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Introduction to Fourier-Mukai and Nahm transforms

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Lecture 1

Introduction to Derived categories

Introduction

In this first lecture we introduce derived categories in connection with Fourier-Mukai transforms. A more comprehensive treatment of this topic may be found in [1]

Derived categories have been present in Fourier-Mukai theory from the very beginning; the paper were Mukai introduced the transform now known as Fourier-Mukai's, was indeed entitled "Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves" [4].

Let us consider the original Mukai transform from a naive point of view: assume that X is an abelian variety and $\mathcal{E}$ a vector bundle on X. We here consider only algebraic (or holomorphic) vector bundles, so we can also think of $\mathcal{E}$ as a smooth hermitian bundle E endowed with an hermitian connection ∇ which is compatible with the complex structure. We now fix an index i and look for the various cohomology spaces $H^i(X, \mathcal{E} \otimes \mathcal{P}_\xi)$ where $\mathcal{P}_\xi$ varies in the space $\hat{X}$ of all flat line bundles on X (the dual abelian variety of X). A natural question to ask is whether the vector spaces $H^i(X, \mathcal{E} \otimes \mathcal{P}_\xi)$ define a vector bundle on $\hat{X}$. In some case this happens; for instance if one has $H^j(X, \mathcal{E} \otimes \mathcal{P}_\xi) = 0$ for any $j \neq i$ and $\xi \in \hat{X}$, there is a vector bundle $\widehat{E}$ on $\hat{X}$ such that $\widehat{E}(\xi) \simeq H^i(X, \mathcal{E} \otimes \mathcal{P}_\xi)$ for any $\xi \in \hat{X}$, or, in algebraic terms, $\widehat{E} \otimes \mathcal{O}_\xi \simeq H^i(X, \mathcal{E} \otimes \mathcal{P}_\xi)$, where $\mathcal{O}_\xi$ is the skyscraper sheaf of length 1 supported at the point ξ.

In general one is no so lucky, and there is no such vector bundles (or more generally sheaves) obtained by collecting together cohomology groups. What we can do is to mimic the construction of the cohomology groups to get objects that play that role. On the product $X \times \hat{X}$ there is a *universal* line bundle $\mathcal{P}$, called the Poincaré bundle, whose restriction to the fibre $\hat{\pi}^{-1}(\xi)$ over ξ of the projection $\hat{\pi}\colon X \times \hat{X} \to \hat{X}$ is the line bundle $\mathcal{P}_\xi$; we normalise $\mathcal{P}$ so that it restrict to the trivial line bundle on the fibre of the origin x_0 of X for the other projection $\pi\colon X \times \hat{X} \to X$. An analogy with the construction of the cohomology groups of a sheaf, we take the sheaf $\mathcal{F} = \pi^*\mathcal{E} \otimes \mathcal{P}$ (whose restriction to $\hat{\pi}^{-1}(\xi)$ is precisely $\mathcal{E} \otimes \mathcal{P}_\xi$), and a resolution

$$0 \to \mathcal{F} \to \mathcal{R}^0 \to \mathcal{R}^1 \to \dots \to \mathcal{R}^n \to \dots$$

by injective sheaves and define the *higher direct images* of $\mathcal{F}$ under $\hat{\pi}$ as the cohomology sheaves of the complex

$$0 \to \pi_*\mathcal{F} \to \pi_*\mathcal{R}^0 \to \pi_*\mathcal{R}^1 \to \dots \to \pi_*\mathcal{R}^n \to \dots\,,$$

that is,

$$R^i\pi_*\mathcal{F} = \mathcal{H}^i(\pi_*\mathcal{R}^\bullet)\,.$$

The relationship between the sheaves $R^i\pi_*(\pi^*\mathcal{E} \otimes \mathcal{P})$ and the cohomology groups $H^i(X, \mathcal{E} \otimes \mathcal{P}_\xi)$ is given by some "cohomology base change" theorems [3, III.12]. This shows that the sheaves $R^i\pi_*(\pi^*\mathcal{E} \otimes \mathcal{P})$ encode more information than the cohomology groups on the fibres. Another classical fact is the the higher direct images are independent of the resolution $\mathcal{R}^\bullet$ of $\mathcal{F}$, that is, if $0 \to \mathcal{F} \to \tilde{\mathcal{R}}^\bullet$ is another acyclic of $\mathcal{F}$ (meaning that the higher direct images $R^i\hat{\pi}^i\tilde{\mathcal{R}}^j$ are zero for every $i, j \geq 0$), then the complexes os sheaves $\pi_*\mathcal{R}^\bullet$ and $\pi_*\tilde{\mathcal{R}}^\bullet$ have the same cohomology sheaves. If we then identify complexes of sheaves when they have the same cohomology sheaves (we say that they are *quasi-isomorphic*, and we write $\mathbf{R}\hat{\pi}_*\mathcal{F}$ for the "class" of any of the complexes $\pi_*\mathbf{R}^\bullet$, the information about the cohomology groups $H^i(X, \mathcal{E} \otimes \mathcal{P}_\xi)$ is encode in the single object

$$\Phi(\mathcal{E}) = \mathbf{R}\hat{\pi}_*\mathcal{F} = \mathbf{R}\hat{\pi}_*(\pi^*\mathcal{E} \otimes \mathcal{P})\,.$$

To make a sense of all of this, we have to construct a category where quasi-isomorphic complexes of sheaves are isomorphic and where we can define "derived functors" such as $\mathbf{R}\hat{\pi}_*$, and also some derived versions $\mathbf{L}\pi^*$ of the pull back π^* and $\mathcal{G} \overset{\mathbf{L}}{\otimes} \mathcal{P}$ of the tensor product, when we are working with sheaves which are not vector bundles.

We are going to describe the construction for the derived category of an abelian variety $\mathfrak{A}$. In our situation $\mathfrak{A}$ will be one of the following:

- The category of modules over a (commutative and unitary) ring A.
- The category $\mathfrak{Mod}(X)$ of sheaves of $\mathcal{O}_X$ modules on an algebraic variety X.

- The category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves of $\mathcal{O}_X$ modules on an algebraic variety X. This case includes the first for rings of the form $A = k[\xi_1, \dots, \xi_r]$ when $X = \operatorname{Spec} A$ is an affine variety.
- The category $\mathfrak{Coh}(X)$ of coherent sheaves of $\mathcal{O}_X$ modules on an algebraic variety X.

1.1 The categories $\mathbf{C}(\mathfrak{A})$ and $K(\mathfrak{A})$

Let $\mathfrak{A}$ be an abelian category. A *complex* $(\mathcal{K}^\bullet, d_{\mathcal{K}^\bullet})$ in $\mathfrak{A}$ is a sequence

$$\cdots \to \mathcal{K}^{n-1} \xrightarrow{d^{n-1}} \mathcal{K}^n \xrightarrow{d^n} \mathcal{K}^{n+1} \to \cdots$$

where the K^n are objects in $\mathfrak{A}$ and the morphisms d^n are morphisms in $\mathfrak{A}$ satisfying the condition $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$. We say that $d_{\mathcal{K}^\bullet}$ is the *differential* of the complex $\mathcal{K}^\bullet$.

Definition 1.1. The category of complexes $\mathbf{C}(\mathfrak{A})$ is the category whose objects are complexes $(\mathcal{K}^\bullet, d_{\mathcal{K}^\bullet})$ in $\mathfrak{A}$ and whose morphisms $f\colon (\mathcal{K}^\bullet, d_{\mathcal{K}^\bullet}) \to (\mathcal{L}^\bullet, d_{\mathcal{L}^\bullet})$ are collections of morphisms $f^n : \mathcal{K}^n \to \mathcal{L}^n$, $n \in \mathbb{Z}$, in $\mathfrak{A}$ such that the diagrams

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{K}^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"] & \mathcal{K}^n \arrow[r, "d^n"] \arrow[d, "f^n"] & \mathcal{K}^{n+1} \arrow[r, "d^{n+1}"] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & \mathcal{L}^{n-1} \arrow[r, "d^{n-1}"] & \mathcal{L}^n \arrow[r, "d^n"] & \mathcal{L}^{n+1} \arrow[r, "d^{n+1}"] & \cdots
\end{tikzcd}$$

commute. △

Given two complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$, their direct sum $\mathcal{K}^\bullet \oplus \mathcal{L}^\bullet$ is defined in the obvious way. One can also describe in a natural way the kernel and the cokernel of a morphism of complexes, and readily check that the category $\mathbf{C}(\mathfrak{A})$ of complexes of an abelian category is an abelian category as well.

