The theory of Vector Bundles on Algebraic curves with some applications

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CHAPTER I
LINE BUNDLES ON A COMPACT RIEMANN SURFACE

1 A historical introduction

The theory of vector bundles has many ramifications. One can study it from number theoretic, algebraic geometric and differential geometric points of view. It has also proved useful to mathematical physicists interested in Conformal Field theory, String theory, etc. In this account, I will mainly deal with the geometric aspects, both algebraic and differential, and will confine myself to just a few remarks on the number theoretic point of view.

The classical theory of abelian class fields seeks to understand Galois extensions of a number field in terms of the number theoretic behaviour of the corresponding integral extensions of the ring of integers in the number field. This has a geometric analogy. Consider any compact Riemann surface. Any covering of the surface gives a (finite) extension of the field of meromorphic functions on it. One may try to understand abelian extensions in terms of geometric data on the Riemann surface. This leads to the theory of *Jacobians* of Riemann surfaces.

A. Weil initiated the theory of vector bundles over an algebraic curve motivated by the desire to develop a 'non-abelian class field theory' for function fields. The number theoretic analogues are more pertinent when the function field have finite field of constants, but the study makes sense and is interesting when the function field in question is the field of meromorphic functions on a compact Riemann surface in the traditional sense.

The study of Jacobians is complex analytic or algebraic and can be understood purely geometrically. Jacobi's work in this respect may be interpreted, as establishing in particular a correspondence between the fundamental group and the Jacobian, that is to say, the variety of divisor classes.

We will start with a brief survey of basic concepts centering around divisors and explain the origins of this aspect and later lead up to the theme of vector bundles.

1.1 Periods of holomorphic differentials.

Let X be a compact Riemann surface, that is to say, a compact, connected, complex manifold of dimension 1. Its topological type is determined by a non-negative integer g, called its *genus*. The genus of the Riemann sphere is 0. The quotient of $\mathbb{C}$ by a discrete subgroup of rank 2, is topologically a two-dimensional torus, but it also inherits a complex structure from $\mathbb{C}$. Riemann surfaces so obtained are called *elliptic curves* and have genus 1. Since it is homeomrphic to $S^1 \times S^1$, its fundamental group is free abelian of rank 2. In general, the first homology group $H_1(X)$ of any Riemann surface is free of even rank and one way to define g is to say that this rank is $2g$. One can actually write out explicitly its fundamental group $\pi(X)$ in terms of the genus. It is isomorphic to a group on $2g$ generators a_i, b_i, $i = 1, \cdots, g$ with a single relation $\prod a_i b_i a_i^{-1} b_i^{-1} = 1$. Of course this implies that its abelianisation, namely $H_1(X, \mathbb{Z})$, is free of rank $2g$.

One has also an analytic interpretation of the invariant g. If g is the genus of X, the space Ω of all holomorphic differentials on X is a vector space of dimension g. The origin of the theory of line bundles from the present point of view is the attempt to integrate a holomorphic differential on X. Let ω be a holomorphic differential on X. We fix a point x_0 and seek to compute the *indefinite* integral $\int_{x_0}^{x} \omega$ as a function of x. In order for this to make sense, we have to integrate the differential along a path c connecting x_0 and x. (For purposes of integration we will assume the path to be a smooth, or piecewise smooth, map $[0, 1] \to X$.) One may think of the integral over c as a linear form on Ω, by varying ω. The integral of course depends on the path c, but the monodromy theorem says that the integral is the same if we replace c by a homotopically equivalent path. In other words, the integral depends only on the *homotopy class* of the path c. In general, if c and c' are any two paths connecting x_0 and x, the two integrals differ by the integral of ω along the loop $c'.c^{-1}$. Therefore we are led to consider the linear forms on Ω obtained by integrating holomorphic forms on loops based at x_0. These special linear forms are called *periods* of holomorphic differentials. We have thus given a *homomorphism* of the fundamental group $\pi(X, x_0)$ into Ω^*. This obviously goes down to a homomorphism of the abelianised fundamental group $H_1(X, \mathbb{Z})$ into Ω^*. It also follows that this homomorphism does not depend on the choice of x_0. The first important result is the following fact.

1.2 Theorem. *The above homomorphism of $H_1(X)$ into Ω^* is injective and the image is a discrete subgroup.*

We will call the image the *period group*. Our remarks above amount to saying that integration of holomorphic differentials leads to a map of X into the *quotient* of Ω^* by the period group. This map is called the *period map*. Here we will denote it by σ.

1.3 The Albanese variety.

Consider now the following situation. Let V be a complex vector space of dimension g and Γ a discrete subgroup of rank $2g$. The quotient $A = V/\Gamma$ has a lot of structure. First of all, it is compact and connected. Topologically it is actually a torus of dimension $2g$. Secondly the natural map $V \to A$ is a local homeomorphism. Using this we can equip A with a complex structure, making it a compact complex manifold of dimension g. On the other hand, it is also an abelian group. The complex structure and the group structure are obviously compatible in the sense that the group operations are holomorphic maps. In other words, it is a *complex Lie group*. Now we have the following fact.

1.4 Theorem. *Any compact connected complex Lie group is isomorphic to the quotient of $\mathbb{C}^g$ by a discrete subgroup.*

Proof. Consider the adjoint representation of the group G in question in its Lie algebra. It is a holomorphic map of the group into the group of linear automorphisms of the Lie algebra. Since any holomorphic function on a compact, connected, complex manifold is a constant, the adjoint representation is the trivial one. It follows that the group G is abelian. The exponential map of the Lie algebra into the Lie group is then a surjective homomorphism of the additive group underlying the Lie algebra into G, and the kernel is a subgroup Γ of $\mathbb{C}^g$ such that $\mathbb{C}^g/\Gamma$ is compact. It is easily seen that this means that Γ is a lattice in $\mathbb{C}^g$.

The complex Lie groups obtained as above are called *complex tori.*

In our case, V is Ω^* and Γ is the period subgroup. The map $X \to A$ that we have defined above is easily seen to be holomorphic, and we have actually the following universal property.

1.5 Theorem. *Any holomorphic map of X into a complex torus T taking x_0 to 0 factors through a unique holomorphic homomorphism of A into T.*

Proof. Note first that the space of holomorphic 1-differentials on T is canonically identified with $\mathbf{t}^*$, where $\mathbf{t}$ is its Lie algebra. The given map of X into T gives rise to a linear map $\mathbf{t}^* \to \Omega$, by simply pulling back holomorphic differentials. Its transpose is a linear map $\Omega^* \to \mathbf{t}$. This takes Γ into $\mathcal{L}$ and induces a map of $\Omega^*/\Gamma \to A/\mathcal{L}$ where $\mathcal{L}$ is the discrete subgroup of $\mathbf{t}$ which is the kernel of the exponential map $\mathbf{t} \to T$.

