb{H}$ is the moduli of polarized Hodge structures of type Φ.

2.4 Period map

The multiplicative group $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$ acts in $\mathbb{C}^2$ in the following way:

$$\lambda \cdot (x, y) = (\lambda^2 x, \lambda^3 y),\ \lambda \in \mathbb{C}^*, (x, y) \in \mathbb{C}^2.$$

This induces an action of $\mathbb{C}^*$ on $\mathbb{U}_0$:

$$\lambda \bullet (t_2, t_3) = (\lambda^4 t_2, \lambda^6 t_3),\ \lambda \in \mathbb{C}^*, (t_2, t_3) \in \mathbb{U}_0$$

and an isomorphism $E_{\lambda\bullet t} \cong E_t$ of elliptic curves in the family (1). Therefore, $E_t,\ t \in \mathbb{P}^{2,3} := \mathbb{C}^*\backslash\mathbb{U}_0$ may be non-isomorphic elliptic curves. Note that $\Delta = 0$ induces a one point set c in $\mathbb{P}^{2,3}$. We have the period map form $\mathbb{P}^{2,3}\backslash\{c\}$ to the moduli of polarized Hodge structures of type Φ. Composing this with the isomorphism obtained in §2.3 and extending it to c we have:

$$\text{(3)} \qquad \mathsf{pm} : \mathbb{P}^{2,3} \to (\Gamma_{\mathbb{Z}}\backslash\mathbb{H}) \cup \{\infty\},\ \mathsf{pm}(t) = [z],\ z := \frac{\int_{\delta_1} \omega_1}{\int_{\delta_2} \omega},\ \mathsf{pm}(c) = \infty,$$

which we call it again the period mapping. The different choice of the cycles δ_1, δ_2 with $\langle \delta_1, \delta_2 \rangle = 1$ will lead to the action of $\mathrm{SL}(2, \mathbb{Z})$ on z and so the above map is well-defined. The point ∞ can be thought of the point obtained by the action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{Q}$. It is called a cusp point.

2.5 Gauss-Manin connection

For practical purposes it is useful to redefine the period map in the following way: Let $\mathcal{L}$ be the set of lattices $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,\ \frac{\omega_1}{\omega_2} \in \mathbb{H}$. The new period map is

$$\text{(4)} \qquad \mathsf{pm} : \mathbb{U}_0\backslash\{\Delta = 0\} \to \mathcal{L},\ \mathsf{pm}(t) = \frac{1}{\sqrt{2\pi i}} \int_{H_1(E_t, \mathbb{Z})} \omega.$$

The previous period map is the quotient of the new one. The derivative of the period map can be calculated using the following argument: For a cycle $\delta \in H_1(E_t, \mathbb{Z})$ define

$$\eta_1 = \int_\delta \frac{dx}{y},\ \eta_2 = \int_\delta \frac{xdx}{y}$$

which are multi-valued holomorphic functions in $\mathbb{U}_0\backslash\{\Delta = 0\}$. Then η_1, η_2 satisfy the following Picard-Fuchs system

$$\text{(5)} \qquad \begin{pmatrix} d\eta_1 \\ d\eta_2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} -\frac{d\Delta}{12} & -\frac{3\delta}{2} \\ -\frac{t_2\delta}{8} & \frac{d\Delta}{12} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},\ \delta = 3t_3 dt_2 - 2t_2 dt_3.$$

(see for instance [45]). The 2×2 matrix in the above equality is called the Gauss-Manin connection matrix of the family (1) with respect to the differential forms $\frac{dx}{y}, \frac{xdx}{y}$. The algorithms which calculate the Gauss-Manin connection can be implemented in any software for commutative algebra (see [39]).

2.6 Eisenstein series and full modular forms

For a given elliptic curve in the Weierstrass family we constructed the associated Hodge structure. Now, is it possible to construct the inverse map? In fact, it turns out that the period map (3) is a biholomorphism whose inverse is given by the $\mathrm{SL}(2,\mathbb{Z})$ invariant j-function:

$$j(z) := q^{-1} + 744 + 196884q + 21493760q^2 + \cdots,\ q = e^{2\pi i z}$$

Let us sketch the proof. The period map in (4) is a local biholomorphic map. This can be check by the formula of its derivative. It satisfies also

$$\mathsf{pm}(\lambda \bullet t) = \lambda^{-1}\mathsf{pm}(t).$$

Therefore, the induced map $\mathbb{P}^{2,3}\backslash\{c\} \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ is a local biholomorphisim. Since $\mathbb{P}^{2,3}\backslash\{c\} \cong \mathbb{C}$, it must be a biholomorphism.

The inverse of the period map (4) composed with the canonical map $\{(\omega_1,\omega_2) \mid \frac{\omega_1}{\omega_2} \in \mathbb{H}\} \to \mathcal{L}$ is given by $(g_4, g_6) := (60E_4, -140E_6)$, where E_i is the Eisenstein series of weight i:

$$E_i(\Lambda) := \sum_{0\neq a\in\Lambda} \frac{1}{a^i} = \sum_{0\neq(n,m)\in\mathbb{Z}^2} \frac{1}{(m\omega_1 + n\omega_2)^i}.$$

This follows from Weierstrass uniformization theorem (see for instance [50]). We have

$$j(z) = \frac{g_4^3(z)}{g_4^3(z) - 27g_6^2(z)}.$$

A holomorphic function g in $\mathcal{L}$ is called a full modular form of weight i, i an even positive integer, if

$$f(k\cdot\Lambda) = k^i f(\Lambda),\ \lambda \in \mathbb{C}^*, \Lambda \in \mathcal{L}$$

and the pull-back $\tilde{f}$ of f by the canonical map $\mathbb{H} \to \mathcal{L}$ has a finite growth at infinity, i.e. $\lim_{\mathrm{Im}(z)\to+\infty} \tilde{f}(z) = a < \infty$. The function $g_i,\ i = 4,6$ is a full modular form of weight i. The fact that the period map (4) is a biholomorphism implies that every full modular form can be written in a unique way as a polynomial in g_4 and g_6. It is easy to see that there is no full modular form of odd weight.

The classical definition of a full modular form is as follows: A holomorphic function $\tilde{f}$ on $\mathbb{H}$ is called a modular form of weight i if it has a finite growth at infinity and

$$(cz+d)^{-i} f(Az) = f(z),\ \forall z \in \mathbb{H},\ A \in \mathrm{SL}(2,\mathbb{Z}).$$

The map $f \mapsto \tilde{f}$ constructed in the previous paragraph is a bijection between the two notions.