Let $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ be complexes; for each $n \in \mathbb{Z}$, we set

$$\operatorname{Hom}(\mathcal{K}^\bullet, \mathcal{L}^\bullet)^n = \prod_i \operatorname{Hom}_{\mathfrak{A}}(\mathcal{K}^i, \mathcal{L}^{i+n})\,.$$

These groups form a complex of abelian groups

$$\operatorname{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet) = \bigoplus_n \operatorname{Hom}(\mathcal{K}^\bullet, \mathcal{L}^\bullet)^n \tag{1.1}$$

endowed with the differential given by

$$\begin{aligned} d^n\colon \operatorname{Hom}(\mathcal{K}^\bullet, \mathcal{L}^\bullet)^n &\to \operatorname{Hom}(\mathcal{K}^\bullet, \mathcal{L}^\bullet)^{n+1} \\ f^i &\mapsto d^{i+n}_{\mathcal{L}^\bullet} \circ f^i + (-1)^{n+1} f^{i+1} \circ d^i_{\mathcal{K}^\bullet} \end{aligned} \tag{1.2}$$

Since our abelian categories admit tensor products, if one assumes that they also admit arbitrary direct sums, we can define the tensor product by:

$$(\mathcal{K}^\bullet \otimes \mathcal{L}^\bullet)^n = \bigoplus_{p+q=n} (\mathcal{K}^p \otimes \mathcal{L}^q)$$

endowed with the differential $d_{\mathcal{K}^\bullet} \otimes \mathrm{Id} + (-1)^p \mathrm{Id} \otimes d_{\mathcal{L}^\bullet}$ over $\mathcal{K}^p \otimes \mathcal{L}^q$. If $\mathfrak{A}$ has no arbitrary direct sum (like the category $\mathfrak{Coh}(X)$), then $\mathcal{K}^\bullet \otimes \mathcal{L}^\bullet$ is defined only if for every n there are only a finite number of summands in $\bigoplus_{p+q=n}(\mathcal{K}^p \otimes \mathcal{L}^q)$.

An important notion is that of shift of a complex by an integer number. For a complex $\mathcal{K}^\bullet$, we define $\mathcal{K}^\bullet[n]$ by setting $\mathcal{K}[n]^p = \mathcal{K}^{p+n}$ with the differential $d_{\mathcal{K}^\bullet[n]} = (-1)^n d_{\mathcal{K}^\bullet}$.

A morphism of complexes $f\colon \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ induces a morphism of complexes $f[n]\colon \mathcal{K}^\bullet[n] \to \mathcal{L}^\bullet[n]$ given by $f[n]^p = f^{p+n}$. In this way, $\mathcal{K}^\bullet \mapsto \mathcal{K}^\bullet[n]$ is an additive functor. Sometimes we shall denote by τ the shifting functor by 1, $\tau(\mathcal{K}^\bullet) = \mathcal{K}^\bullet[1]$, so that $\tau^n(\mathcal{K}^\bullet) = \mathcal{K}^\bullet[n]$ for any integer n. One has canonical isomorphisms:

$$\mathcal{K}^\bullet[n] \otimes \mathcal{L}^\bullet \simeq (\mathcal{K}^\bullet \otimes \mathcal{L}^\bullet)[n] \simeq \mathcal{K}^\bullet \otimes \mathcal{L}^\bullet[n]$$
$$\mathrm{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet[n]) \simeq \mathrm{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)[n] \simeq \mathrm{Hom}^\bullet(\mathcal{K}^\bullet[-n], \mathcal{L}^\bullet)$$

The *n-th cohomology* of a complex $\mathcal{K}^\bullet$ is the object

$$\mathcal{H}^n(\mathcal{K}^\bullet) = \ker d^n / \operatorname{Im} d^{n-1}\,.$$

We say that $\mathcal{Z}^n(\mathcal{K}^\bullet) = \ker d^n$ are the *n-cycles* of $\mathcal{K}^\bullet$ and $\mathcal{B}^n(\mathcal{K}^\bullet) = \operatorname{Im} d^{n-1}$ are the *n-boundaries* of $\mathcal{K}^\bullet$.

A morphism of complexes $f\colon \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ induces morphisms between the cycles and the boundaries, and then it passes to cohomology yielding morphisms

$$\mathcal{H}^n(f)\colon \mathcal{H}^n(\mathcal{K}^\bullet) \to \mathcal{H}^n(\mathcal{L}^\bullet)\,,$$

for every n. One has $\mathcal{H}^n(\mathcal{K}^\bullet[m]) \simeq \mathcal{H}^{n+m}(\mathcal{K}^\bullet)$ and $\mathcal{H}^n(f[m]) \simeq \mathcal{H}^{n+m}(f)$.

We say that a complex $\mathcal{K}^\bullet$ is said to be *acyclic* or *exact* if $\mathcal{H}(\mathcal{K}^\bullet) = 0$; we also say that a morphism of complexes $f\colon \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ is called a *quasi-isomorphism* if $\mathcal{H}(f)\colon \mathcal{H}(\mathcal{K}^\bullet) \to \mathcal{H}(\mathcal{L}^\bullet)$ is an isomorphism. The composition of two quasi-isomorphisms is a quasi-isomorphism.

We now introduce the important notion of homotopy equivalence, which will enable us to build a new category — the homotopy category — out of the category of complexes.

Let $f\colon \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ a morphism of complexes. We say that f is *homotopic to zero* if there is a collection of morphisms $h^n\colon \mathcal{K}^n \to \mathcal{L}^{n-1}$ such that $f^n = h^{n+1} \circ d^n_{\mathcal{K}^\bullet} + d^{n-1}_{\mathcal{L}^\bullet} \circ h^r$ for every n. A complex $\mathcal{K}^\bullet$ is said to be homotopic to zero if its identity morphism is homotopic to zero. Finally, two morphisms $f, g\colon \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ are said to be homotopic if $f - g$ is homotopic to zero.

It is clear that the sum of two morphisms homotopic to zero is homotopic to zero. Moreover, the composition $f \circ g$ is homotopic to zero whenever either f or g is homotopic to zero. Let us denote by $\mathrm{Ht}(\mathcal{K}^\bullet, \mathcal{L}^\bullet)$ the set of the morphisms of complexes $f\colon \mathcal{K}^\bullet \to \mathcal{R}^\bullet$ which are homotopic to zero.

Definition 1.2. The homotopy category $K(\mathfrak{A})$ is the category whose objects are the objects of $\mathbf{C}(\mathfrak{A})$ and whose morphisms are

$$\mathrm{Hom}_{K(\mathfrak{A})}(\mathcal{K}^\bullet, \mathcal{L}^\bullet) = \mathrm{Hom}_{\mathbf{C}(\mathfrak{A})}(\mathcal{K}^\bullet, \mathcal{L}^\bullet) / \, \mathrm{Ht}(\mathcal{K}^\bullet, \mathcal{L}^\bullet)\,.$$

$\triangle$

From Equation (1.2) we see that the n-cycles of the complex of homomorphisms $\mathrm{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)$ coincide with the morphisms of complexes $\mathcal{K}^\bullet \to \mathcal{L}^\bullet[n]$, while the n-boundaries coincide with morphisms homotopic to zero. Therefore,

$$\mathcal{H}^n(\mathrm{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)) = \mathrm{Hom}_{K(\mathfrak{A})}(\mathcal{K}^\bullet, \mathcal{L}^\bullet[n])\,.$$

What is important to us here is that if a morphism of complexes $f\colon \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ is homotopic to zero, then it induces the zero morphism in cohomology, $\mathcal{H}(f) = 0$; hence, *two homotopic morphisms induce the same morphism in cohomology.* In particular, *if a complex* $\mathcal{K}^\bullet$ *is homotopic to zero, then it is acyclic.*

We now define the cone of a morphism; this notion comes from classical homotopy theory, and its is a way of overcoming the fact that there are no kernels and cokernels in the homotopy category $K(\mathfrak{A})$.

Definition 1.3. The cone of a morphism of complexes $f\colon \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ is the complex $\mathrm{Cone}(f)$ such that $\mathrm{Cone}(f)^n = \mathcal{K}^{n+1} \oplus \mathcal{L}^n$ and the differential is defined as

$$d^n_{\mathrm{Cone}(f)} = \begin{pmatrix} -d^{n+1}_{\mathcal{K}^\bullet} & 0 \\ f^{n+1} & d^n_{\mathcal{L}^\bullet} \end{pmatrix}$$

$\triangle$

Although one has $\mathrm{Cone}(f)^n = (\mathcal{K}^\bullet[1])^n \oplus \mathcal{L}^n$ for every n, $\mathrm{Cone}(f)$ is not isomorphic as a complex with the direct sum $\mathcal{K}^\bullet[1] \oplus \mathcal{L}^\bullet$, because the differential of the latter is the direct sum of the differentials of the factors. There are functorial morphisms :

$$\begin{aligned} \beta\colon\ \mathrm{Cone}(f) &\to \mathcal{K}^\bullet[1], & \alpha\colon \mathcal{L}^\bullet &\to \mathrm{Cone}(f) \\ (k, l) &\mapsto k & l &\mapsto (0, l) \end{aligned}$$

and an exact sequence of complexes $0 \to \mathcal{L}^\bullet \to \mathrm{Cone}(f) \to \mathcal{K}^\bullet[1] \to 0$. Let us consider the sequence

$$\mathcal{K}^\bullet \xrightarrow{f} \mathcal{L}^\bullet \xrightarrow{\alpha} \mathrm{Cone}\, f \xrightarrow{\beta} \mathcal{K}^\bullet[1]\,. \tag{1.3}$$