This is called the *Albanese property* of the period map. Using the group structure of A, we may analyse the period map a little further. As we observed above, the period group does not depend on the base point x_0 that we chose. But the map σ does depend on the choice. We wish to do away with this dependence. We take, instead of a single point, any finite set of points x_i. In order to allow repetitions, we will assign multiplicities m_i to each x_i. We do not insist that these integers m_i be positive. Such a datum is called a *divisor* in X. (The name has its origin in Number theory. Any nonzero rational number can be thought of as assigning some multiplicities to primes). If indeed all the m_i are non-negative, we call it an *effective divisor.* A concise way of saying this is that a divisor is an element of the free abelian group $Div(X)$ with the underlying set X as basis. Given a divisor $D = \sum m_i x_i$, we can define an element of A as follows. Take $\sum m_i \sigma(x_i)$ in the sense of the group addition in A. The point is that this map is *independent of* x_0 *if* $\sum m_i = 0$. The integer $\sum m_i$ is called *the degree* of the divisor $\sum m_i x_i$. Degree is then obviously a homomorphism of $Div(X)$ into $\mathbb{Z}$. What we asserted above is that we have a canonical homomorphism of the group of divisors of degree 0 into A. We will denote this map by α. Actually this homomorphism is surjective. One would like to understand the kernel of this map.

1.6 Theorem *The kernel of α consists of those divisors $\sum m_x x$ which have the property that there exists a nonzero meromorphic function f on X with $ord_x(f) = m_x$ for all $x \in X$.*

We do not prove this theorem here. See [**?**].

1.7 Divisor Classes.

Any nonzero meromorphic function f gives rise to a divisor as follows. For any $x \in X$ associate the integer $ord_x(f)$, namely the integer i such that $f = t^i.g$ where g is a nonzero holomorphic function in a neigbourhood of x and t is a local coordinate. Since the zeros and poles of f are finite in number, $\sum_{x \in X} ord_x(f)x$ is actually a divisor $div(f)$. This is called *the divisor associated to f*. Divisors of meromorphic functions are called *principal divisors*. This in fact gives a homomorphism of the multiplicative group of nonzero meromorphic functions into the group of divisors. It is an easy consequence of the integral formula that the degree of any principal divisor is zero. If two divisors are considered *equivalent* whenever their difference is a principal divisor, then the equivalence classes are called *divisor classes*. Since principal divisors obviusly form a subgroup of the divisor group, divisor classes form a group. What the above theorem asserts therefore is that the the map α induces an isomorphism of the group of divisor classes of degree 0, onto the Albanese variety Ω^*/Γ.

1.8 Invertible sheaves and Line bundles.

Suppose $D = \sum m_x x$ is a divisor. To every open set U in X, we associate the vector space of all meromorphic functions f in U such that $div(f) + D$ is effective. In long hand, this means that the order of the function at any point x is at least $-m_x$. For example, if the divisor is simply a for some point $a \in X$, then the above space consists of all meromorphic functions in U, which have at most a simple pole at a. This assignment gives a sheaf on X. Indeed it is a sheaf of $\mathcal{O}$-Modules, since multiplication by a holomorphic function preserves the above property. We denote this sheaf by $\mathcal{O}(D)$. Note that if U is a coordinate neighbourhood of a point x, with the coordinate t, then any function with the above property has the Laurent expansion at x of the form $t^{-m_x}.g$ where g is a holomorphic function. Thus locally this sheaf is isomorphic to $\mathcal{O}$. However globally it is not in general isomorphic to $\mathcal{O}$.

1.9 Definition. A sheaf of $\mathcal{O}$-Modules which is locally isomorphic to $\mathcal{O}$ is called *an invertible sheaf* on X. The invertible sheaf $\mathcal{O}$ on X is said to be *trivial.*

The set of all invertible sheaves on X is an abelian group under tensor product as group operation, the trivial sheaf $\mathcal{O}$ being the identity element. We have observed above that every divisor gives rise to an invertible sheaf. This yields an isomorphism of the group of divisor classes onto the group of invertible sheaves. An important invertible sheaf is the sheaf of holomorphic differentials on X. We will generally denote this by K_X or ω_X.

The notion of invertible sheaves is entirely equivalent to that of *line bundles.* A line bundle consists of a variety L and a holomorphic map onto X, with fibres equipped with the structure of a vector space of rank 1. It is supposed to be *locally trivial* in the sense that every point of X has an open neighbourhood U over which it is isomorphic to the product $U \times \mathbb{C}$. The sheaf of sections of this gives an invertible sheaf, and this assignment of invertible sheaves to line bundles makes the two notions interchangeable.

1.10 Cohomological interpretation.

If we trivialise a line bundle on an open set U in two different ways, they 'differ' by an automorphism of the trivial bundle on this open set. Since an automorphism of a one-dimensional vector space consists only of nonzero scalars, we conclude that the two trivialisations differ (multiplicatively) by a function on U with values in $\mathbb{C}^*$. If we cover the manifold with open sets U_i with trivialisations of the line bundle over each of these, we obtain on pairs of intersections $U_i \cap U_j$, functions $m_{i,j} : U_i \cap U_j \to \mathbb{C}^*$ as above. These satisfy the compatibility requirement

$$m_{i,j} m_{j,k} = m_{i,k}$$

for all triples i, j, k. These can be interpreted as cocycles in the Cech sense with respect to the given covering. Hence one gets an element of $H^1(X, \mathcal{O}^*)$, where $\mathcal{O}^*$ is the sheaf of nonzero holomorphic functions. Assuming the theory of cohomology of sheaves, one can check that this gives an isomorphism of the divisor class group with $H^1(X, \mathcal{O}^*)$.

1.11 Linear systems.

To an invertible sheaf L (and indeed to any sheaf of abelian groups) is associated *cohomology groups* $H^i(X, L)$. If L is the invertible sheaf associated to a divisor D, the vector space $H^0(X, L)$ is simply the space of (global) holomorphic functions f with $div(f) + D$ effective. When X is a curve, these groups vanish for $i \geq 2$. Towards the computation of H^0 and H^1, which are finite dimensional vector spaces, we have the following famous result, called the *Riemann-Roch theorem*.