One can interpret the modular forms of weight i (in the second sense) as the sections $K^{\frac{i}{2}}$, where K is the canonical bundle of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$(see §6). The literature of modular forms and their arithmetic properties is huge. For instance, the Fermat last theorem is proved to be equivalent to Shimura-Taniyama conjecture which asks for the existence of certain modular forms. The moonshine conjecture interprets the coefficients of the j-function in terms of the representation of Monster group.

3 Hodge theory of projective manifolds

Let $M \subset \mathbb{P}^N$ be a projective (compact) manifold of dimension n. This means that in the homogeneous coordinates of $\mathbb{P}^N$, M is given by the zeros of homogeneous polynomials. In M we consider the canonical orientation induced by its complex structure. Any Hermitian metric in M induces such an orientation. In the following by $H^m(M, \mathbb{Z})$ (resp. $H_m(M, \mathbb{Z})$) we mean the image of the classical singular cohomology (resp. homology) in $H^m(M, \mathbb{C})$ (resp. $H_m(M, \mathbb{C})$). Therefore, we have killed the torsion.

3.1 De Rham cohomology

The de-Rham cohomology of M is given by

$$H^m_{\mathrm{dR}}(M) := \frac{Z^m(M)}{dA^{m-1}(M)}$$

where $A^m(M)$ (resp. $Z^m(M)$) is the set of C^∞ complex valued differential m-forms (resp. closed m-forms) on M. From another side we have

$$H^m_{\mathrm{dR}}(M) = H^m(M, \mathbb{C}) := H^m(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

We look $H^m(M, \mathbb{Z})$ as a lattice in $H^m_{\mathrm{dR}}(M)$.

3.2 Intersection form

We have the Poincaré duality

$$P : H_m(M, \mathbb{Z}) \to H^{2n-m}(M, \mathbb{Z}), \ \int_{\delta'} P(\delta) = \langle \delta, \delta' \rangle$$

under which for $n = m$ the non-degenerate bilinear intersection map $\langle \cdot, \cdot \rangle$ in homology corresponds to the bilinear map

$$\psi(\omega_1, \omega_2) = c \int_M \omega_1 \wedge \omega_2, \ \omega_1, \omega_2 \in H^m_{\mathrm{dR}}(M),$$

where c is a positive real number depending only on the M (note that any two Hermitian metric induce the same orientation in M). It assures us that for $\omega_1, \omega_2 \in H^m(M, \mathbb{Z})$ we have $\int_M \omega_1 \wedge \omega_2 \in \mathbb{Z}$. If we use the notion of singular p-chains (see [21], p. 43) for the definition of integral on manifolds we can assume that $c = 1$.

3.3 Hodge decomposition

We have the Hodge decomposition

$$H^m_{\mathrm{dR}}(M) = H^{m,0} \oplus H^{m-1,1} \oplus \cdots \oplus H^{1,m-1} \oplus H^{0,m}, \tag{6}$$

where $H^{p,q} \cong \frac{Z^{p,q}(M)}{dA^{p+q-1}(M) \cap Z^{p,q}(M)}$ and $Z^{p,q}_d$ is the set of C^∞ closed (p,q)-forms on M. We have the canonical inclusions $H^{p,q} \to H^m_{\mathrm{dR}}(M)$ and one can prove (6) using harmonic forms, see M. Green's lectures [20], p. 14. We have the conjugation mapping

$$H^m_{\mathrm{dR}}(M) \to H^m_{\mathrm{dR}}(M), \ \omega \mapsto \bar{\omega}$$

which lefts $H^m(M,\mathbb{R})$ invariant and maps $H^{p,q}$ isomorphically to $\bar{H}^{q,p}$. By taking local charts it is easy to verify that

$$\psi(H^{i,m-i}, H^{m-j,j}) = 0 \text{ unless } i = j, \tag{7}$$

$$(-1)^{i+\frac{m}{2}}\psi(a,\bar{a}) > 0,\ \forall a \in H^{i,m-i},\ a \neq 0. \tag{8}$$

3.4 Hodge conjecture

One of the central conjectures in Hodge theory is the so called Hodge conjecture. Let m be an even natural number and $Z = \sum_{i=1}^{s} r_i Z_i$, where $Z_i,\ i = 1,2,\ldots,s$ is a compact algebraic subvariety of M of complex dimension $\frac{m}{2}$ and $r_i \in \mathbb{Z}$. By Chow theorem every compact analytic subvariety of M is algebraic, i.e. it is given by zeros of polynomials (see [21]). Using a resolution map $\tilde{Z}_i \to M$, where $\tilde{Z}_i$ is a compact complex manifold, one can define an element $\sum_{i=1}^{s} r_i[Z_i] \in H_m(M,\mathbb{Z})$ which is called an algebraic cycle (see [4]). Since the restriction to Z of a (p,q)-form with $p+q = m$ and $p \neq \frac{m}{2}$ is identically zero, an algebraic cycle δ has the following property:

$$\int_\delta H^{p,m-p} = 0,\ \forall p \neq \frac{m}{2}.$$

A cycle $\delta \in H_m(M,\mathbb{Z})$ with the above property is called a Hodge cycle. The assertion of the Hodge conjecture is that for any Hodge cycle $\delta \in H_n(M,\mathbb{Z})$ there is an integer $a \in \mathbb{N}$ such that $a \cdot \delta$ is an algebraic cycle, i.e. there exist subvarieties $Z_i \subset M$ of dimension $\frac{m}{2}$ and rational numbers r_i such that $\delta = \sum r_i[Z_i]$. The difficulty of this conjecture lies in constructing varieties just with their homological information. The conjecture is false with $a = 1$ (see [25]). In the literature one usually finds the notion of a Hodge class which is an element in $H^{\frac{m}{2},\frac{m}{2}} \cap H^m(M,\mathbb{Q})$. The Poincaré duality gives a bijection between the $\mathbb{Q}$-vector space generated by Hodge cycles and the $\mathbb{Q}$-vector space of Hodge classes (see [54] §11).