The composition $\alpha \circ f$ is homotopic to zero. If we consider the sequence (1.3) in the homotopy category $K(\mathfrak{A})$, the composition of any two consecutive morphisms is zero. The sequence (1.3) is called *a distinguished (or exact) triangle* in $K(\mathfrak{A})$ and is also written in the form

$$\begin{tikzcd} \mathcal{K}^\bullet \arrow[rr, "f"] & & \mathcal{L}^\bullet \arrow[dl, "\alpha"] \\ & \operatorname{Cone} f \arrow[ul, dashed, "\beta"] & \end{tikzcd}$$

where the dashed arrow stands for a morphism $\operatorname{Cone} f \to \mathcal{K}^\bullet[1]$. The following property is readily checked:

Proposition 1.4. *Given an exact triangle in* $K(\mathfrak{A})$*,*

$$\mathcal{K}^\bullet \xrightarrow{f} \mathcal{L}^\bullet \xrightarrow{\alpha} \operatorname{Cone} f \xrightarrow{\beta} \mathcal{K}^\bullet[1]\,,$$

there is an exact sequence of cohomology groups

$$\mathcal{H}^n(\mathcal{K}^\bullet) \xrightarrow{\mathcal{H}^n(f)} \mathcal{H}^n(\mathcal{L}^\bullet) \xrightarrow{\mathcal{H}^n(\alpha)} \mathcal{H}^n(\operatorname{Cone} f) \xrightarrow{\mathcal{H}^n(\beta)} \mathcal{H}^n(\mathcal{K}^\bullet[1]) \simeq \mathcal{H}^{n+1}(\mathcal{K}^\bullet)$$

for every integer n. □

Putting all these exact sequences together we have the so-called *cohomology long exact sequence*:

$$\dots \xrightarrow{\mathcal{H}^{n-1}(\beta)} \mathcal{H}^n(\mathcal{K}^\bullet) \xrightarrow{\mathcal{H}^n(f)} \mathcal{H}^n(\mathcal{L}^\bullet) \xrightarrow{\mathcal{H}^n(\alpha)} \mathcal{H}^n(\operatorname{Cone} f) \xrightarrow{\mathcal{H}^n(\beta)} \mathcal{H}^{n+1}(\mathcal{K}^\bullet) \dots \tag{1.4}$$

Proposition 1.4 tell us that the the functors $\mathcal{H}^n\colon K(\mathfrak{A}) \to \mathfrak{A}$ are cohomological. Actually, if $\mathfrak{B}$ is another abelian category, an additive functor $F\colon K(\mathfrak{A}) \to \mathfrak{B}$ is *cohomological* if for every exact triangle $\mathcal{K}^\bullet \xrightarrow{f} \mathcal{L}^\bullet \xrightarrow{\alpha} \operatorname{Cone} f \xrightarrow{\beta} \mathcal{K}^\bullet[1]$ the sequence $F(\mathcal{K}^\bullet) \xrightarrow{F(f)} F(\mathcal{L}^\bullet) \xrightarrow{\alpha} F(\operatorname{Cone} f) \xrightarrow{F(\beta)} F(\mathcal{K}^\bullet)[1]$ is exact.

An important consequence of the cohomology long exact sequence (1.4) is the following:

Corollary 1.5. *A morphism of complexes* $f\colon \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ *is a quasi-isomorphism if and only if* $\operatorname{Cone}(f)$ *is acyclic.* □

Cones are good substitutes for exact sequences of complexes.

If $0 \to \mathcal{K}^\bullet \xrightarrow{f} \mathcal{L}^\bullet \xrightarrow{g} \mathcal{N}^\bullet \to 0$ be an exact sequence of complexes (in $\mathbf{C}(\mathfrak{A})$), there is a morphism of complexes : $\operatorname{Cone}(f) \to \mathcal{N}^\bullet$ defined in degree n by

$$\begin{aligned} \mathcal{K}^{n+1} \oplus \mathcal{L}^n &\to \mathcal{N}^n \\ (a_{n+1}, b_n) &\mapsto g(b_n)\,. \end{aligned}$$

One easily checks that it is a quasi-isomorphism. Combining this with the the cohomology long exact sequence (1.4) we get directly the more usual cohomology sequence: There exist functorial morphisms

$$\delta^n \colon H^n(\mathcal{N}^\bullet) \to H^{n+1}(\mathcal{L}^\bullet)$$

such that one has an exact sequence:

$$\cdots \xrightarrow{\delta^{n-1}} H^n(\mathcal{L}^\bullet) \to H^n(\mathcal{M}^\bullet) \to H^n(\mathcal{N}^\bullet) \xrightarrow{\delta^n} H^{n+1}(\mathcal{L}^\bullet) \to$$
$$\to H^{n+1}(\mathcal{M}^\bullet) \to H^{n+1}(\mathcal{N}^\bullet) \xrightarrow{\delta^{n+1}} \cdots$$

1.2 Derived Category

In our way to defining a category where quasi-isomorphic complexes are actually isomorphic, we have first identify homotopic morphisms, and thus move form the category of complexes $\mathbf{C}(\mathfrak{A})$ to the homotopy category $K(\mathfrak{A})$. A second step is to "localize" by (classes of) quasi-isomorphims. This localization is a fraction calculus for categories if we just think of the composition of morphisms as a product. Recall that if one has a ring A (like the integer numbers) and we cant to make the elements s is a part S of A invertible, so that a fraction a/s makes sense, this can be done if S is a multiplicative system, namely, if it contains the unity and is closed under products. Then one can define the localised ring $S^{-1}A$ whose elements are equivalence classes a/s of pairs $(a, s) \in A \times S$ where $(a, s) \sim (a', s')$ (or $a/s = a'/s'$ if there is $t \in S$ such that $t(as' - a's) = 0$. Any element $s \in S$ becomes invertible in the fractions ring $S^{-1}A$ because $s/1 \cdot 1/s = 1$.

A similar thing can be done for morphisms of complexes, since quasi-isomorphisms verify the conditions for being a nice set of denominators (or a multiplicative system as before), namely, the identity is a quasi-isomorphism and the composition of two quasi-isomorphisms is a quasi-isomorphism. We now take a fraction as a diagram of (homotopy classes of) complex morphisms

$$\begin{tikzcd} & \mathcal{R}^\bullet \arrow[dl, "\phi"'] \arrow[dr, "f"] & \\ \mathcal{K}^\bullet & & \mathcal{L}^\bullet \end{tikzcd}$$

in the homotopy category $K(\mathfrak{A})$ where ϕ is a quasi-isomorphism. We denote such a diagram by f/ϕ. A second diagram g/ψ

$$\begin{tikzcd} & \mathcal{S}^\bullet \arrow[dl, "\psi"'] \arrow[dr, "g"] & \\ \mathcal{K}^\bullet & & \mathcal{L}^\bullet \end{tikzcd}$$

is said to be equivalent to the former, if there are quasi-isomorphisms $\mathcal{R}^\bullet \leftarrow \mathcal{T}^\bullet \rightarrow \mathcal{S}^\bullet$ such that the diagram

$$\begin{tikzcd} & & \mathcal{T}^\bullet \arrow[dl] \arrow[dr] & & \\ & \mathcal{R}^\bullet \arrow[dl, "\phi"'] \arrow[drrr, "f"' near end] & & \mathcal{S}^\bullet \arrow[dlll, "\psi" near end] \arrow[dr, "g"] & \\ \mathcal{K}^\bullet & & & & \mathcal{L}^\bullet \end{tikzcd}$$

is commutative in $K(\mathfrak{A})$. One can prove that equivalence of fractions is actually an equivalence relation using the following result

Proposition 1.6. *Given a diagram*

$$\begin{tikzcd} & \mathcal{R}^\bullet \arrow[d, "g"] \\ \mathcal{M}^\bullet \arrow[r, "f"] & \mathcal{N}^\bullet \end{tikzcd}$$

in $K(\mathfrak{A})$, there are morphisms of complexes $\mathcal{M}^\bullet \xleftarrow{g'} \mathcal{Z}^\bullet \xrightarrow{f'} \mathcal{R}^\bullet$ such that the diagram

$$\begin{tikzcd} \mathcal{Z}^\bullet \arrow[r, "f'"] \arrow[d, "g'"'] & \mathcal{R}^\bullet \arrow[d, "g"] \\ \mathcal{M}^\bullet \arrow[r, "f"] & \mathcal{N}^\bullet \end{tikzcd}$$

is commutative in $K(\mathfrak{A})$. Moreover, f' (respectively, g') is a quasi-isomorphism if and only if f (respectively, g) is so. □

Though we are not giving a proof here, it is important to note that the proof is based on the properties of the cone of a morphism.