1.12 Theorem. $dim H^0(X, L) - dim H^1(X, L) = deg(L) + 1 - g.$

Ideally we would have liked a formula for computing $dim\ H^0(X, L)$, but this number can vary when L is perturbed a little. So we cannot have a formula which computes $dim H^0(X, L)$ in terms of discrete invariants of L and X. On the other hand, the left side of the above formula is indeed a *deformation invariant*. It is called the *Euler characteristic* of L and is usually denoted by $\chi(X, L)$.

The Riemann-Roch theorem and the following duality theorem are important tools in the study of curves.

1.13 Theorem. *$H^0(X, L)$ and $H^1(X, K \otimes L^{-1})$ are canonically dual.*

If a line bundle admits a nonzero section, then its degree is non-negative. For if L is represented by a divisor D and f a meromorphic function on X with $div(f) + D$ effective, then since degree of $div(f)$ is zero, we ought to have $deg(L) > 0$. Thus if $deg\ (L) < 0$, then $H^0(X, L) = 0$. By duality we have, as a corollary, $H^1(X, L) = 0$ if $deg(L) > deg(K)$.

If we take L is the trivial bundle, then $H^0(X, L)$ is the space of holomorphic functions on X and is therefore one dimensional. Hence $H^1(X, K)$ is also one dimensional. On the other hand, if we take $L = K$ and use the fact that $\Omega = H^0(X, K)$ has rank g, we get $deg(K) = 2g - 2$. In other words, any holomorphic differential vanishes on a divisor of degree $2g - 2$. Also we can restate the vanishing theorem as follows.

If $deg(L) > 2g - 2$, then $H^1(X, L) = 0$.

Thus the Riemann Roch theorem computes $dim(H^0(X, L))$ if the degree is greater than $2g - 2$.

Using these facts, one can show for example that the complex manifold X is indeed isomorphic to a submanifold of some complex projective space $\mathbb{CP}^n$. This is accomplished as follows. Let L be a line bundle which has 'lots of sections'. Let $s_0, \cdots, s_n$ be a basis for $H^0(X, L)$. Locally these can be regarded as functions. Then the assignment $x \mapsto (s_0(x), \cdots, s_n(x))$ gives a holomorphic map into $\mathbb{CP}^n$ locally. It is easy to see that indeed this map is global, and gives an imbedding under our assumption that there are sufficiently many sections. Canonically speaking, the projective space $\mathbb{CP}^n$ is the projective space associated to the vector space of linear forms on $H^0(X, L)$, and to each x we associate the one dimensional subspace of sections vanishing at x.

¿ From the Riemann Roch formula we can deduce that any line bundle L of large degree (for example $> 2g$), serves this purpose. This is because for any two $x, y \in X$, we can compute the dimension of $H^0(L)$, $H^0(X, L \otimes \mathcal{O}(-x))$ and $H^0(L \otimes \mathcal{O}(-x-y))$ by the Riemann Roch theorem to be $deg(L)+1-g$, $(deg(L)-1)+1-g$ and $(deg(L)-2)+1-g$, respectively. Since these are respectively the space of sections of L, the space of sections of L vanishing at x, and the space of sections vanishing at both x and y, it follows that for any two points x and y, there is a section of L vanishing at x but not at y. This proves that the map we defined above is injecive and taking y to be x in the above computation, one can show that the differential of the map at x is also injective. In other words, X can be imbedded in a complex projective space as a submanifold. Indeed, it can be shown that any such submanifold can also be defined by the vanishing of homogeneous polynomials, thus showing that X is a *projective algebraic curve*.

What we explained above is a general procedure, applicable to higher dimensional manifolds as well. If a compact manifold admits a line bundle with a lot of sections, then the manifold is actually a projective algebraic manifold. This is almost tautologous, but it concentrates the effort of imbedding a manifold in a projective space to finding such line bundles, which we accomplished above in the case of compact Riemann surfaces.

1.14 Polarisation.

Let us now go back to our starting point. We gave an isomorphism of group of divisors classes of degree 0 onto Ω^*/Γ. We noticed that the divisor class group can be identified with the group of line bundles. We have also seen that the latter group can be identified in turn with the cohomology group $H^1(X, \mathcal{O}^*)$. Consider the exact sequence of sheaves:

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

The connecting homomorphism $H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$ associates to each line bundle its degree. The image is the *Chern class* of the line bundle. So the kernel, namely the group of divisor classes of degree 0, is isomorphic to $H^1(X, \mathcal{O})/H^1(X, \mathbb{Z})$. This is thus again the quotient of a complex vector space by a lattice. As a matter of fact, there is a duality between $H^1(X, \mathcal{O})$ and $H^0(X, K) = \Omega$. Thus the two spaces $H^1(X, \mathcal{O})$ and Ω^* are canonically isomorphic. The lattice $H^1(X, \mathbb{Z})$ goes into the period subgroup under this isomorphism, thereby yielding an isomorphism of these complex tori.

Now Hodge theory gives a decomposition

$$H^1(X, \mathbb{C}) = H^1(X, \mathcal{O}) \oplus \Omega$$

and an anti-isomorphism of $H^1(X, \mathcal{O})$ with Ω given by complex conjugation on 1-forms. Using these, we get a positive definite hermitian form on the vector space $H^1(X, \mathcal{O})$. Its imaginary part restricts to $H^1(X, \mathbb{Z})$ as the Poincaré pairing $H^1(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}) = \mathbb{Z}$.

Let us turn to the abstract situation of the quotient of a complex vector space V by a lattice Γ. We have already remarked that it is a complex torus. Suppose in addition that there is a positive definite hermitian form on V with imaginary part α. Assume that the restriction e of α to Γ is integral and nondegenerate (in the sense that $e(x, y) = 0$ for all $y \in \Gamma$ if and only if $x = 0$). Then one can actually show that $A = V/\Gamma$ is a projective variety, i.e. a submanifold of $\mathbb{CP}^n$ for some n, given by the vanishing of some homogeneous polynomials. In such a case, we call $A = V/\Gamma$ an *Abelian variety.*

We will now elaborate on this a little bit. The Chern class of any line bundle is an element of $H^2(A, \mathbb{Z})$. For a torus A this can be identified with the space of alternating forms on $H_1(A, \mathbb{Z}) = \Gamma$. Thus the imaginary part of the hermitian form we hypothesized above, can be interpreted to be an element of $H^2(A, \mathbb{Z})$. One can show that it is actually the Chern class of a holomorphic line bundle on A. Some poaitive tensor power of this line bundle possesses enough sections to imbed A in a suitable projective space. We will collect all these remarks in the following statement.