4 The classifying space of polarized Hodge structure

In this section we construct the classifying space of Hodge structures $D = D(m,h,\psi)$ with a fixed weight m, Hodge numbers $h = (h^{m-i,i},\ i = 0,1,\ldots,m)$ and a polarization ψ on a fixed freely generated $\mathbb{Z}$-module $H(\mathbb{Z})$. One can look D in two ways: First, as the space of Hodge filtrations and second as a quotient G/K, where G is a real Lie group and K is a compact subgroup (not necessarily maximal). For a subring $\mathbb{K}$ of $\mathbb{C}$ we define

$$H(\mathbb{K}) := H(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{K}.$$

4.1 Polarized Hodge structures

Let us be given a freely generated $\mathbb{Z}$-module $H(\mathbb{Z})$. A Hodge structure of weight m, $m \in \mathbb{N}$ and type $h = (h^{m,0}, h^{m-1,1}, \ldots, h^{0,m}) \in \mathbb{N}^{m+1}$ on $H(\mathbb{Z})$ is a decomposition

$$H(\mathbb{C}) := H^{m,0} \oplus H^{m-1,1} \oplus \cdots \oplus H^{0,m}$$

with $\bar{H}^{i,m-i} = H^{m-i,i}$ and $h^{i,m-i} = dim_{\mathbb{C}} H^{i,m-i}$. We call $h^{m-i,i}$ Hodge numbers and define $h^i := h^{m,0} + h^{m-1,1} + \cdots , h^{m-i,i}$. When $H(\mathbb{Z})$ has a Hodge structure of type m we denote it by $H^m(\mathbb{Z})$.

Instead of Hodge structures, we will use Hodge filtrations. The main reason is that one can define de Rham cohomologies of algebraic varieties over a field k and the associated Hodge filtrations in such a way that they coincide with the classical notions in the case of $k = \mathbb{C}$ (see [24]). For $0 \leq i \leq m$ we define

$$F^i = F^i H^m(\mathbb{C}) := H^{m,0} \oplus H^{m-1,1} \oplus \cdots \oplus H^{i,m-i}.$$

To recover the Hodge filtration we define $H^{i,m-i} = F \cap \bar{F}^{m-i}$.

When we have a family of Hodge structures parameterized by $\alpha \in I$ then we denote the Hodge structure associated to $\alpha \in I$ by $H^m(\alpha, \mathbb{Z})$. This notation is also used replacing $\mathbb{Z}$ with $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and so on. We write also $F^i = F^i(\alpha, \mathbb{C}) = F^i(\alpha)$, $H^{m-i,i} = H^{m-i,i}(\alpha, \mathbb{C}) = H^{i,m-i}(\alpha)$ and so on.

A polarization $\psi = \langle \cdot, \cdot \rangle$ for $H(\mathbb{Z})$ is a non-degenerate bilinear map $H(\mathbb{Z}) \times H(\mathbb{Z}) \to \mathbb{Z}$ symmetric if m is even and skew if m is odd

$$\psi(a, b) = (-1)^m \psi(b, a), \ a, b \in H^m(\mathbb{Z})$$

such that its complexification (we denote it again by ψ) satisfies (7) and (8). All the data above is called a polarized Hodge structure of type $\Phi = (m, h, \psi)$.

4.2 Hodge structure in cohomologies with real coefficients

Every element $x \in H^m(\mathbb{R})$ can be written in the form

$$x = x_{m,0} + x_{m-1,1} + \cdots + \overline{x_{m-1,1}} + \overline{x_{m,0}}, \ x_{m-i,i} \in H^{m-i,i}$$

with $\overline{x_{\frac{m}{2},\frac{m}{2}}} = x_{\frac{m}{2},\frac{m}{2}}$ if m is even. We define $H^i \subset H^m(\mathbb{R}), i < \frac{m}{2}$ to be the set of all $x_{m-i,i} + \overline{x_{m-i,i}}, \ x \in H^m(\mathbb{C})$ and $H^{\frac{m}{2}}$ to be the set of all $x_{\frac{m}{2},\frac{m}{2}}, x \in H^m(\mathbb{C})$. These are real sub vector spaces of $H^m(\mathbb{R})$ and we have the following decomposition of $H^m(\mathbb{R})$:

$$H^m(\mathbb{R}) = H^0 \oplus H^1 \oplus \cdots \oplus H^{\frac{m}{2}}, \tag{9}$$

$$x = (x_{m,0} + \overline{x_{m,0}}) + (x_{m-1,1} + \overline{x_{m-1,1}}) + \cdots + x_{\frac{m}{2},\frac{m}{2}}.$$

For $i < \frac{m}{2}$ the map $x_{m-i,i} \to x_{m-i,i} + \overline{x_{m-i,i}}$ induces an isomorphism of $\mathbb{R}$-vector spaces $H^{m-i,i}$ and H^i. Multiplication by $\sqrt{-1}$ in $H^{m-i,i}$ induces a map

$$J_i : H^i \to H^i, \ J_i^2 = -\mathrm{Id}, \tag{10}$$

$$x = x_{m-i,i} + \bar{x}_{m-i,i}, \ J_i x := i x_{m_i,i} + \overline{i x_{m_i,i}} = i(x_{m-i,i} - \bar{x}_{m-i,i}).$$

If m is even then we define $J_{\frac{m}{2}}$ to be the identity. A $\mathbb{C}$-linear map in $H^{m-i,i}$ corresponds to a $\mathbb{R}$-linear map in H^i which commutes with J_i.

Proposition 1. *(Riemann relations) For $i, j \leq \frac{m}{2}$ we have:*

1. *$\psi(H^i, H^j) = 0$ unless $i = j$;*

2. *$\psi(J_i x, J_i y) = \psi(x, y)$ for all $x, y \in H^i$;*

3. If m is odd then $(-1)^{\frac{m-1}{2}+i}\psi(x, J_i x) > 0$ for all $x \in H^i,\ x \neq 0$ ($\psi(x,x) = 0$);

4. If m is even then $\psi(x, J_i x) = 0$ and $(-1)^{\frac{m}{2}+i}\psi(x,x) > 0$ for all $x \in H^i,\ x \neq 0$.

Proof. 1. It is a direct consequence of (7). 2. Write $x = a + \bar{a}, y = b + \bar{b}, a, b \in H^{m-i,i}$.