Definition 1.7. The derived category $D(\mathfrak{A})$ of $\mathfrak{A}$ is the category whose objects are the objects of $K(\mathfrak{A})$ (that is, they are complexes of objects of $\mathfrak{A}$), and whose morphisms are equivalence classes $[f/\phi]$ of diagrams. △

In order this definition to make sense we have to say how to compose morphisms. This can be done thanks to Proposition 1.6. Now, given two morphisms $[f/\phi]$ and $[g/\psi]$ in $D(\mathfrak{A})$, corresponding to diagrams

$$\begin{tikzcd} & \mathcal{R}^\bullet \arrow[dl, "\phi"'] \arrow[dr, "f"] & & \mathcal{S}^\bullet \arrow[dl, "\psi"'] \arrow[dr, "g"] & \\ \mathcal{K}^\bullet & & \mathcal{L}^\bullet & & \mathcal{M}^\bullet \end{tikzcd}$$

their composition is defined through the diagram

$$\begin{tikzcd}[column sep=small]
 & & \mathcal{T}^\bullet \arrow[dl, "\psi'"'] \arrow[dr, "f'"] & & \\
 & \mathcal{R}^\bullet \arrow[dl, "\phi"'] \arrow[dr, "f"] & & \mathcal{S}^\bullet \arrow[dl, "\psi"'] \arrow[dr, "g"] & \\
\mathcal{K}^\bullet & & \mathcal{L}^\bullet & & \mathcal{M}^\bullet
\end{tikzcd}$$

Hence, we set $[g/\psi] \circ [f/\phi] = [(g \circ f')/(\phi \circ \psi')]$, which makes sense because the above construction is independent of the representatives.

A morphism $f\colon \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ in $K(\mathfrak{A})$ defines the morphism $f/\mathrm{Id}_{\mathcal{K}^\bullet}\colon \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ in the derived category, which we shall denote simply by f. Hence, we have a functor $K(\mathfrak{A}) \to D(\mathfrak{A})$.

The derived category $D(\mathfrak{A})$ is an additive category and the functor $K(\mathfrak{A}) \to D(\mathfrak{A})$ is additive.

A morphism $f/\phi\colon \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ in the derived category induces a morphism in cohomology $\mathcal{H}(f/\phi)\colon \mathcal{H}(\mathcal{K}^\bullet) \to \mathcal{H}(\mathcal{L}^\bullet)$, defined as the composition

$$\mathcal{H}(\mathcal{K}^\bullet) \xrightarrow{\mathcal{H}(\phi)^{-1}} \mathcal{H}(\mathcal{R}^\bullet) \xrightarrow{\mathcal{H}(f)} \mathcal{H}(\mathcal{L}^\bullet)\,,$$

which is independent of the representative f/ϕ of the class and is compatible with compositions.

Definition 1.8. Two complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ are quasi-isomorphic if there is a complex $\mathcal{Z}^\bullet$ and quasi-isomorphisms $\mathcal{K}^\bullet \leftarrow \mathcal{Z}^\bullet \to \mathcal{L}^\bullet$. △

It follows from the Lemma 1.6 that the notion of quasi-isomorphism induces an equivalence relation between complexes. One can also prove that two complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ are quasi-isomorphic if there is a complex $\mathcal{Z}^\bullet$ and quasi-isomorphisms $\mathcal{K}^\bullet \to \mathcal{Z}^\bullet \leftarrow \mathcal{L}^\bullet$.

We now have the result we were looking for:

Proposition 1.9. *A morphism of complexes $f\colon \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ is a quasi-isomorphism if and only the induced morphism in the derived category is an isomorphism. Moreover, two complexes are quasi-isomorphic if and only if they are isomorphic in the derived category.*

The derived category can be also defined by means of a universal property.

Proposition 1.10. *Let $\mathfrak{C}$ be an additive category. An additive functor $F\colon K(\mathfrak{A}) \to \mathfrak{C}$ factors through an additive functor $D(\mathfrak{A}) \to \mathfrak{C}$ if and only if it maps quasi-isomorphisms to isomorphisms. If $\mathfrak{B}$ is an abelian category, an additive functor*

$G\colon K(\mathfrak{A}) \to K(\mathfrak{B})$ mapping quasi-isomorphisms into quasi-isomorphisms induces an additive functor $G\colon D(\mathfrak{A}) \to D(\mathfrak{B})$ such that the diagram

$$\begin{tikzcd} \mathbf{C}(\mathfrak{A}) \arrow[r, "G"] \arrow[d] & \mathbf{C}(\mathfrak{B}) \arrow[d] \\ D(\mathfrak{A}) \arrow[r, "G"] & D(\mathfrak{B}) \end{tikzcd}$$

is commutative. □

We can also defined categories out of some subcategories of $\mathbf{C}(\mathfrak{A})$, as long as all the operations we have done can be reproduced in the new situation, namely, we have to be able to construct the corresponding homotopy category, and to localise by the quasi-isomorphims; for this, we need to be able to define the cone of a morphism inside the new category (cf. for instance Proposition 1.6, whose proof requires the cone construction).

The most natural examples are the following:

- Let us consider the category $\mathbf{C}^+(\mathfrak{A})$ of *bounded below* complexes in $\mathfrak{A}$, that is, complexes $\mathcal{K}^\bullet$ for which there is n_0 such that $\mathcal{K}^{\bullet n} = 0$ for all $n \le n_0$. We can define the homotopy category $K^+(\mathfrak{A})$ and a "derived" category $D^+(\mathfrak{A})$ by following an analogous procedure as the one for arbitrary complexes. Due to Proposition 1.10, the natural functor $K^+(\mathfrak{A}) \to D(\mathfrak{A})$ induces a functor $\gamma\colon D^+(\mathfrak{A}) \to D(\mathfrak{A})$. This functor is fully faithful and its essential image is the faithful subcategory of $D(\mathfrak{A})$ consisting of complexes in $\mathfrak{A}$ with bounded below cohomology. (The essential image of the functor γ is the subcategory of the objects which are isomorphic to objects of the form $\gamma(\mathcal{K}^\bullet)$ for some $\mathcal{K}^\bullet$ in $D^+(\mathfrak{A})$). In a similar way, the categories $\mathbf{C}^-(\mathfrak{A})$ of *bounded above* complexes (i.e. complexes for which there is n_0 such that $\mathcal{K}^{\bullet n} = 0$ for all $n \ge n_0$) and $\mathbf{C}^b(\mathfrak{A})$) of *bounded* on both sides complexes, give rise to "derived" categories $D^-(\mathfrak{A})$ and $D^b(\mathfrak{A})$, which are characterised as faithful subcategories of $D(\mathfrak{A})$ as above.

- Let $\mathfrak{A}'$ be a thick abelian subcategory of $\mathfrak{A}$, that is, any extension in $\mathfrak{A}$ of two objects of $\mathfrak{A}'$ is also in $\mathfrak{A}'$. If $\mathbf{C}_{\mathfrak{A}'}(\mathfrak{A})$ is the category of complexes whose cohomology objects are in $\mathfrak{A}'$, we can construct its homotopy category $K_{\mathfrak{A}'}(\mathfrak{A})$ and its derived category $D_{\mathfrak{A}'}(\mathfrak{A})$. The functor $K_{\mathfrak{A}'}(\mathfrak{A}) \to D(\mathfrak{A})$ induces by Proposition 1.10 a functor $D_{\mathfrak{A}'}(\mathfrak{A}) \to D(\mathfrak{A})$, which is fully faithful; its essential image is the subcategory of $D(\mathfrak{A})$ whose objects are the complexes with cohomology objects in $\mathfrak{A}'$.

- Combining the two procedures we also have the homotopy categories $K^+_{\mathfrak{A}'}(\mathfrak{A})$, $K^-_{\mathfrak{A}'}(\mathfrak{A})$ and $K^b_{\mathfrak{A}'}(\mathfrak{A})$ of complexes bounded below, above and on both sides, respectively, and whose cohomology objects are in the subcategory $\mathfrak{A}'$ of $\mathfrak{A}$. We also have the corresponding derived categories $D^+_{\mathfrak{A}'}(\mathfrak{A})$, $D^-_{\mathfrak{A}'}(\mathfrak{A})$ and $D^b_{\mathfrak{A}'}(\mathfrak{A})$

Let us write $\star$ for any of the symbols $+$, $-$, b or for no symbol at all. Since the functor natural functor $K^\star(\mathfrak{A}') \to D(\mathfrak{A})$ maps quasi-isomorphisms to isomorphisms, it yields a functor $D^\star(\mathfrak{A}') \to D^\star_{\mathfrak{A}'}(\mathfrak{A})$. In general, it may fail to be an equivalence of categories.