1.15 Theorem. *The complex torus V/Γ is isomorphic to a projective variety if and only if there exists a positive definite hermitian form on V whose imaginary part is integral on Γ.*

1.16 Definition. When V/Γ is a projective variety, it is called *an Abelian variety*. The Abelian variety Ω/Γ associated to X is called the *Jacobian* of X.

1.17 Theta functions.

We have remarked above that there is a line bundle L whose Chern class is the alternating form e, namely, the imaginary part of the given positive definite hermitian form h. This line bundle is not unique, but is determined up to a translation. The classical proof of the theorem quoted above is via the theory of *theta functions*.

In fact, the line bundle when pulled up to V itself is trivial. Let us choose a trivialization. Sections of L when pulled up to V become holomorphic functions. The fact that the line bundle came from A would impose some conditions on these functions, under translation by elements of Γ. Explicitly, these are functions that satisfy the functional equation:

$$f(v+\gamma) = \alpha(\gamma) exp(\pi h(v,\gamma) + \frac{\pi}{2} h(\gamma,\gamma)) f(v)$$

for all $v \in V$ and $\gamma \in \Gamma$. Here, α is a fixed map of Γ into $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, satisfying

$$\alpha(\gamma_1 + \gamma_2) = exp(i\pi e(\gamma_1, \gamma_2)) \alpha(\gamma_1) \alpha(\gamma_2)$$

for all $\gamma_1 . \gamma_2 \in \Gamma$. Note that α is not uniquely determined by e, but can only be altered multiplicatively by a unitary character on Γ. This of course depends on the particular line bundle L we take with e as its Chern class.

By explicit analysis of these equations, one can actually write down a basis for these. These are called *theta functions*. The choice of the hermitian form (and therefore e) is referred to as a *polarisation*. It can be shown that some positive power of L – indeed Lefschetz showed that L^3 would do – has enough sections to imbed A in a projective space.

1.18 Poincaré Bundle.

We have seen above that the group of divisor classes of degree 0 on X is naturally bijective with points of $H^1(X, \mathcal{O})$ modulo the period group. In particular this group has been provided with the structure of an Abelian variety. This is called the *Jacobian* of X and denoted by $J(X)$ or simply J.

In what sense is this structure natural? In order to answer this, one needs to understand what is meant by *classification* of line bundles.

Let T be any (parameter) variety. A *family* of line bundles on X, parametrised by T, should be a line bundle L on $T \times X$. For each point $t \in T$, we may restrict L to $\{t\} \times X$ and obtain a line bundle L_t on X. We would like to think of the family to be the collection $\{L_t\}_{t \in T}$. But then two non-isomorphic line bundles L and L' on $X \times T$ can give rise to the same family in this sense. For example, one can take a line bundle ξ on T and take $L' = L \otimes p_T^* \xi$. Fortunately, this is is only thing that can go wrong! In any case, we will consider such families to be *equivalent*. Given any such family L of line bundles of degree 0, one gets a map of T into $J(X)$ which associates to each $t \in T$, the isomorphism class of L_t considered as a point of J. We call it the *classifying map* $\varphi_L : T \to J$. We require that this map be holomorphic. In fact, this nails down the structure of J as a complex manifold. Since both X and J are actually algebraic varieties we may equally well work with algebraic varieties, or even schemes.

We may actually wish to construct a (universal) family of line bundles on X parametrised by J. In other words, we seek to construct a line bundle P on $X \times J$ such that for any $\xi \in J$, the equivalence class of the line bundle P_ξ, considered as a point of J is ξ. Such a family does exist and is called the *Poincaré* bundle P. Given any family L parametrised by T we have the classifying map $\theta_L : T \to J$. We can thus use the map $Id_X \times \theta_L : T \times X \to J \times X$ to pull back P. This and L are obviously equivalent families.

We have explained the construction of a variety structure on the set of divisor classes of degree 0. We could also do the same for the set of divisor classes of any fixed degree d. This is of course not a group, but a coset of J in the divisor class group. As such, the group J acts simply transitively on it and so we can transfer the strucure of a projective variety on it. An appropriate notation for this variety would be $J^d(X)$.

1.19 Algebraic Geometric point of view.

Any compact Riemann surface is a projective variety, and its Jacobian also turned out to be a projective variety, but all our constructions have been transcendental. It is a natural question therefore whether all these can be carried out in the context of algebraic geometry. Indeed the Riemann-Roch theorem assures us that any line bundle of degree at least g has at least one nonzero section. In other words, any such line bundle is isomorphic to

$\mathcal{O}(D)$ where D is an effective divisor. Clearly the set of effective divisors of a given degree d can be identified with the d-fold *symmetric* power $S^d(X)$ of the curve. Hence there is a morphism from $S^d(X)$ into J^d which takes D to $\mathcal{O}(D)$. The Riemann-Roch theorem implies that when $d \geq g$ this map is surjective, for any line bundle L of degree d would then have nonzero H^0. The divisor of zeros of a nonzero section is then an effective divisor D such that $\mathcal{O}(D) \simeq L$. If $d = g$, this is in fact a birational morphism and André Weil used this to give a purely algebraic construction of the Jacobian.

If $d = g-1$, we get a morphism of S^{g-1} into J^{g-1} and one can check that the image is a divisor θ in J^{g-1} and consequently defines a line bundle on it. Its Chern class is the hermitian form which we explained purely in analytic terms. Thus we might redefine *theta functions* to be sections of powers of the line bundle $\mathcal{O}(\theta)$.

When $g = 1$, this gives an isomorphism of X with J^1. In other words, X is itself an abelian variety of dimension 1. One-dimensional abelian varieties are called *elliptic curves*.

The theory of line bundles, Jacobians and theta functions has a long history and has been developed intensely from the geometric, arithmetic and analytic points of view. We have given above a short account of those aspects which are relevant to the following lectures.

CHAPTER II
VECTOR BUNDLES

2 Locally free sheaves and Vector Bundles

The notion of locally free sheaves (resp. line bundles) is entirely similar to that of invertible sheaves (resp. vector bundles). We will run through some of the easily proved facts on vector bundles which are proved either by reducing them to, or imitating the proofs of, analogous theorems on line bundles.

2.1 Definition. A sheaf of $\mathcal{O}$-Modules which is locally isomorphic to $\oplus^n \mathcal{O}$ is called a *locally free sheaf* on X.