$$\psi(J_i x, J_i y) = \psi(\sqrt{-1}(a - \bar{a}), \sqrt{-1}(b - \bar{b})) = -\psi(a - \bar{a}, b - \bar{b}) = \psi(a, \bar{b}) + \psi(\bar{a}, b) =$$

$$\psi(a + \bar{a}, b + \bar{b}) = \psi(x, y).$$

3,4. We use (7) to obtain

$$\psi(x, J_i x) = \psi(a + \bar{a}, \sqrt{-1}(a - \bar{a})) = \sqrt{-1}((-1) + (-1)^m)\psi(a, \bar{a}),$$

$$\psi(x, x) = \psi(a + \bar{a}, a + \bar{a}) = \psi(a, \bar{a}) + \psi(\bar{a}, a) = (1 + (-1)^m)\psi(a, \bar{a}),$$

and then use (8). □

Let $C : H^m(\mathbb{R}) \to H^m(\mathbb{R})$ be defined in the following way

$$Cx := \begin{cases} (-1)^{\frac{m-1}{2}+i} J_i x & m \text{ odd}, \\ (-1)^{\frac{m}{2}+i} x & m \text{ even}, \end{cases} \quad x \in H^i.$$

Now $\psi(x, Cy)$ is a positive form on $H^m(\mathbb{R})$ (see [13]). We will call the decomposition (9) of $H^m(\mathbb{R})$ and the data (10) with the properties 1,2,3 and 4 of Proposition 1 the polarized Hodge structure on $H^m(\mathbb{R})$. A polarized Hodge structure on $H^m(\mathbb{R})$ gives in a canonical way a polarized Hodge structure on $H^m(\mathbb{C})$. To see this, for $i = \frac{m}{2}$ we put $H^{\frac{m}{2},\frac{m}{2}} = H^{\frac{m}{2}} \otimes \mathbb{C}$ and for $i < \frac{m}{2}$ we put $H^{m-i,i}$ (resp. $H^{i,m-i}$) the vector space generated by the eigenvectors of J_i with the eigenvalue $\sqrt{-1}$(resp. $-\sqrt{-1}$). Now one can check (7) and (8).

Proposition 2. *For a polarized Hodge structure α on $H^m(\mathbb{Z})$,*

(11) $$V_\alpha := \{A \in GL(H^m(\mathbb{R})) \mid A \textit{ respects the Hodge structure and}$$

$$\psi(Ax, Ay) = \psi(x, y),\ AJ_i = J_i A, i = 0, 1, \ldots\}$$

is a compact subgroup of $GL(H^m(\mathbb{R}))$.

Proof. Define $P(x) = \psi(x, Cx),\ x \in H^m(\mathbb{R})$. We have $P(Ax) = P(x), \forall A \in V_\alpha$. Therefore, elemrnts of V_α leaves the fibers $P^{-1}(c), c \in \mathbb{R}^+$ invariant. The fibers of P are compact sets and this finish the proof of the proposition. □

4.3 Generalized Jacobians

Let us suppose that m is odd and $i = \frac{m+1}{2}$. Then we have a canonical isomorphism $H^m(\mathbb{R}) \to F^{\frac{m+1}{2}}$ of $\mathbb{R}$-vector spaces. Therefore, the projection L of $H^m(\mathbb{Z})$ in $F^{\frac{m+1}{2}}$ is a lattice of rank $2\dim_{\mathbb{C}} F^{\frac{m+1}{2}}$ and so we can define the compact complex torus

$$J_{\frac{m+1}{2}} H^m(\mathbb{Z}) := F^{\frac{m+1}{2}}/L.$$

This is called the $\frac{m+1}{2}$-Jacobian variety of $H^m(\mathbb{Z})$. In the case for which $H^m(\mathbb{Z})$ is the integral cohomology of a smooth projective variety of dimension m, the torus $J_1 H^1(\mathbb{Z}),\ m = 1$ (resp. $J_n H^{2n-1}(\mathbb{Z}),\ m = 2n-1$) is also called the Albanese variety (resp. Picard variety).

4.4 Griffiths domain

We fix m, Hodge numbers $h = (h^{m-i,i},\ i = 0, 1, \dots, m)$ and a non-degenerate bilinear map $\psi = \langle \cdot, \cdot \rangle$ in $H^m(\mathbb{Z})$. The Griffiths domian D is the space of all decompositions $H^m(\mathbb{C}) := H^{m,0} \oplus H^{m-1,1} \oplus \cdots \oplus H^{0,m},\ \dim H^{m-i,i} = h^{m-i,i}$ resulting a polarized Hodge structure on $H^m(\mathbb{Z})$. We define the compact dual $\check{D}$ of D similar to D but without the condition (8). Note that D is an open subset of $\check{D}$. The compact dual $\check{D}$ of D is an analytic variety in the following way: First note that a Hodge structure on $H^m(\mathbb{Z})$ is completely determined by the data:

$$F^{[\frac{m}{2}]+1} = H^{m,0} \oplus H^{m-1,1} \oplus \cdots \oplus H^{[\frac{m}{2}]+1, m-[\frac{m}{2}]-1}, \tag{12}$$

$$F^{[\frac{m}{2}]+1} \cap \overline{F^{[\frac{m}{2}]+1}} = \{0\},\ \psi(H^{m-i,i}, H^{m-j,j}) = 0,\ \forall i, j \leq m - [\frac{m}{2}] - 1. \tag{13}$$

($H^{i,m-i}$ is defined to be $\overline{H^{m-i,i}}$ and in the case m even $H^{\frac{m}{2},\frac{m}{2}}$ is the ψ-orthogonal complement of $F^{[\frac{m}{2}]+1} + \overline{F^{[\frac{m}{2}]+1}}$). The decomposition (12) determines a point in the complex grassmannian manifold $\mathrm{Gr} := \mathrm{Gr}(h^{m,0}, H^m(\mathbb{C})) \times \cdots \times \mathrm{Gr}(h^{[\frac{m}{2}]+1, m-[\frac{m}{2}]-1}, H^m(\mathbb{C}))$. The first condition in (13) determines an open subset of Gr and the second condition an analytic subvariety of Gr.

4.5 D as a quotient of real Lie groups

For a subring $\mathbb{K}$ of $\mathbb{R}$ define

$$\Gamma_{\mathbb{K}} := \mathrm{Aut}(H(\mathbb{K}), \langle \cdot, \cdot \rangle) :=$$

$$\{A : H(\mathbb{K}) \to H(\mathbb{K}) \mid A \text{ is } \mathbb{K}\text{-linear and } \forall x, y \in H(\mathbb{K}),\ \langle Ax, Ay \rangle = \langle x, y \rangle\}.$$

The group $\Gamma_{\mathbb{R}}$ acts from left on D in a canonical way. For a fixed point $p_0 \in D$ define

$$V = V_{p_0} := \{a \in \Gamma_{\mathbb{R}} \mid a.p_0 = p_0\}.$$

According to Proposition 2, V is a compact subgroup of $\Gamma_{\mathbb{R}}$. The map

$$\alpha : \Gamma_{\mathbb{R}}/V \to D,\ \alpha(a) = a \cdot p_0$$

is an isomorphism and so we may identify D with $\Gamma_{\mathbb{R}}/V$. Note that V may not be maximal.