We have special notations for the abelian categories we are most interested in:

- If $\mathfrak{A}$ is the category of modules over a commutative ring A, we use the notations $D(A)$, $D^+(A)$, $D^-(A)$, and $D^b(A)$.
- If $\mathfrak{A} = \mathfrak{Mod}(X)$ is the category of sheaves of $\mathcal{O}_X$-modules on an algebraic variety X, the corresponding derived categories are denoted by $D(X)$, $D^+(X)$, $D^-(X)$, and $D^b(X)$.
- If $\mathfrak{A} = \mathfrak{Mod}(X)$ as above and $\mathfrak{A}' = \mathfrak{Qco}(X)$ is the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules on X, the derived category $D_{\mathfrak{A}'}(\mathfrak{A})$ of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves is denoted $D_{qc}(X)$. In a similar we have the categories $D^+_{qc}(X)$, $D^-_{qc}(X)$ and $D^b_{qc}(X)$.
- If $\mathfrak{A} = \mathfrak{Mod}(X)$ as above and $\mathfrak{A}' = \mathfrak{Coh}(X)$ is the category of coherent sheaves of $\mathcal{O}_X$-modules on X, the derived category $D_{\mathfrak{A}'}(\mathfrak{A})$ of complexes of $\mathcal{O}_X$-modules with coherent cohomology sheaves is denoted $D_c(X)$. One also has the derived categories $D^+_c(X)$, $D^-_c(X)$ and $D^b_c(X)$.
- If $\mathfrak{A} = \mathfrak{Qco}(X)$ and $\mathfrak{A}' = \mathfrak{Coh}(X)$, we have the derived categories $D_c(\mathfrak{Qco}(X))$ $D^+_c(\mathfrak{Qco}(X))$, $D^-_c(\mathfrak{Qco}(X))$ and $D^b_c(\mathfrak{Qco}(X))$.

As we already mentioned, the natural functors $D^\star_{qc}(X) \to D^\star(\mathfrak{Qco}(X))$ may fail to be equivalences of categories. However, one has equivalences of categories:

$$D^+_{qc}(X) \simeq D^+(\mathfrak{Qco}(X))\,, \quad D^b_{qc}(X) \simeq D^b(\mathfrak{Qco}(X))\,.$$

The first equivalence is a consequence of the fact that every quasi-coherent sheaf on an algebraic variety can be embedded as a subsheaf of an *injective* quasi-coherent sheaf. One also has $D^+_c(X) \simeq D^+_c(\mathfrak{Qco}(X))$ and $D^b_c(X) \simeq D^b_c(\mathfrak{Qco}(X))$.

When X is smooth, the same is true for unbounded complexes as well, so that $D^\star_{qc}(X) \simeq D^\star(\mathfrak{Qco}(X))$ and $D^\star(\mathfrak{Coh}(X)) \simeq D^\star_c(\mathfrak{Qco}(X)) \simeq D^\star_c(X)$ for any value of $\star$

The derived category as a triangulated category

The derived category $D(\mathfrak{A})$ (and any of the derived categories $D^{\star}_{\mathfrak{A}'}(\mathfrak{A})$) is an example of a triangulated category. We are not giving here the definition of triangulated category, but just point out some of the features of the derived category that make it into a triangulated category.

The first one is the existence of a shift functor $\tau\colon D(\mathfrak{A}) \to D(\mathfrak{A})$, $\tau(\mathcal{K}^\bullet) = \mathcal{K}^\bullet[1]$, which is an equivalence of categories. The second is the existence of "triangles", and among them a class of "distinguished triangles" fulfilling some properties we are not describing here. In the case of the derived category, distinguished triangles are defined in the following way.

A *triangle* in $D(\mathfrak{A})$ is a sequence of morphisms

$$\mathcal{K}^\bullet \xrightarrow{u} \mathcal{L}^\bullet \xrightarrow{v} \mathcal{M}^\bullet \xrightarrow{w} \mathcal{K}^\bullet[1]$$

which we also write in the form:

$$\begin{tikzcd} \mathcal{K}^\bullet \arrow[rr, "u"] & & \mathcal{L}^\bullet \arrow[dl, "v"] \\ & \mathcal{M}^\bullet \arrow[ul, dashed, "w"] & \end{tikzcd}$$

where the dashed arrow stands for the morphism $\mathcal{M}^\bullet \xrightarrow{w} \mathcal{K}^\bullet[1]$. A morphism of triangles is defined in the obvious way, and we say that a triangle is *distinguished or exact* if it is isomorphic to the triangle defined by the cone of a morphism $f\colon \mathcal{K}^\bullet \to \mathcal{L}^\bullet$, which is the triangle (cf. (1.3))

$$\mathcal{K}^\bullet \xrightarrow{f} \mathcal{L}^\bullet \xrightarrow{\alpha} \mathrm{Cone}(f) \xrightarrow{\beta} \mathcal{K}^\bullet[1]\,.$$

From Proposition 1.4, an exact triangle in $D(\mathfrak{A})$ induces a long exact sequence in cohomology

$$\begin{aligned} \cdots \to \mathcal{H}^i(\mathcal{A}^\bullet) \xrightarrow{\mathcal{H}^i(u)} \mathcal{H}^i(\mathcal{B}^\bullet) \xrightarrow{\mathcal{H}^i(v)} \mathcal{H}^i(\mathcal{C}^\bullet) \xrightarrow{\mathcal{H}^i(w)} \\ \mathcal{H}^{i+1}(\mathcal{A}^\bullet) \xrightarrow{\mathcal{H}^{i+1}(u)} \mathcal{H}^{i+1}(\mathcal{B}^\bullet) \xrightarrow{\mathcal{H}^{i+1}(v)} \mathcal{H}^{i+1}(\mathcal{C}^\bullet) \xrightarrow{\mathcal{H}^{i+1}(w)} \cdots \end{aligned}$$

If $\mathfrak{B}$ is another abelian category, an additive functor $F\colon D^{\star}_{\mathfrak{A}'}(\mathfrak{A}) \to D(\mathfrak{B}$ is said to be *exact* if it commutes with the shift functor, $F(\mathcal{K}^\bullet[1]) \simeq F(\mathcal{K}^\bullet)[1]$, and maps exact triangles to exact triangles. Then, for any exact triangle $\mathcal{K}^\bullet \xrightarrow{u} \mathcal{L}^\bullet \xrightarrow{v} \mathcal{M}^\bullet \xrightarrow{w} \mathcal{K}^\bullet[1]$ we have a long exact sequence

$$\begin{aligned} \cdots \to \mathcal{H}^i(F(\mathcal{K}^\bullet) \to \mathcal{H}^i(F(\mathcal{L}^\bullet)) \to \mathcal{H}^i(F(\mathcal{M}^\bullet)) \to \\ \mathcal{H}^{i+1}(F(\mathcal{K}^\bullet)) \to \mathcal{H}^{i+1}(F(\mathcal{L}^\bullet)) \to \mathcal{H}^{i+1}(F(\mathcal{M}^\bullet)) \to \cdots \end{aligned}$$

1.2.1 Derived Functors

We know that he cohomology groups of a sheaf $\mathcal{F}$ on an algebraic variety X are the cohomology objects of the complex of global sections $\Gamma(X, \mathcal{I}^\bullet)$ of a resolution $\mathcal{I}^\bullet$ of $\mathcal{F}$ by injective sheaves. Moreover, the De Rham theorem says that two different resolutions give rise to the same cohomology groups. Now one easily checks that if $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ are two injective resolutions of a sheaf $\mathcal{F}$, the complexes $\Gamma(X, \mathcal{I}^\bullet)$ and $\Gamma(X, \mathcal{J}^\bullet)$ are quasi-isomorphic, so that they define isomorphic objects in the derived category of the category of abelian groups. We can then associate with $\mathcal{F}$ a single object $\mathbf{R}\Gamma(X, \mathcal{F}); = \Gamma(X, \mathcal{I}^\bullet) \simeq \Gamma(X, \mathcal{J}^\bullet)$.

This is the procedure we mimic to define derived functors on the derived category. Let $\mathfrak{A}$ be an abelian category with enough injectives. Hence, any object $\mathcal{M}$ in $\mathfrak{A}$ has an injective resolution

$$\mathcal{M} \to I^0(\mathcal{M}) \to I^1(\mathcal{M}) \to \dots$$

functorial in $\mathcal{M}$. One can prove using bicomplexes, a notion we have not introduced in this notes, that for any complex $\mathcal{M}^\bullet$ there is a complex of injective objects $I(\mathcal{M}^\bullet)$ and a quasi-isomorphism

$$\mathcal{M}^\bullet \to I(\mathcal{M}^\bullet)\,,$$

which defines a functor $I\colon K(\mathfrak{A}) \to K(\mathfrak{A})$.

Let now $\mathfrak{B}$ be another abelian category and $F\colon \mathfrak{A} \to \mathfrak{B}$ a left-exact functor. Then F induces a functor $\mathbf{R}F\colon K^+(\mathfrak{A}) \to D^+(\mathfrak{B})$ by $\mathbf{R}F(\mathcal{M}^\bullet) = F(I(\mathcal{M}^\bullet))$. Moreover, if $\mathcal{J}^\bullet$ is an acyclic complex of injective objects then $F(\mathcal{J}^\bullet)$ is acyclic, because $\mathcal{J}^\bullet$ splits. This implies that $\mathbf{R}F$ maps quasi-isomorphisms to isomorphisms and then (cf. Proposition 1.10) yields a functor

$$\mathbf{R}F\colon D^+(\mathfrak{A}) \to D^+(\mathfrak{B})\,,$$

which is the *right derived functor* of F.

We can also derive on the right functors from $K(\mathfrak{A})$ to $K(\mathfrak{B})$ what are not induced by a left-exact functor. We shall give some examples, without the complete theory.