Similarly, let E be a complex manifold (or a variety) with a morphism $\pi : E \to X$ with the structure of vector spaces of rank n on all fibres $\pi^{-1}(x), x \in X$ satisfying the following condition. Every point $x \in X$ has a neighbourhood U such that $\pi^{-1}(U)$ is isomorphic to the product $U \times C^n$, the isomorphism being linear on all the fibres. Then we say E is a vector bundle of rank n over X. The sheaf of sections of E is a locally free sheaf and this gives a bijection between the sets of isomorphism classes of vector bundles and those of locally free sheaves. We generally make no distinction between the two.

2.2 Duality and Riemann-Roch theorems.

All the linear algebraic operations that one performs on a vector space may be performed on vector bundles as well. For example, if E_1 and E_2 are two vector bundles, then one can form its *direct sum* $E_1 \oplus E_2$, by taking the fibre product of $E_1 \to X$ and $E_2 \to X$ and equipping the fibres with the direct sum structure. We may also construct the tensor product of two bundles and hence also the *tensor power* $\otimes^n E$ of any bundle E. Also, the *symmetric power* and the *exterior power* of a vector bundle are defined similarly. Clearly the rank of $E_1 \oplus E_2$ (resp. $E_1 \otimes E_2$)is $rk(E_1) + rk(E_2)$ (resp. $rkE_1.rkE_2$) and so on. In particular, if n is the rank of E, then the n-th exterior power of E, which we call the *determinant* of E and denote by $det(E)$, is of rank 1, that is to say, a line bundle. As such one can talk of its *Chern class*, or over curves, of its degree.

2.3 Definition. The *degree* of a vector bundle E is the degree of the line bundle $det(E)$.

We can now state the vector bundle analogues of the results on line bundles. Firstly it is true that $H^i(X, E)$ are 0 for all $i/geq2$. Secondly we have the duality theorem for vector bundles exactly as for line bundles.

2.4 Theorem. *There is a natural duality between $H^0(X, E)$ and $H^1(X, K \otimes E^*)$.*

We also have the following more general Riemann-roch theorem.

2.5 Theorem. $dim\ H^0(X, E) - dim\ H^1(E) = deg(E) + rk(E)(1 - g)$

2.6 Extensions.

The above operations give methods of construction of other vector bundles, starting with one. But we have not given thus far examples of vector bundles, other than those obtained by taking direct sums of line bundles. We wish now to discuss more interesting constructions of vector bundles.

Consider short exact sequences of sheaves of $\mathcal{O}$-Modules of the form

$$0 \to E' \to E \to E'' \to 0$$

with E', E'' locally free. Then one can deduce that E is also locally free. So, starting with E' and E'', if we can construct an exact sequence as above, then we obtain a new vector bundle$E$, which is said to be an *extension* of E'' by E'. The direct sum $E' \oplus E''$ is then a particular case of this. The set of all such extensions can be put in bijective correspondence with the vector space $Ext^1(E'', E') = H^1(X, E''^* \otimes E')$. By taking nonzero elements of this space, we may construct new vector bundles.

In fact, we will show that starting with line bundles, by successively taking extensions, we may obtain *all* vector bundles. Indeed, let E be *any* vector bundle. One may tensor it with a line bundle L of large degree and ensure that $H^0(E \otimes L) \neq 0$. Take a nonzero section s. Suppose it vanishes at some point $x \in X$. This means that it is actually a section of $\mathcal{M}_x \otimes E \otimes L$, where $\mathcal{M}_x$ is the ideal sheaf of functions vanishing at the point x. But $\mathcal{M}_x$ is an invertible sheaf and is indeed isomorphic to $\mathcal{O}(-x)$. Thus, ultimately s may be regarded as an everywhere nonzero section of $\mathcal{O}(-D) \otimes E \otimes L$ where D is the divisor of zeros of s. Thus we get an exact sequence

$$0 \to \mathcal{O} \to E \otimes M \to F \to 0$$

of locally free sheaves, denoting by M the line bundle $\mathcal{O}(-D) \otimes L$. Tensoring this with M^{-1} we see that E is obtained as an extension of a vector bundle of rank $n-1$ by a line bundle. Iterating it, one concludes that *every vector bundle* is obtained as successive extension of line bundles. Incidentally the Riemann-Roch theorem for vector bundles, stated above can be reduced to the same for line bundles from this fact.

2.7 Remarks. If we take elements $v, \lambda v \in H^1(E''^* \otimes E) = V$ (with $\lambda \in \mathbb{C}^*$), the extensions obtained are distinct, but the vector bundles obtained as extensions are isomorphic. Now one can easily construct a *family* of vector bundles on X parametrised by the affine space V, namely E_v is the extension corresponding to $v \in V$. By restricting this family to the one-dimensional subspace $\mathbb{C}v$, we get a subfamily. Note that every nonzero vector in this subspace corresponds to the 'same' vector bundle (i.e. upto isomorphism) while the zero vector corresponds to the direct sum of E' and E'', which is generally speaking not isomorphic to the non-trivial extension. This strange fact (called the 'jump' phenomenon) gives the following negative result. *There cannot be a variety structure on the set of isomorphic classes of vector bundles of rank ≥ 2 with even the minimal naturality assumptions, such as we explained in our discussion of the Poincaré bundle.* For if there were such a structure, there would be a morphism from $\mathbb{C}v$ into that space which would take the open set of nonzero vectors into a single point and the zero vector to some other point.

2.8 Elementary transformations.

Here we start with a vector bundle and 'alter' it at a point to get a new vector bundle of the same rank. For example, one may start with the trivial line bundle $\mathcal{O}$ and alter it at a point $x \in X$ to obtain the line bundle $\mathcal{O}(-x)$. Since the latter consists of functions which vanish at x, the way to go about this construction is to take a surjection $\mathcal{O} \to \mathcal{O}_x$ where the latter is the structure sheaf of the single point x. The kernel is $\mathcal{O}(-x)$. The same procedure can be adopted in general. Take a locally free sheaf E and take any surjection to $\mathcal{O}_x$. It is clear that the kernel is locally free. Since the sheaf $\mathcal{O}_x$ is concentrated at x, in order to give a surjection of E onto $\mathcal{O}_x$, we need take only a nonzero linear form on the fibre E_x of the vector bundle E at x.