Since $\Gamma_{\mathbb{Z}}$ is discrete in $\Gamma_{\mathbb{R}}$, i.e. it has the discrete topology as a subset of $\Gamma_{\mathbb{R}}$, the group $\Gamma_{\mathbb{Z}}$ acts discontinuously on D, i.e. for any two compact subset K_1, K_2 in D the set

$$\{A \in \Gamma_{\mathbb{Z}} \mid A(K_1) \cap K_2 \neq \emptyset\}$$

is finite. The set $\Gamma_{\mathbb{Z}} \backslash D$ is the moduli of polarized Hodge structures of type $\Phi = (m, h, \langle \cdot, \cdot \rangle)$ and it has a canonical structure of a complex analytic space.

4.6 Tangent space of D

Let

$$\mathfrak{g}_{\mathbb{K}} = \mathrm{L}ie(\Gamma_{\mathbb{K}}) := \{N \in \mathrm{End}_{\mathbb{K}}(H^m(\mathbb{K})) \mid \langle Nx, y \rangle + \langle x, Ny \rangle = 0,\ \forall x, y \in H^m(\mathbb{K})\}.$$

To each $\alpha \in D$, there is a natural filtration in $\mathfrak{g}_{\mathbb{C}}$

$$F^i \mathfrak{g}_{\mathbb{C}} = \{N \in \mathfrak{g}_{\mathbb{C}} \mid N(F^p) \subset F^{p+i},\ \forall p \in \mathbb{Z}\},\ i = 0, -1, -2, \ldots,$$

where $F^\bullet$ is the Hodge filtration associated to α. We get a natural filtration of the tangent bundle

$$T^h_\alpha D := F^{-1}(\mathfrak{g}_{\mathbb{C}})/F^0(\mathfrak{g}_{\mathbb{C}}) \subset F^{-2}(\mathfrak{g}_{\mathbb{C}})/F^0(\mathfrak{g}_{\mathbb{C}}) \subset \cdots \subset \mathfrak{g}_{\mathbb{C}}/F^0(\mathfrak{g}_{\mathbb{C}}) = T_\alpha D.$$

One usually calls $T^h_\alpha D$ the horizontal tangent bundle. When $m = 1$ then the horizontal and usual tangent bundles are the same.

The subgroup V is connected and is contained in a unique maximal compact subgroup K of $\Gamma_{\mathbb{R}}$. When $K \neq V$, then there is a fibration of $D = \Gamma_{\mathbb{R}}/V \to \Gamma_{\mathbb{R}}/K$ with compact fibers isomorphic to K/V which are complex subvarieties of D. In this case we have $T_\alpha(D) = T^h_\alpha(D) \oplus T^v_\alpha(D)$, where $T^v(D)$ restricted to a fibre of π coincides with the tangent bundle of that fibre. For more information see [11] and the reference there.

4.7 Few good cases

Here is the classification of all cases in which V is a maximal compact subgroup of $\Gamma_{\mathbb{R}}$, i.e. it is compact and there is no more compact subgroup of $\Gamma_{\mathbb{R}}$ containing V.

Proposition 3. *The group V is maximal in $\Gamma_{\mathbb{R}}$ only in one of the following cases:*

1. $m = 2a + 1, h^{p,n-p} = 0$ *if* $p \neq a, a + 1$*;*
2. $m = 2a, h^{a+1,a-1} \leq 1$ *and* $h^{p,m-p} = 0$ *if* $p \neq a - 1, a, a + 1$.

See [32] page 4 or [10]. In the first case D is the Siegel domain (see [17, 34]).

A subgroup of $\Gamma_{\mathbb{R}}$ is called arithmetic if it is commensurable with $\Gamma_{\mathbb{Z}}$. Here two subgroups A and B of a group are commensurable when their intersection has finite index in each of them. For the case in which V is maximal in $\Gamma_{\mathbb{R}}$, the quotient $D := \Gamma_{\mathbb{R}}/V$ is a Hermitian symmetric domain (see for instance [37]). For an arithmetic subgroup Γ of $\Gamma_{\mathbb{R}}$, the compactification of $\Gamma\backslash D$ is done by I. Satake, A. Borel and W. Baily ([3, 46]). In fact the compactification due to Borel and Baily gives us an algebraic structure on $\Gamma\backslash D$. For the case in which V is not maximal, partial compactifications are done by E. Cattani, A. Kaplan, A. Ash, D. Mumford, M. Rapoport, Y. S. Tai, K. Kato and S. Usui (see the reference of [31, 32]).

5 Period map

Roughly speaking the period map associate to each variety its polarized Hodge structure and hence a point in the Griffiths domain.

5.1 Ehresmann fibration theorem

For a family of algebraic varieties depending on a parameter one can always find a Zariski open subset U in the parameter space in such a way that the varieties with corresponding parameter in U are topologically the same (see for instance [53], corollary 5.1). However, they may have different analytic structures and Hodge structures.

Theorem 1. *(Ehresmann's Fibration Theorem [16]). Let $f : Y \to B$ be a proper submersion between the manifolds Y and B. Then f fibers Y locally trivially i.e., for every point $b \in B$ there is a neighborhood U of b and a C^∞-diffeomorphism $\phi : U \times f^{-1}(b) \to f^{-1}(U)$ such that $f \circ \phi = \pi_1 =$ the first projection. Moreover if $N \subset Y$ is a closed submanifold such that $f\mid_N$ is still a submersion then f fibers Y locally trivially over N i.e., the diffeomorphism ϕ above can be chosen to carry $U \times (f^{-1}(b) \cap N)$ onto $f^{-1}(U) \cap N$.*

The map ϕ is called the fiber bundle trivialization map. Ehresmann's theorem can be rewriten for manifolds with boundary and also for stratified analytic sets. In the last case the result is due to R. Thom, J. Mather and J. L. Verdier (see [36, 53]).

Let λ be a path in B connecting b_0 to b_1 and defined up to homotopy. Ehresmann's Theorem gives us a unique map $h_\lambda : f^{-1}(b_0) \to f^{-1}(b_1)$ defined up to homotopy. In particular, for $b := b_0 = b_1$ we have the action of $\pi_1(Y, b)$ on the homology group $H_n(f^{-1}(b), \mathbb{Z})$. The image of $\pi_1(Y, b)$ in $\mathrm{Aut}(H_n(f^{-1}(b), \mathbb{Z}))$ is called the monodromy group.