We shall denote $\mathbf{R}^iF(\mathcal{M}^\bullet) = H^i(\mathbf{R}F(\mathcal{M}^\bullet))$. The restriction of the functor $\mathbf{R}^iF\colon D(\mathfrak{A}) \to \mathfrak{B}$ to $\mathfrak{A}$ is the "classical" right i-th derived functor of F.

The right derived functor $\mathbf{R}F$ is exact, that is, it maps exact triangle to exact triangles. In particular, an exact triangle in $K(\mathfrak{A})$

$$\mathcal{M}'^\bullet \to \mathcal{M}^\bullet \to \mathcal{M}''^\bullet \to \mathcal{M}'^\bullet[1]$$

induces a long exact sequence

$$\cdots \to \mathbf{R}^i F(\mathcal{M}'^\bullet) \to \mathbf{R}^i F(\mathcal{M}^\bullet) \to \mathbf{R}^i F(\mathcal{M}''^\bullet) \to \\ \mathbf{R}^{i+1} F(\mathcal{M}'^\bullet) \to \mathbf{R}^{i+1} F(\mathcal{M}^\bullet) \to \mathbf{R}^{i+1} F(\mathcal{M}''^\bullet) \to \cdots$$

For any bounded below complex $\mathcal{K}^\bullet$ there is a natural morphism

$$\mathcal{M}^\bullet \to \mathbf{R}F(\mathcal{M}^\bullet)\,.$$

in the derived category. The complex $\mathcal{M}^\bullet$ is said to be *F-acyclic* is this morphism is an isomorphism, that is, $\mathcal{M}^\bullet \simeq \mathbf{R}F(\mathcal{M}^\bullet)$ in $D^+(\mathfrak{B})$.

The right derived functor $\mathbf{R}F$ satisfies a version of the "De Rham theorem", namely, if a complex $\mathcal{M}^\bullet$ is isomorphic in the derived category $D^+(\mathfrak{A}$ to a F-acyclic complex $\mathcal{J}^\bullet$, then

$$\mathbf{R}F(\mathcal{M}^\bullet) \simeq F(\mathcal{J}^\bullet)\,,$$

in $D^+(\mathfrak{B})$.

Let $\mathfrak{C}$ be a third abelian category and $G\colon \mathfrak{B} \to \mathfrak{C}$ another left-exact functor.

Proposition 1.11 (Composite functor theorem of Grothendieck)**.** *If F transforms complexes of injective objects into G-acyclic complexes, one has a natural isomorphism of derived functors*

$$\mathbf{R}(G \circ F) \xrightarrow{\sim} \mathbf{R}G \circ \mathbf{R}F\,.$$

The theory of right derived functors can be applied when $\mathfrak{A}$ is one of the categories $\mathfrak{Mod}(X)$ or $\mathfrak{Qco}(X)$ because both have enough injectives.

One can develop in a similar way a theory for deriving left exact functors on the left if we assume that $\mathfrak{A}$ has enough projectives, so that any object $\mathcal{M}$ has a functorial projective resolution

$$\dots P^1(\mathcal{M}) \to P^0(\mathcal{M}) \to \mathcal{M} \to 0\,.$$

Then for every bounded above complex $\mathcal{M}^\bullet$ there exists a bounded above complex $P(\mathcal{M}^\bullet)$ of projective objects which defines a functor $P\colon K^-(\mathfrak{A}) \to K^-(\mathfrak{A})$. Then the functor $\mathbf{L}F\colon K^-(\mathfrak{A}) \to K^-(\mathfrak{B}$ given by $\mathbf{L}F(\mathcal{M}^\bullet) = F(P(\mathcal{M}^\bullet))$ defines as above a *left derived functor*

$$\mathbf{L}F\colon D^-(\mathfrak{A}) \to D^-(\mathfrak{B})\,.$$

Analogous properties to those proved for right derived functors holds for left derived functors.

One should note that the categories $\mathfrak{Mod}(X)$, $\mathfrak{Qco}(X)$ and $\mathfrak{Coh}(X)$ do not have enough projectives. However if we restrict ourselves to the case when X

is a projective or quasi-projective variety, then any quasi-coherent sheaf admits a resolution by locally free sheaves (possibly of infinite rank), and the problem is circumvented by considering complexes $\mathcal{P}^\bullet$ of locally free sheaves. For every bounded above complex $\mathcal{M}^\bullet$ of quasi-coherent sheaves there exist a bounded above complex $P(\mathcal{M}^\bullet)$ of locally free sheaves and a quasi-isomorphism $P(\mathcal{M}^\bullet) \to \mathcal{M}^\bullet$, and we can still define the left derived functor by $\mathbf{L}F(\mathcal{M}^\bullet) = F(P(\mathcal{M}^\bullet))$. For a general algebraic variety (not necessarily projective or quasi-projective) this is not possible, though some functors can still be derived. We shall come again to this point in Section

1.2.2 A:pullback

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Derived Direct Image

Let $f\colon X \to Y$ be a morphism of algebraic varieties. The direct image functor $f_*\colon \mathfrak{Mod}(X) \to \mathfrak{Mod}(Y)$ is left-exact, so it induces a right derived functor

$$\mathbf{R}f_*\colon D^+(X) \to D^+(Y)$$

described as $\mathbf{R}f_*\mathcal{M}^\bullet \simeq f_*(\mathcal{I}^\bullet)$ where $\mathcal{I}^\bullet$ is a complex of injective $\mathcal{O}_X$-modules quasi-isomorphic to $\mathcal{M}^\bullet$. Under very mild conditions, the direct image of a quasi-coherent sheaf is also quasi-coherent (f has to be quasi-compact and locally of finite type); in this case, $\mathbf{R}f_*$ maps complexes with quasi-coherent cohomology to complexes with quasi-coherent cohomolgy, thus defining a functor

$$\mathbf{R}f_*\colon D^+_{qc}(X) \to D^+_{qc}(Y)\,,$$

that we denote with the same symbol. When f is proper, so that the higher direct images of a coherent sheaf are coherent as well (cf. [2, Thm.3.2.1] or [3, Thm. 5.2] in the projective case), we also have a functor

$$\mathbf{R}f_*\colon D^+_c(X) \to D^+_c(Y)\,.$$

Finally, since the dimension of X bounds the number of higher direct images of a sheaf of $\mathcal{O}_X$-modules), $\mathbf{R}f_*$ maps complexes with bounded cohomology to complexes with bounded cohomology, thus defining a functor

$$\mathbf{R}f_*\colon D^b_{qc}(X) \to D^b_{qc}(Y)\,.$$

Moreover, in this case $\mathbf{R}f_*$ can be extended to a functor

$$Rf_*\colon D_{qc}(X) \to D_{qc}(Y)$$

between the whole derived categories, which actually map $D^b_{qc}(X)$ to $D^b_{qc}(Y)$ and $D^b_c(X)$ to $D^b_c(Y)$.

If Y is a point, then $\mathfrak{A}_Y$ is the category Ab of abelian groups and f_* is the functor of global sections $\Gamma(X,\ )$. In this case, $\mathbf{R}f_*\mathcal{M}^\bullet = \mathbf{R}\Gamma(X, \mathcal{M}^\bullet)$ and $\mathbf{R}^i f_*\mathcal{M}^\bullet$ is called the i-th hypercohomology group $\mathbb{H}^i(X, \mathcal{M}^\bullet)$ of the complex $\mathcal{M}^\bullet$. If coincides with the cohomology group $H^i(X, \mathcal{M})$ when the complex reduces to a single sheaf.

If $f\colon X \to Y$ is a continuous map, then $\Gamma(X,\ ) = \Gamma(Y,\ ) \circ f_*$ and one may apply the composite functor theorem of Grothendieck (since f_* transforms injective sheaves into injective sheaves) obtaining

$$\mathbf{R}\Gamma(X,\ ) = \mathbf{R}\Gamma(Y,\ ) \circ \mathbf{R}f_*$$

The derived inverse image

Let $f\colon X \to Y$ be a morphism of algebraic varieties. We want to derive on the left the inverse image (or pull-back) functor $f^*\colon \mathfrak{Mod}(Y) \to \mathfrak{Mod}(X)$, which is left exact.

One can prove quite easily that any sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ is a quotient of a *flat* sheaf of $\mathcal{O}_X$-modules $P(\mathcal{M})$ and that we can choose $P(\mathcal{M})$ depending functorially on $\mathcal{M}$. One then prove that for any bounded above complex $\mathcal{M}^\bullet$ there is a complex $P(\mathcal{M}^\bullet)$ of flat sheaves and a quasi-isomorphism $P(\mathcal{M}^\bullet) \to \mathcal{M}^\bullet$ which defines a functor $K^-(\mathfrak{Mod}(Y)) \to K^-(\mathfrak{Mod}(Y))$. One then proves that $\mathbf{L}f^*(\mathcal{M}^\bullet) = f^*(P(\mathcal{M}^\bullet))$ yields a left derived functor

$$\mathbf{L}f^*\colon D^-(Y) \to D^-(X)$$

It is very easy to check that $\mathbf{L}f^*$ induces functors $\mathbf{L}f^*\colon D^-_{qc}(X) \to D^-_{qc}(Y)$ and $\mathbf{L}f^*\colon D^-_c(X) \to D^-_c(Y)$. In some cases it induces a functor

$$\mathbf{L}f^*\colon D_c(X) \to D_c(Y)\,,$$

that maps $D^b_c(X)$ to $D^b_c(Y)$. One such case holds when every coherent sheaf $\mathcal{G}$ on Y admits a *finite* resolution by coherent locally free sheaves, a condition which is equivalent to the smoothness of Y by Serre's criterion. In such a case, every object in $D^b_c(Y)$ can be represented as a bounded complex of coherent locally free sheaves, i.e. it is a *perfect* complex. Another is when f is *of finite homological dimension*, that is, when for every coherent sheaf $\mathcal{G}$ on Y there are only a finite number of non-zero derived inverse images $\mathbf{L}_j f^*(\mathcal{G}) = \mathcal{H}^{-j}(\mathbf{L}f^*(\mathcal{G}))$; in particular, flat morphisms are of finite homological dimension.