2.9 Direct Images.

Let Y be another compact Riemann surface and $\pi : Y \to X$ a surjective morphism. Take any locally free sheaf on Y and take its direct image on X. It is a locally free sheaf on X and this is another way of constructing new vector bundles. For example, if we start with a line bundle L on Y, its direct image is a vector bundle of rank equal to the degree of the map π. Also since the Euler characteristic is invariant under direct images we can compute the degree of the direct image. In fact, we have

$$\begin{aligned} \chi(E) &= deg(E) + rk(E)(1 - g(Y)) \\ &= deg(\pi_*(E)) + rk(E)deg(\pi)(1 - g(X)) \end{aligned}$$

This computes the degree of $\pi_*(E)$ in terms of that of $deg(E), deg(\pi)$ and the genera of X and Y.

2.10 Representations of the Fundamental group.

A transcendental construction of vector bundles on X is as follows. Let ρ be a linear representation of the fundamental group $\pi(X)$ in a vector space V. Now π acts (conventionally on the right) as deck transformations on the universal covering space $\tilde{X}$ of X on the one hand and on V by linear transformations on the other. Let E_ρ be the quotient of the space $\tilde{X} \times V$ by the action of π on the product by the prescription

$$g(z, v) = (zg^{-1}, \rho(g)v)$$

for $g \in \pi, z \in \tilde{X}$ and $v \in V$. There is a natural morphism of $E_\rho \to X$ given by the second projection. This actually makes it a vector bundle. This is called the *vector bundle associated to the representation* π. Obviously, the rank of this vector bundle is the same as that of V. It degree is 0 as we shall see presently.

If we take characters (i.e. one-dimensional representations) on π, we would of course get line bundles by this procedure. Since we have seen that line bundles are classified by $H^1(X, \mathcal{O}^*)$, we have a natural homomorphism of the group of characters of π into $H^1(X, \mathcal{O}^*)$. This homomorphism is easy to explain. The natural homomorphism of the constant sheaf $\mathbb{C}^*$ into the sheaf $\mathcal{O}^*$ induces a homomorphism of $H^1(X, \mathbb{C}^*)$ into $H^1(X, \mathcal{O}^*)$. The former can be identified with the group of characters on π and the latter with the group of isomorphism classes of line bundles. Now consider the following commutative diagram of sheaves (with exact rows).

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \to & \mathbb{C} & \to & \mathbb{C}^* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{Z} & \to & \mathcal{O} & \to & \mathcal{O}^* & \to & 0 \end{array}$$

Here the maps $\mathbb{C} \to \mathbb{C}^*$ and $\mathcal{O} \to \mathcal{O}^*$ are both given by the exponential map. We are interested in the associated cohomology sequences. Since $H^2(X, \mathbb{Z}) = \mathbb{Z} \to H^2(X, \mathbb{C}) = \mathbb{C}$ is an inclusion, it follows that the homomorphism $H^1(X, \mathbb{C}^*) \to H^2(X, \mathbb{Z})$ is zero. This implies that the associated line bundle, as an element of $H^1(X, \mathcal{O}^*)$ gets mapped onto zero by the connecting homomorphisminto $H^2(X, \mathbb{Z})$. In other words, the degree of the associated line bundle is always zero. But then since the map $H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O})$ is the Hodge projection, it follows that any line bundle of degree zero, which is the image of an element of $H^1(X, \mathcal{O})$ is associated to a suitable character. Indeed, by using the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$$

instead and noting that by virtue of Hodge decomposition, $H^1(X, \mathbb{R})$ is mapped isomorphically on $H^1(X, \mathcal{O})$, we can actually conclude that any line bundle of degree zero comes from an element of $H^1(X, S^1)$, namely a *unitary* character of π.

In fact, the above argument gives the following result.

2.11 Theorem. *There is a natural isomorphism between unitary character group of the fundamental group and the group of line bundles of degree zero.*

What is the corresponding result in respect of vector bundles? If ρ is a representation of π then the determinant of the associated bundle E_ρ is clearly the line bundle associated to the character $det(\rho)$. Hence we conclude that $deg(E_\rho)$ is zero. If we restrict ourselves to unitary representations, then we have the following easily proved, but very useful, statement.

2.12 Proposition. *Any homomorphism of E_ρ into $E_{\rho'}$ with ρ and ρ' unitary, is induced by a π-homomorphism of the representation spaces. In particular, E_ρ and $E_{\rho'}$ are isomorphic if and only if the representations ρ and ρ' are equivalent.*

However, it is *not true* that every vector bundle of degree 0 arises from a unitary representation of π. For example, take a nontrivial extension E of $\mathcal{O}$ by $\mathcal{O}$. Such an extension exists since these extensions are classified by $H^1(X, \mathcal{O})$ which is of dimension $g \geq 1$ by assumption. If it is associated to a unitary representation of π, the inclusion of $\mathcal{O}$ in E arises, by the above result, by a homomorphism of the trivial representation space into the representation of ρ. Since unitary representations are completely reducible, this subrepresentation would split and hence so would the inclusion of $\mathcal{O}$ in E. By assumption this is not the case.

One might of course ask whether there exists a possibly non-unitary representation to which any given bundle of degree 0 is associated. This is also false. In fact, we have a very precise theorem in this regard, due to A. Weil.

2.13 Definition. A vector bundle E is said to be *decomposable* if it is isomorphic to a nontrivial direct sum of subbundles. If not, it is said to be *indecomposable.*

It is obvious that any vector bundle is a direct sum of indecomposable bundles. Such a decomposition is not unique but in any two decompositions the components are isomorphic, upto order. Hence one can talk of *indecomposable components* of a vector bundle.

2.14 Theorem. *A vector bundle is associated to a representation of the fundamental group, if and only if every indecomposable component is of degree zero.*

Thus a direct sum of a line bundle of degree 1 with one of degree -1 has degree zero, but is not associated to any representation of π.

All these show that it is not possible to construct a reasonable structure of a variety on the set of isomorphism classes of *all* vector bundles of a given degree. We may however restrict ourselves to a big subset of vector bundles and then give an affirmative answer to this question. We will now turn to these considerations.

CHAPTER III MODULI SPACE OF VECTOR BUNDLES

In order to construct a variety whose points correspond to isomorphism classes of vector bundles, one would first like to fix some numerical invariants. The rank r of the bundle is one such invariant and the degree is another.

One might first consider a rough classification and then pass to the equivalence in order to construct the moduli variety in question. When we do this we encounter the problem of passing to the quotient by a group action on a projective variety.

In the sixties, Mumford studied group actions on projective varieties and this led to the notion of *stability* under group actions. He applied his theory to many constructions in Algebraic geometry and in particular to the construction of the *moduli space of* vector bundles.

2.15 Stable and Semistable Vector bundles.

The key to the construction of the moduli of vector bundles is the definition of a stable (resp. semistable) vector bundle.