5.2 period map

Let us be given a holomorphic map between projective (compact) varieties $f : X \to S$. Let S' be the locus of points $t \in S$ such that f is not a submersion. According to Ehresmann's theorem the mapping f restricted to $T := S \backslash S'$ is a C^∞ fiber bundle. Fix a point $t_0 \in T$ and identify $H^m(f^{-1}(t_0), \mathbb{Z}), \langle \cdot, \cdot \rangle$ with the polarized $\mathbb{Z}$-module in the definition of D. Let

$$\Gamma = \mathrm{Im}(\pi_1(T, t_0) \mapsto \Gamma_\mathbb{Z})$$

be the monodromy group associated to f. We have the well-defined period map

$$\mathsf{pm} : T \to \Gamma \backslash D.$$

I believe that the monodromy group Γ is arithmetic but I do not know any reference or proof for it. To avoid this problem we may use $\Gamma_\mathbb{Z} \backslash D$ instead of $\Gamma \backslash D$.

5.3 Griffiths transversality theorem

The Griffiths transversality theorem is originally stated using the Gauss-Manin connection on the m-th cohomology bundle on T. It implies that the image of the derivation of the period map is in the horizontal tangent space $T^h_{\mathsf{pm}(t)} D$ of D.

6 Automorphic functions

Automorphic functions, from the point of view of this text, are connecting functions between coefficient spaces of algebraic varieties and the corresponding space of integrals/Hodge structures.

6.1 Positive line bundles and Kodaira embedding theorem

A line bundle L on a compact complex manifold A is called negative if the zero section A_0 of L can be contracted to a point in the context of analytic varieties, i.e there is an analytic map from a neighborhood of A_0 in L to a singularity $(X, 0)$ such that it is a biholomorphism outside A_0 and the inverse image of 0 is A_0. Naturally, a line bundle is called positive if its dual is negative.

Theorem 2. *Let L be a positive line bundle on a complex compact manifold A. Then there is an integer n and global holomorphic sections $s_0, s_1, \dots, s_N$ of L^n such that the mapping*

$$A \to \mathbb{P}^N, \ x \mapsto [s_0(x); s_1(x); \dots : s_N(x)]$$

is an embedding.

As far as I know this is the only way for giving an algebraic structure to a complex manifold. For further information the reader is referred to [19, 7].

6.2 Automorphy factors

Let D be a complex manifold and Γ be a subgroup of biholomorphisms of D. A holomorphic automorphy factor for Γ on D is a map $\mathrm{j} : D \times \Gamma \to \mathbb{C}^*$ which for fixed $A \in \Gamma$ is holomorphic in $x \in D$ and which satisfies the identity

$$\mathrm{j}(x, A \cdot B) = \mathrm{j}(x, A)\mathrm{j}(A \cdot x, B), \ \forall A, B \in \Gamma, \ x \in D.$$

Two automorphic factors j_1 and j_2 are called equivalent if there is a group homomorphism $a : \Gamma \to (\mathbb{C}^*, \cdot)$ such that $\mathrm{j}_1(x, \cdot) = a(\cdot)\mathrm{j}_2(x, \cdot)$ for all $x \in D$. If both D and $D' := \Gamma\backslash D$ are smooth varieties then the equivalence class of an automorphy factor j corresponds to a line bundle L_{j} on D' . Conversely, every line bundle on D' is obtained by an automorphy factor. A global holomorphic section of L_{j}^n corresponds to a holomorphic function s in D such that

$$s(A \cdot x) = \mathrm{j}(x, A)^n s(x), \forall \ x \in D, \ A \in \Gamma. \tag{14}$$

In an arbitrary case in which D' may not be a complex manifold it is natural to say that a holomorphic function on D is an automorphic function of weight n if it satisfies (14) and a certain growth condition (depending on the situation). Usually, for a holomorphic function s in D one defines the slash operator

$$s|_n A(\cdot) := \mathrm{j}(\cdot, A)^{-n} s(\cdot), \ A \in \Gamma$$

which satisfies

$$s|_n (AB) = (s|_n A)|_B, \ A, B \in \Gamma$$

and one rewrites (14) in the form

$$s|_n A = s, \ \forall A \in \Gamma. \tag{15}$$

If we wish to find a canonical embedding of D' in some projective space and we want to use the idea behind Kodaira embedding theorem then we have to find an automorphy factor j, a positive integer n and automorphic functions $s_0, s_1, \dots, s_N$ of weight n such that the map

$$D \to \mathbb{P}^N, \ x \mapsto [s_0(x); s_1(x); \dots : s_N(x)]$$

is an embedding.

If D is a domain in $\mathbb{C}^n$ then the determinant of the Jacobian of $h \in \Gamma$ at x, denote it by $\det \mathrm{jacob}(x, h)$ is an automorphy factor. The corresponding line bundle in D' is the canonical line bundle in D', i.e. the wedge product of the cotangent bundle of D' n-times.

6.3 Poincaré series

Let us consider a holomorphic function s in D which may not satisfy the equality (14). To s one can associate the formal series

$$\tilde{s}(x) := \sum_{A \in \Gamma} s|_n A.$$

It satisfies the property (15) but it may not be convergent. The Borel-Baily theorem says that for D a Hermitian symmetric domains and for suitable s the above series is convergent.

6.4 Borel-Baily Theorem

Theorem 3. *(Borel-Baily) Let $\Gamma\backslash D$ be the quotient of a Hermitian symmetric domain by a torsion free arithmetic subgroup of $\mathrm{Hol}(D)^+$ then there is a positive integer n such that the automorphic forms of weight n gives us an embedding of $\Gamma\backslash D$ in some projective space.*

The automorphy factor in the above theorem comes, as usual, from the canonical bundle of $\Gamma\backslash D$. The proof is mainly based on the study of automorphic functions and convergence of the Poincaré series (see [3], §5).

Shimura Varieties are special cases of quotients $\Gamma\backslash D$. They represent certain moduli spaces in Algebraic geometry. For further information on this the reader is referred to [37].