Derived homomorphism Functor

As before, X is an algebraic variety over a field k. We denote simply by $\operatorname{Hom}_X(\mathcal{K},\mathcal{L})$ the k-vector space of homomorphism of $\mathcal{O}_X$-modules, and we wish to construct a "derived functor" of the complex of homomorphisms $\mathcal{L}^\bullet \mapsto \operatorname{Hom}^\bullet_X(\mathcal{K}^\bullet,\mathcal{L}^\bullet)$, for a fixed complex $\mathcal{K}^\bullet$. Although this functor is not induced by a left-exact functor $F\colon \mathfrak{Mod}(X) \to \mathfrak{Mod}(X)$, we still can derive the complex of homomorphisms by mimicking the procedure used to defined $\mathbf{R}F$.

The key observation is that if $\mathcal{I}^\bullet$ be a bounded below complex of injective objects and $\mathcal{K}^\bullet$ any complex, then if either $\mathcal{K}^\bullet$ or $\mathcal{I}^\bullet$ is acyclic, the complex $\operatorname{Hom}^\bullet_X(\mathcal{K}^\bullet,\mathcal{I}^\bullet)$ is acyclic as well.

It follows that if we fix a complex $\mathcal{K}^\bullet$, the functor

$$\mathbf{R}_{II}\operatorname{Hom}^\bullet_X(\mathcal{K}^\bullet,\_)\colon K^+(\mathfrak{Mod}(X)^0) \to D(k)\,,$$

defined by

$$\mathbf{R}_{II}\operatorname{Hom}^\bullet_X(\mathcal{K}^\bullet,\mathcal{L}^\bullet) = \operatorname{Hom}^\bullet_X(\mathcal{K}^\bullet,\mathcal{I}(\mathcal{N}^\bullet)^\bullet)\,,$$

maps quasi-isomorphisms to isomorphisms. (The subscript "II" reflects the fact that we are deriving with respect to the second variable). By Proposition 1.10 it induces a right derived functor

$$\mathbf{R}_{II}\operatorname{Hom}^\bullet_X(\mathcal{K}^\bullet,\_)\colon D^+(X) \to D(k)\,.$$

One then proves that for any fixed object $\mathcal{L}^\bullet \in D^+(X)$, the functor

$$\mathbf{R}_{II}\operatorname{Hom}^\bullet_X(\_,\mathcal{N}^\bullet)\colon K(\mathfrak{Mod}(X))^0 \to D(k)$$

maps quasi-isomorphisms to isomorphisms so that it induces, again by Proposition 1.10, a functor $\mathbf{R}_I\mathbf{R}_{II}\operatorname{Hom}^\bullet_X(\_,\mathcal{L}^\bullet)\colon D(X)^0 \to D(k)$. Hence one obtains a bifunctor

$$\mathbf{R}\operatorname{Hom}^\bullet_X\colon D(X)^0 \times D^+(X) \to D(k)$$

defined as $\mathbf{R}\operatorname{Hom}^\bullet_X(\mathcal{K}^\bullet,\mathcal{L}^\bullet) = \mathbf{R}_I\mathbf{R}_{II}\operatorname{Hom}^\bullet(\mathcal{K}^\bullet,\mathcal{L}^\bullet) \simeq \operatorname{Hom}^\bullet(\mathcal{K}^\bullet,\mathcal{I}^\bullet))$, where $\mathcal{I}^\bullet$ is complex of injective sheaves quasi-isomorphic to $\mathcal{L}^\bullet$. We shall denote

$$\operatorname{Ext}^i_X(\mathcal{K}^\bullet,\mathcal{L}^\bullet) = \mathbf{R}^i\operatorname{Hom}^\bullet_X(\mathcal{K}^\bullet,\mathcal{L}^\bullet) = H^i(\mathbf{R}\operatorname{Hom}^\bullet(\mathcal{K}^\bullet,\mathcal{L}^\bullet))\,.$$

If X is projective or quasi-projective one can also derive first the homomorphisms in the reverse order, so that for any complex $\mathcal{L}^\bullet$ we have a right derived functor $\mathbf{R}_I\operatorname{Hom}^\bullet_X(\ ,\mathcal{N}^\bullet)\colon D^-(\mathfrak{Qco}(X))^0 \to D(k)$ given by $\mathbf{R}_I\operatorname{Hom}^\bullet(\ cplxM,\mathcal{N}^\bullet) \simeq \operatorname{Hom}^\bullet(P(\mathcal{M}^\bullet),\mathcal{N}^\bullet)$, where $P(\mathcal{M}^\bullet)$ is a complex of locally free sheaves quasi-isomorphic to $\mathcal{K}^\bullet$- Moreover, this functor induces a bifunctor

$$\mathbf{R}_{II}\mathbf{R}_I\operatorname{Hom}^\bullet\colon D^-(\mathfrak{Qco}(X))^0 \times D(\mathfrak{Qco}(X)) \to D(k)\,,$$

or, what's amount to the same

$$\mathbf{R}_{II}\mathbf{R}_I\mathrm{Hom}^\bullet\colon D^-_{qc}(X)^0\times D_{qc}(X)\to D(k)\,,$$

The functors $\mathbf{R}_I\mathbf{R}_{II}\mathrm{Hom}^\bullet_X$ and $\mathbf{R}_{II}\mathbf{R}_I\mathrm{Hom}^\bullet_X$ coincide over $D^-_{qc}(X)^0\times D^+_{qc}(X)$.

One has the following property, known as the *Yoneda formula.*

Proposition 1.12. *Let $\mathcal{M}^\bullet$ be a complex of $\mathcal{O}_X$-modules and $\mathcal{N}^\bullet$ a bounded below complex. Then*

$$\mathrm{Ext}^i(\mathcal{M}^\bullet,\mathcal{N}^\bullet)\simeq\mathrm{Hom}^i_X(\mathcal{M}^\bullet,\mathcal{N}^\bullet):=\mathrm{Hom}_X(\mathcal{M}^\bullet,\mathcal{N}^\bullet[i])$$

□

One can also consider the complex of sheaves of homomorphisms, which we denote by $\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{M}^\bullet,\mathcal{N}^\bullet)$, and is given by

$$\mathcal{H}om^n(\mathcal{M}^\bullet,\mathcal{N}^\bullet)=\prod_i\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}^i,\mathcal{N}^{i+n})$$

with the differential $df=f\circ d_{\mathcal{M}^\bullet}+(-1)^{n+1}d_{\mathcal{N}^\bullet}\circ f$.

Proceeding as above we can define a derived sheaf homomorphism

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}=\mathbf{R}_I\mathbf{R}_{II}\mathcal{H}om^\bullet_{\mathcal{O}_X}\colon D(X)^0\times D^+(X)\to D(X)$$

described as

$$\begin{aligned}\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{K}^\bullet,\mathcal{L}^\bullet)&\simeq\mathbf{R}_I\mathbf{R}_{II}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{K}^\bullet,\mathcal{L}^\bullet)\\&\simeq\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{K}^\bullet,\mathcal{I}^\bullet)\,,\end{aligned}$$

where $\mathcal{I}^\bullet$ is a bounded below complex of injective objects quasi-isomorphic to $\mathcal{L}^\bullet$.

We can apply Grothendieck's composite functor theorem to the composition $\Gamma(U,\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{K}^\bullet,\mathcal{L}^\bullet))\simeq\mathrm{Hom}^\bullet_{\mathcal{O}_U}(\mathcal{K}^\bullet{}_{|U},\mathcal{L}^\bullet{}_{|U})$, to obtain an isomorphism in the derived category $D(U)$:

$$\mathbf{R}\Gamma(U,\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{K}^\bullet,\mathcal{L}^\bullet))\simeq\mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{O}_U}(\mathcal{K}^\bullet{}_{||U},\mathcal{L}^\bullet{}_{||U})\,.$$

One can derive the local homomorphisms with respect to the first argument, obtaining a derived functor

$$\mathbf{R}_{II}\mathbf{R}_I\mathcal{H}om^\bullet_{\mathcal{O}_X}\colon D^-_c(X)^0\times D(X)\to D(X)$$

where $D^-_c(X)$ is the derived category constructed from the bounded above complexes of $\mathcal{O}_X$-modules with coherent cohomology sheaves. Both derived functors $\mathbf{R}_I\mathbf{R}_{II}\mathcal{H}om^\bullet_{\mathcal{O}_X}$ and $\mathbf{R}_{II}\mathbf{R}_I\mathcal{H}om^\bullet_{\mathcal{O}_X}$ coincide on $D^-_c(X)^0\times D(X)$.