2.16 Definition. The *slope* $\mu(E)$ of a vector bundle E is the rational number $deg(E)/rk(E)$. A vector bundle on X is *stable* if for every (proper) subbundle F of E, we have the inequality

$$\mu(F) < \mu(E).$$

If the inequality is replaced by $\mu(F) \leq \mu(E)$ then we get the notion of a *semistable* vector bundle.

It is trivial to check that the above condition is equivalent to any one of the following.

1) $\mu(E/F) > \mu(E)$;

2) $\mu(F) < \mu(E/F)$;

3) $\chi(F)/rk(F) < \chi(E)/rk(E)$;

4) $\chi(E/F)/rk(E/F) > \chi(E)/rk(E)$;

5) $\chi(F)/rk(F) < \chi(E/F)/rk(E/F)$.

2.17 Remarks. Note that if n and d are coprime, equality is not possible in the definition of semistable bundles. Hence there is no distinction between stable and semistable bundles in this case.

2.18 Elementary Properties.

Since line bundles do not have any proper subbundles, they are all stable. Moreover, for any line bundle L, we have $det(E \otimes L) = det(E) \otimes L^{rk(E)}$ and hence $deg\ (E \otimes L) = deg\ E + rk(E).deg\ (L)$. In other words, $\mu(E \otimes L) = \mu(E) + deg(L)$. From this we see that E is stable or semi-stable if and only if $E \otimes L$ is so.

2.19 Remarks. Indeed, we will eventually show that the tensor product of two semistable bundles is semistable. But it is not very easy to prove at this stage.

Thanks to the above theorem, in order to study stability of bundles and their properties, we can and (often will) assume that its slope is sufficiently large.

Stable vector bundles behave like line bundles in many ways. To start with, a semistable bundle of negative degree cannot admit any nonzero section. In fact, we have remarked (2.6) that any section can be considered a section of $E \otimes \mathcal{O}(-D)$ which does not vanish anywhere, where D is an effective divisor. In other words, $\mathcal{O}$ is a subbundle of $E \otimes \mathcal{O}(-D)$. Since the latter is semistable, we have $0 \leq \mu(E) - degD$, but by assumption, $\mu(E) < 0$ and $deg\ (D) \geq 0$. Moreover, using the duality theorem, we can derive a vanishing theorem for H^1 as well.

2.20 Proposition. *If E is a semistable vector bundle of negative degree, then $H^0(E) = 0$. The same conclusion subsists if E is stable, and nontrivial of non-negative degree. If E is a semistable bundle of slope greater than $2g-2$, then $H^1(X, E) = 0$. The same vanishing is true if E is stable of slope at least $2g-2$ and not isomorphic to K.*

Using this, we can also give a criterion for the stability of a vector bundle in terms of the dimensions of the space of sections of subbundles instead of the Euler characteristics.

2.21 Proposition. *If the slope of E is at least $2g-2$, then E is stable if $rk\ H^0(X,F)/rk(F) < rk\ H^0(X,E)/rk(E)$ for every proper subbundle F.*

Proof. If E is not stable, then take a subbundle F of maximal slope. (Using the fact that every vector bundle is an extension of line bundles one can check the existence of such a bundle). Then F is clearly semistable with $\mu(F) > \mu(E)/geq2g-2$. Hence $H^1(X,F) = 0$ and consquently $\chi(F)/rk(F) = rkH^0(F)/rk(F)$ so that the given inequality gives‘ $\chi(X,F)/rk(F) = \mu(F) + 1 - g < \mu(E) + 1 - g = \chi(E)/rk(E)/leqrkH^0(X,E)/rk(E)$, a contradiction.

2.22 Remarks. Since given any vector bundle we may tensor it with a line bundle to make H^1 vanish and inflate the slope to $2g-2$ or more, this is indeed a criterion for stability.

Consider any semistable bundle E of slope greater than $2g-1$. Then for any $x \in X$, we have both $H^1(E)$ and $H^1(E \otimes \mathcal{O}(-x))$ are zero. Hence we conclude from Riemann-Roch theorem that the dimension of the space of sections of E that vanish at x (which we identified with $H^0(E \otimes \mathcal{O}(-x))$) is n less than the dimension of $H^0(E)$. This shows that sections of E generate the fibre of E at all points. In other words, all these bundles occur as quotients of the trivial line bundle (of rank $R = d + n(1-g)$). Now there is a variety which parametrises all quotients of a fixed bundle (the trivial bundle of rank R in our case), of a fixed rank and degree. Moreover the group $GL(R)$ acts in an obvious way on this quotient and points of the quotient will correspond to isomorphism classes of semistable bundles.

However it turns out that the problem of taking quotients is more delicate than simply taking the orbits. At stable points for the action which form an open invariant set, things are ok. In this way, Mumford proved the following theorem.

2.23 Theorem. *There exists a natural structure of a nonsingular variety on the space of isomorphism classes of stable vector bundles of rank r and degree d.*

2.24 Theorem of Narasimhan and Seshadri.

¿ From the definition, it follows easily that a stable bundle is indecomposable. For if $E = E_1 \oplus E_2$ then $degE = deg(E_1) + deg(E_2)$ and also $rk(E) = rk(E_1) + rk(E_2)$. This contradicts the stability of E, since we cannot have both $\mu(E_1) < \mu(E)$, $\mu(E_2) < \mu(E)$ at the same time and the above equalities.

In particular, by Weil's theorem, if E is stable of degree 0, then it is given by a representation of the fundamental group (2.14). It turns out that this is not so very significant. There is a deeper fact, due to Narasimhan and Seshadri, (see 3.10 below) which is more vital.

In this respect stable bundles behave again like line bundles. In order to state the theorem neatly, we will introduce a related definition.

2.25 Definition. A vector bundle is said to be *polystable* if it is a direct sum of stable bundles all of which have the same slope.

2.26 Theorem. *A vector bundle of degree 0 is polystable if and only if it is associated to a unitary representation of the fundamental group. A vector bundle of degree 0 is stable if and only if it is associated to an irreducible unitary representation of* π*.*

We have already seen (2.12) that the unitary representation which gives rise to a vector bundle is uniquely determined up to equivalence. So this implies that the set of all polystable bundles can be naturally topologised as follows. Consider the $2g$-fold product $U(n)^{2g}$ of the unitary group $U(n)$. In view of the presentation of π that we have described in (1.1), the space of all unitary matrix representations can be identified with matrices (A_i, B_i), $i = 1, \cdots, g$ with the single relation $\prod A_i B_i A_i^{-1} B_i^{-1} = 1$. In other words, it is the inverse image of Id under the commutator map $U(n)^{2g}$ into the special unitary group $SU(n)$. This is therefore a compact space R. The quotient of this space by the action of the (special) unitary group, acting by (diagonal) conjugation gives then a topological model for the set of equivalence classes of n-dimensional unitary representations of π.