6.5 Final note on the moduli of polarized Hodge structures

Let D be the Griffiths domain and Γ be an arithmetic subgroup of $\Gamma_{\mathbb{R}}$. We have the following vector bundle on $D' := \Gamma\backslash D$:

$$H^m := \cup_{\alpha \in D'} H^m(\alpha, \mathbb{C}).$$

For $0 \leq i \leq m$ it has the subbundle $F^i := \cup_{\alpha \in D'} F^i(\alpha, \mathbb{C})$. The wedge product of F^i, $\mathrm{rank}(F^i)$ times, gives us a line bundle in D' and hence an automorphic factor in D. Except in the few cases mentioned in §4.7 these line bundles are not positive, in the sense that they have not enough holomorphic sections in order to embed D' in some projective space.

7 A new point of view

In the construction of the Griffiths domain D, we have considered many Hodge structures that may not come from geometry. In the simplest way we may define that a Hodge structure comes from geometry if it arises in the m-th cohomology of some smooth algebraic variety. However, A. Grothendieck in [23] p. 260 gives an example in which a Hodge structure comes in a certain way from geometry but it is not included in our premature definition. In the case of Hodge structures arising from Riemann surfaces of genus $g \geq 2$ the Griffiths domain is of dimension $\frac{g(g+1)}{2}$ and its subspace consisting of Hodge structures coming from Riemann surfaces is is of dimension $3g - 3$. The conclusion is that it would not be a reasonable idea to look for certain algebraic structures for $\Gamma\backslash D$. Instead, we propose a point of view which is explained in this section. For simplicity, we explain it in the case of hypersurfaces.

7.1 Kodaira-Spencer Theorem on deformation of hypersurfaces

For a given smooth hypersurface M of degree d in $\mathbb{P}^{n+1}$ is there any deformation of M which is not embedded in $\mathbb{P}^{n+1}$? The answer to our question is no, except for some few cases. It is given by Kodaira-Spencer Theorem which we are going to explain it in this section. For the proof and more information on deformation of complex manifolds the reader is referred to [35], Chapter 5.

Let M be a complex manifold and $M_t, t \in B := (\mathbb{C}^s, 0),\ M_0 = M$ be a deformation of M_0 which is topologically trivial over B. We say that the parameter space B is effective if the Kodaira-Spencer map

$$\rho_0 : T_0B \to H^1(M, \Theta)$$

is injective, where Θ is the sheaf of vector fields on M. It is called complete if other families are obtained from $M_t, t \in B$ in a canonical way (see [35], p. 228).

Theorem 4. *If ρ_0 is surjective at 0 then $M_t, t \in B$ is complete.*

Let $m = \dim_{\mathbb{C}} H^1(M, \Theta)$. If one finds an effective deformation of M with $m = \dim B$ then ρ_0 is surjective and so by the above theorem it is complete.

Let us now M be a smooth hypersurface of degree d in the projective space $\mathbb{P}^{n+1}$. Let T be the projectivization of the coefficient space of smooth hypersurfaces in $\mathbb{P}^{n+1}$. In the definition of M one has already $\dim T = \binom{n+1+d}{d} - 1$ parameters, from which only

$$m := \binom{n+1+d}{d} - (n+2)^2$$

are not obtained by linear transformations of $\mathbb{P}^{n+1}$.

Theorem 5. *Assume that $n \geq 2$, $d \geq 3$ and $(n, d) \neq (2, 4)$. There exists a m-dimensional smooth subvariety of T through the parameter of M such that the Kodaira-Spencer map is injective and so the corresponding deformation is complete.*

For the proof see [35] p. 234. Let us now discuss the exceptional cases. For $(n, d) = (2, 4)$ we have 19 effective parameter but $\dim H^1(M, \Theta) = 20$. The difference comes from a non algebraic deformation of M (see [35] p. 247). In this case M is a $K3$ surface. For $n = 1$, we are talking about the deformation theory of a Riemann surface. According to Riemann's well-known formula, the complex structure of a Riemann surface of genus $g \geq 2$ depends on $3g - 3$ parameters which is again $\dim H^1(M, \Theta)$ ([35] p. 226).

7.2 Tame polynomials

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \in \mathbb{N}^{n+1}$ and assume that the greatest common divisor of all α_i's is one. We consider a parameter ring $\mathsf{R} := \mathbb{C}(t),\ t = (t_1, t_2, \ldots, t_s)$. We also consider the polynomial ring $\mathsf{R}[x] := \mathsf{R}[x_1, x_2, \ldots, x_{n+1}]$ as a graded algebra with $deg(x_i) = \alpha_i$. A polynomial $f \in \mathsf{R}[x]$ is called a quasi-homogeneous polynomial of degree d with respect to the grading α if f is a linear combination of monomials of the type $x^\beta := x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n+1}^{\beta_{n+1}}, \alpha.\beta := \sum_{i=1}^{n+1} \alpha_i \beta_i = d$. For an arbitrary polynomial $f \in \mathsf{R}[x]$ one can write in a unique way $f = \sum_{i=0}^{d} f_i,\ f_d \neq 0$, where f_i is a quasi-homogeneous polynomial of degree i. The number d is called the degree of f.

Let us be given a polynomial $f \in \mathsf{R}[x]$. We assume that f is a tame polynomial. In this text this means that there exist natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in \mathbb{N}$ such that the Milnor vector space

$$V_g := \mathsf{R}[x]/ < \frac{\partial g}{\partial x_i} \mid i = 1, 2, \ldots, n+1 >$$

is a finite dimensional R-vector space, where $g = f_d$ is the last quasi-homogeneous piece of f in the graded algebra $\mathsf{R}[x],\ \deg(x_i) = \alpha_i$.

We choose a basis $x^I = \{x^\beta \mid \beta \in I\}$ of monomials for the Milnor R-vector space and define the Gelfand-Leray n-forms

$$\omega_\beta := \frac{x^\beta dx}{df},\ \beta \in I, \tag{16}$$

where $dx = dx_1 \wedge \cdots \wedge dx_n \wedge dx_{n+1}$ (see [1]). Let $\mathbb{U}_0 = \mathbb{C}^s$, $\Delta \in \mathbb{C}(t)$ be the discriminant of f and $T := \mathbb{U}_0 \backslash (\mathrm{Zero}(\Delta) \cup \mathrm{Pole}(\Delta))$. It turns out that $L_t := \{f_t = 0\},\ t \in T$ is a topologically trivial family of affine hypersurfaces, where f_t is obtained from f by fixing the value of t. The differential forms $\omega_\beta, \beta \in I$ restricted to $L_t,\ t \in T$ form a basis of $H^n_{\mathrm{dR}}(L_t)$. The reader is referred to [44, 41] for all unproved statements in this section. In these articles we have also given algorithms which calculate a basis of the de Rham cohomology compatible with the mixed Hodge structure of the affine variety L_t. Mixed Hodge structures generalize the classical Hodge structures for arbitrary varieties which may be non compact and singular. The reader is referred to [12, 14] on this subject.