Derived Tensor Product

Let $(X, \mathcal{O})$ be an algebraic variety. Given a complex $\mathcal{N}^\bullet$, there exist a complex $P(\mathcal{N}^\bullet)$ of flat sheaves and a quasi-isomorphism $P(\mathcal{N}^\bullet) \to \mathcal{N}^\bullet$. Moreover, if a complex $\mathcal{P}^\bullet$ of flat sheaves is acyclic, then the tensor product $\mathcal{N}^\bullet \otimes \mathcal{P}^\bullet$ of the two complexes is acyclic for any complex $\mathcal{N}^\bullet$.

It follows that if we fix a complex $\mathcal{M}^\bullet$, the functor "tensor product complex"

$$\mathcal{M}^\bullet \otimes \_ \colon K(\mathfrak{Mod}(X)) \to K(\mathfrak{Mod}(X))$$

has a "left derived functor", denoted by $(\mathcal{M}^\bullet \underline{\otimes}\ )\colon D(X) \to D(X)$. Now, fixed $\mathcal{N}^\bullet$, the functor $(\ \underline{\otimes} \mathcal{N}^\bullet)\colon \mathbf{C}(X) \to D(X)$ induces a bifunctor, called derived tensor product

$$\overset{\mathbf{L}}{\otimes} : D(X) \times D(X) \to D(X)$$

whose description is $\mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{N}^\bullet = \mathcal{M}^\bullet \otimes P(\mathcal{N}^\bullet)$, where $P(\mathcal{N}^\bullet) \to \mathcal{N}^\bullet$ is a quasi-isomophism and $P(\mathcal{N}^\bullet)$ is a complex of flat sheaves.

One can derive the tensor product reversing the sense of the derivations and obtaining the same result, i.e., $\mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{N}^\bullet \simeq \mathcal{N}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{M}^\bullet$. It is also easy to prove that

$$(\mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{N}^\bullet) \overset{\mathbf{L}}{\otimes} \mathcal{P}^\bullet \simeq \mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes} (\mathcal{N}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{P}^\bullet)$$

1.2.3 Some remarkable formulas in the derived category

We finish this lecture with a relation of some formulas which will be useful for the theory of integral functors and Fourier-Mukai transforms.

The first one is known as the adjunction formula between the derived inverse a direct images.

Proposition 1.13. *Let $f\colon X \to Y$ be a morphism of algebraic varieties. There is a functorial morphism*

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathbf{L}f^*\mathcal{M}^\bullet, \mathcal{N}^\bullet) \to \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{M}^\bullet, \mathbf{R}f_*\mathcal{N}^\bullet)\,.$$

for $\mathcal{M}^\bullet$ in $D^-(X)$ and $\mathcal{N}^\bullet$ in $D^+(Y)$, which is an isomorphism if $\mathcal{M}^\bullet$ has bounded cohomology. Moreover in this case we have a group isomorphism

$$\operatorname{Hom}_{D(X)}(\mathbf{L}f^*\mathcal{M}^\bullet, \mathcal{N}^\bullet) \simeq \operatorname{Hom}_{D(Y)}(\mathcal{M}^\bullet, \mathbf{R}f_*\mathcal{N}^\bullet)\,.$$

If f has finite homological dimension, so that $\mathbf{L}f^*$ maps bounded complexes to bounded complexes, this formula says that $\mathbf{L}f^*\colon D^b(Y) \to D^b(X)$ is left adjoint to $\mathbf{R}f_*\colon D^b(X) \to D^b(Y)$. □

The second formula is known as the *projection formula*.

Proposition 1.14. *Let $f\colon X\to Y$ be a morphism of algebraic varieties. There is a functorial morphism*

$$\mathbf{R}f_*(\mathcal{M}^\bullet)\overset{\mathbf{L}}{\otimes}\mathcal{N}^\bullet\to\mathbf{R}f_*(\mathcal{M}^\bullet\overset{\mathbf{L}}{\otimes}\mathbf{L}f^*\mathcal{N}^\bullet)$$

in $D(Y)$ the derived category, with $\mathcal{M}^\bullet$ a complex of $\mathcal{O}_X$-modules and $\mathcal{N}^\bullet$ a complex of $\mathcal{O}_Y$-modules, which is an isomorphism if $\mathcal{N}^\bullet$ has quasi-coherent cohomology.

We also have a compatibility between the derived tensor product and the derived inverse image.

Proposition 1.15. *Let $f\colon X\to Y$ be a morphism of algebraic varieties. If $\mathcal{M}^\bullet,\mathcal{N}^\bullet\in D(Y)$, one has a functorial isomorphism*

$$(\mathbf{L}f^*\mathcal{M}^\bullet)\overset{\mathbf{L}}{\otimes}(\mathbf{L}f^*\mathcal{N}^\bullet)\xrightarrow{\sim}\mathbf{L}f^*(\mathcal{M}^\bullet\overset{\mathbf{L}}{\otimes}\mathcal{N}^\bullet)\,.$$

One of the most useful formulas related with the Fourier-Mukai transform is the*base change formula in the derived category*:

Proposition 1.16. *Let us consider a cartesian diagram of morphisms of algebraic varieties*

$$\begin{tikzcd} X\times_Y\widetilde{Y} \arrow[r, "\tilde g"] \arrow[d, "\tilde f"] & X \arrow[d, "f"] \\ \widetilde{Y} \arrow[r, "g"] & Y \end{tikzcd}$$

with f and g of finite homological dimension. For any complex $\mathcal{M}^\bullet$ of $\mathcal{O}_X$-modules there is a natural morphism

$$\mathbf{L}g^*\mathbf{R}f_*\mathcal{M}^\bullet\to\mathbf{R}\tilde f_*\mathbf{L}\tilde g^*\mathcal{M}^\bullet$$

Moreover, if $\mathcal{M}^\bullet$ has quasi-coherent cohomology and either f or g is flat, then the above morphism is an isomorphism.

We list here some more remarkable formulas.

- One has a functorial isomorphism in $D(X)$

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{M}^\bullet,\mathcal{N}^\bullet)\overset{\mathbf{L}}{\otimes}\mathcal{H}^\bullet\xrightarrow{\sim}\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{M}^\bullet,\mathcal{N}^\bullet\overset{\mathbf{L}}{\otimes}\mathcal{H}^\bullet)$$

  for $\mathcal{M}^\bullet$ in $D^b_c(X)$, $\mathcal{N}^\bullet$ in $D^+(X)$ and $\mathcal{H}^\bullet$ is in $D^b_c(X)$ and is a perfect complex.

- If X is projective or quasi-projective so that any coherent sheaf is a quotient of a locally free sheaf of finite rank, one has a functorial isomorphism (in the derived category)

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{M}^\bullet, \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{N}^\bullet, \mathcal{H}^\bullet)) \xrightarrow{\simeq} \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{N}^\bullet, \mathcal{H}^\bullet)$$

  with $\mathcal{M}^\bullet$, $\mathcal{N}^\bullet$ in $D^-_c X$ and $\mathcal{H}^\bullet$ in $D^+(X)$.

- If $\mathcal{K}^\bullet$ is a bounded complex of coherent sheaves of finite homological dimension, then its derived dual $\mathcal{K}^{\bullet\vee} = \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{K}^\bullet, \mathcal{O}_X)$ is also of finite homological dimension and one has $\mathcal{K}^\bullet \simeq \mathcal{K}^{\bullet\vee\vee}$ in $D^b_c(X)$. Moreover the functor $(-) \overset{\mathbf{L}}{\otimes} \mathcal{K}^{\bullet\vee} \colon D^b_c(X) \to D^b_c(X)$ is both left and right adjoint to the functor $(-) \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet \colon D^b_c(X) \to D^b_c(X)$.

We need another formula, which is the statement of Grothendieck duality

Proposition 1.17. *Let $f\colon X \to Y$ be a proper morphism of algebraic varieties. The derived direct image functor $\mathbf{R}f_*\colon D_{qc}(X) \to D_{qc}(Y)$ has a right adjoint $f^!\colon D_{qc}(Y) \to D_{qc}(X)$, so that there is a functorial isomorphism*

$$\mathrm{Hom}_{D(Y)}(\mathbf{R}f_{X*}\mathcal{M}^\bullet, \mathcal{G}^\bullet) \simeq \mathrm{Hom}_{D(X)}(\mathcal{M}^\bullet, f^!\mathcal{G}^\bullet)\,. \tag{1.5}$$

When f is smooth (i.e., it is flat and has smooth fibres) of relative dimension n, then the functor $f^!$ is given by

$$f^!\mathcal{G}^\bullet \simeq f^*\mathcal{G}^\bullet \otimes \omega_{X/Y}[n]\,,$$

where $\omega_{X/Y}$ is the line bundle of the relative n-differentials.

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