This, together with the remarks in (2.12), means that the category of polystable vector bundles and that of unitary representations of π are equivalent.

2.27 Remarks. For polystable bundles with other (fixed slopes) there are analogous theorems, but we pass them here for simplicity.

2.28 Openness of Stability.

One can show that the property of stability of bundles is open in the following sense. Let E be a *family* of vector bundles over X, parametrised by a variety T. As we did in the case of line bundles, this has to be viewed as $\{E_t\}_{t \in T}$, where E_t is the restriction of E to $\{t\} \times X$. Consider the subset of T consisting of $t \in T$ with E_t stable. What we mean is that this subset is (Zariski) open. We will content ourselves by indicating how it is proved in the case of bundles of rank 2.

Consider on $J \times T \times X$ the bundle $p_{23}^*(E) \otimes p_{13}^*(P)$ where P is the Poincaré bundle of appropriate degree. Then nonstable bundles in our family correspond to points $t \in T$ such that there exist $j \in J$ with $H^0(j \otimes E_t) \neq 0$. But we know by the semiconinuity principle that such pairs (j, t) form a closed subset in $J \times T$. The map $J \times T \to T$ being proper, its image is closed.

2.29 Connectedness.

If E is a vector bundle of rank 2, then by tensoring it with a suitable line bundle we can ensure that it has an everywhere nonvanishing section. In other words E occurs in an exact sequence

$$0 \to \mathcal{O} \to E \to L \to 0$$

Taking determinants we conclude that $L = det(E)$.

If E_1 and E_2 are any two vector bundles of the same degree, then both of them occur as extensions of a line bundle by $\mathcal{O}^{\oplus n-1}$. As we have seen, if L is given, these form a family parametrised by a vector space, namely $H^1(X, L^* \otimes \mathcal{O}^{\oplus n})$. Since L is any element of $J^d, d = deg(E)$, it follows that all these extensions form a family parametrised by a vector bundle T over J^d! In particular, if they are both stable, then they lie in a Zariski open subset of T. Thus we conclude that E_1 and E_2 occur in a connected family. Therefore, eventually if one constructs a moduli space of stable bundles, it would be connected and indeed irreducible! Besides, if E is *any* bundle, we have seen that it is obtainable as an extension of the above type. Since we may also take a stable bundle and get it as such an extension, it implies that there is a Zariski open set of the affine space, which corresponds to stable bundles, and so in a sense, any bundle can be approximated by stable bundles.

2.30 Moduli space of vector bundles.

Since polystable bundles form a compact set as we have remarked, although the parameter variety was not a complex variety, one might even hope to construct a *projective variety* whose points correspond to polystable bundles of a given rank and degree. This is essentially true and was proved by Seshadri. However, there are some delicate points here, which we will presently explain.

2.31 Theorem. *There exists a natural structure of a normal complex projective variety of dimension $n^2(g-1)+1$ on the set of isomorphism classes of polystable vector bundles of rank n and degree d.*

¿ From the fact that we have mentioned normality in the above statement, it may be surmised that it is not smooth. Indeed, it is only smooth if n and d are coprime, *and when $g = 2, n = 2$ for any degree.* See NR1].

Although we have said above that it parametrises polystable bundles, there is a better way to think of it. Consider any *semistable* vector bundle. If it is not stable, then it admits a proper subbundle which is also of the same slope. If F is a subbundle of E of least rank and same slope, then it follows that F is stable. By induction then, we obtain a flag of subbundles

$$F_0 = 0 \subset F_1 \subset \cdots \subset F_r = E$$

where all the subbundles F_i have the same slope and F_i/F_{i-1} are stable. A flag with this property is not unique, much as the Jordan-Hölder series of a module is not unique. However, again as in Jordan-Hölder theorem, the successive quotients are however isomorphic upto order. In other words, the polystable bundle $GrE = \sum F_i/F_{i-1}$ is uniquely determined, upto isomorphism, by E.

2.32 Definition. Two semistable vector bundles E_1 and E_2 of the same slope are said to be *S-equivalent* if $Gr(E_1)$ and $Gr(E_2)$ are isomorphic.

2.33 Remarks. The notion of S-equivalence is relevant only for nonstable polystable bundles. On stable bundles, it reduces to isomorphism.

The openness, valid for stable and semistable bundles is not valid for polystable bundles. For example, we may take extensions of $\mathcal{O}$ by itself. All the extensions are S-equivalent to the trivial bundle of rank 2 but the trivial extension is the only one which gives a polystable bundle. We gave this as an example to show that a 'good' structure of moduli cannot exist. But with the notion of S-equivalence, all the extensions are S-equivalent to the trivial bundle, so that it is no longer a negative example!

Thus it is more fruitful to think of the moduli space as the set of S-equivalence classes of semistable bundles, rather than as the set of isomorphism classes of polystable bundles.

2.34 Universal Property.

Let us denote the moduli space we referred to above by $U_X(n,d)$, or simply $U(n,d)$. It has the following universal properties.

2.35 Theorem. *If E is any family of semistable vector bundles on X, parametrised by a variety T, then the (classifying) map φ_E, which maps $t \in T$ on the S-equivalence class of E_t, is a morphism $T \to U(n,d)$.*

2.36 Theorem. *Let M be a variety. Assume given for every family E of semistable bundles, parametrised by T, a morphism $f_E : T \to M$ and that the morphisms f_E are compatible with pull backs in the sense that if E' is a family obtained by pulling back E by the morphism $(Id \times g) : T' \times X \to T \times X$ where $g : T' \to T$ is a morphism, then we have $f_{E'} = f_E \circ g$. Then there is a unique morphism $f : U(n,d) \to M$ with the property that $f \circ \varphi_E = f_E$ for any family E.*

The most optimistic expectation would be that there exists a Poincaré bundle on the lines of (1.16) at least on the open set of stable points. However this turns out to be false in general. The precise theorem that I proved was:

2.37 Theorem. **[R]** *If there exists a Poincaré bundle on any Zariski open set of $U(n,d)$, then n and d are coprime. If they are indeed coprime, there exists a Poincaré bundle on the whole of $U(n,d) \times X$ with the obvious universal property.*