The reader may have already noticed that the main reason for us to use arbitrary weights α_i, is to put the example (1) and the hypersurfaces discussed in §7.1 into one context.

7.3 Poincaré series

Despite the fact that the differential forms $\omega_\beta,\ \beta \in I$ may not have any compatibility with the mixed Hodge structure of L_t, we may still ask for the convergence of Poincaré type series explained bellow. Define the period matrix

$$\mathsf{pm}(t) = \begin{pmatrix} \int_{\delta_1} \omega_1 & \int_{\delta_1} \omega_2 & \cdots & \int_{\delta_1} \omega_\mu \\ \int_{\delta_2} \omega_1 & \int_{\delta_2} \omega_2 & \cdots & \int_{\delta_2} \omega_\mu \\ \vdots & \vdots & \vdots & \vdots \\ \int_{\delta_\mu} \omega_1 & \int_{\delta_\mu} \omega_2 & \cdots & \int_{\delta_\mu} \omega_\mu \end{pmatrix},$$

where $\delta = (\delta_i,\ i = 1, 2, \ldots, \mu)$ is a basis of the freely generated $\mathbb{Z}$-module $H_n(L_t, \mathbb{Z})$ and $\{\omega_i \mid i = 1, 2, \ldots, \mu\} = \{\omega_\beta \mid \beta \in I\}$. Assume that the intersection matrix in the basis δ is Ψ. Different choices of the basis δ will lead to the action of

$$\Gamma_{\mathbb{Z}} := \{A \in \mathrm{Mat}(\mu \times \mu, \mathbb{Z}) \mid A \Psi A^{\mathsf{t}} = \Psi\}$$

from the left on $\mathsf{pm}(t)$. The Poincaré series in the context of this section is defined to be

$$\check{P}(t) := \sum_{A \in \Gamma_{\mathbb{Z}}} P(A \cdot \mathsf{pm}(t)),$$

where $P : \mathrm{Mat}(\mu \times \mu, \mathbb{C}) \to \mathbb{C}$ is a holomorphic function, for instance take it a rational function. If $\check{P}$ is convergent then by definition it is one valued in T. The main question we pose in this text is the convergence of $\check{P}$ and the description of the sub algebra of $\mathbb{C}(t)$ generated by those convergent $\check{P}$ which extend meromorphically to $\mathbb{U}_0$. For the discussion of these problems in the case

$$f = y^2 - 4t_0(x - t_1)^3 + t_2(x - t_1) + t_3,\ t \in \mathbb{C}^4 \tag{17}$$

the reader is referred to [42, 43].

7.4 A convergence criterion

In this section we describe a method which is used to prove that certain Poincaré series are convergent. Similar methods can be found in [17] Satz 4.3, [18] Lemma 5.1 p. 55 and [3] Theorem 5.3 p. 49. In the third reference the authors associate their convergence theorem to Harish-Chandra.

Let M be a complex manifold and ds^2 be a Hermitian metric in M and let dz be the associated volume form. Let $K \subset U \subset M$, where K is a compact set and U is an open set. There is a real positive constant C depending only on (M, ds^2) and K such that for all holomorphic functions $f : M \to \mathbb{C}$ we have:

$$|f(a)|^2 < C \int_U |f(z)|^2 dz,\ \forall a \in K,$$

(see [17] Satz 4.3, Hilfsatz 2, [18] Lemma 5.1). Note that every Hermitian form can be written in a local holomorphic coordinate system $(z_1, z_2, \ldots, z_n)$ as $dz_1 \otimes d\bar{z}_1 + \cdots + dz_n \otimes d\bar{z}_n$.

Let $\Gamma \subset \mathrm{Aut}(M)$ such that Γ acts discontinuously on M. This implies that for an arbitrary $a \in M$ the stabilizer $\Gamma_a := \{A \in \Gamma \mid Aa = a\}$ of a is finite and there exists an open neighborhood U of a such that

$$A \in \Gamma,\ A(U) \cap U \neq \emptyset \Rightarrow A \in \Gamma_a,$$

$$A \in \Gamma_a \Rightarrow A(U) = U.$$

We assume that the Hermitian metric of M is invariant under the action of Γ. Let us take a holomorphic function f on M. We claim that if

$$\int_M |f|^2 dz < \infty$$

then the Poincaré series $\check{f}(z) = \sum_{A \in \Gamma} f(Az)$ converges uniformely in z. This follows from the equalities:

$$\sum_{A \in \Gamma} |f(Az)|^2 \leq C \sum_{A \in \Gamma} \int_U |f(Az)|^2 dz = C \sum_{A \in \Gamma} \int_{A^{-1}U} |f(z)|^2 dz \leq C \cdot \#\Gamma_a \int_M |f|^2 dz < \infty.$$

7.5 References for further investigation

In this section I give a list of articles and books which may be useful for further development of the ideas explained in the present text. Of course the reader will find much more literature, if he/she looks for the reference citation or review citations of the mentioned works in Mathematical Review or Zentralblatt Mathematik.

For the literature on arithmetic and algebraic groups the book [5] is a good source of information. We have also the book [51] on algebraic groups. The original paper of Baily and Borel [3] can be served as a source for Poincaré/Eisenstein series on Hermitian symmetric domains. Some simplifications are done in [8]. An explicit construction of resolutions of the Borel-Baily compactification is given in [2, 33]. Further developments in the compactification problem is sketched in [6]. For the arithmetic point of view for the quotients of Hermitian symmetric domains by arithmetic groups the reader is referred to the original papers of Shimura [48, 49] and Deligne's papers [13, 15]. For the study of the cohomology of Shimura varieties one can mention the article [26] and two other papers with the same title. The text [37] can be served as an up to date exposition of the subject.

In the Hodge theory side of the subject the expository article of Griffiths [23] is still a good source of information. See also [22] for Deligne's report on Griffiths works. The compactification problem in Hodge theory can be seen as the determining the limit of Hodge structures. For this problem see [52, 47, 10, 11, 27, 9]. Recently, there have been attempts to look at the compcatification problem from log-geometry point of view (see [31, 32]). For the literature on log geometry the reader is referred to [28, 30, 29]

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