Local solvability of second order differential operators with double characteristics on Heisenberg groups

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Monastir Lectures 2005: Local solvability of second order differential operators with double characteristics on Heisenberg groups

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Abstract

Consider doubly characteristic differential operators of the form

$$L = \sum_{j,k=1}^{m} \alpha_{jk}(x) V_j V_k + \text{lower order terms}\,,$$

where the V_j are smooth real vector fields and the α_{jk} are smooth complex coefficients forming a symmetric matrix $\mathcal{A}(x) := \{\alpha_{jk}(x)\}_{j,k}$. We say that L is essentially dissipative at x_0, if there is some $\theta \in \mathbb{R}$ such that $e^{i\theta} L$ is dissipative at x_0, in the sense that $\operatorname{Re}\left(e^{i\theta}\mathcal{A}(x_0)\right) \geq 0$. For a large class of doubly characteristic operators L of this form, one can show that a necessary condition for local solvability at x_0 is essential dissipativity of L at x_0. By means of Hörmander's classical necessary condition for local solvability, the proof is reduced to the following question:

Suppose that Q_A and Q_B are two real quadratic forms on a finite dimensional symplectic vector space, and let $Q_C := \{Q_A, Q_B\}$ be given by the Poisson bracket of Q_A and Q_B. Then Q_C is again a quadratic form, and we may ask: When can we find a common zero of Q_A and Q_B at which Q_C does not vanish?

In view of these results, there essentially remains the problem of giving sufficient conditions for local solvability of essential dissipative operators L. The lectures focus on this problem for left-invariant operators of this type on Heisenberg groups, for which a fairly complete answer to the question of local solvability can be given.

These latter results are based on tools from representation theory, in particular on Howe's work on the oscillator semigroup, which in return is related to the metaplectic representation of Shale and Weil. A major part of these tools has been developped in long standing collaborations, in particular with Fulvio Ricci.
[1]

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Contents

1 Introduction: A necessary condition

Consider a second-order linear differential operator of order k with smooth coefficients

$$L = \sum_{|\alpha|\le k} c_\alpha(x) D^\alpha$$

on an open subset Ω of $\mathbb{R}^n$, where $D^\alpha := \left(\frac{\partial}{2\pi i\,\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{2\pi i\,\partial x_n}\right)^{\alpha_n}$.

L is said to be *locally solvable* at $x_0 \in \Omega$, if there exists an open neighborhood U of x_0 such that the equation $Lu = f$ admits a distributional solution $u \in \mathcal{D}'(U)$ for every $f \in C_0^\infty(U)$ (for a slightly more general definition, see [7]).

Around 1956, Malgrange and Ehrenpreis proved that every non-trivial linear differential operator with constant coefficients is locally solvable.

Shortly later H. Lewy produced the following example of a nowhere solvable operator on $\mathbb{R}^3$:

$$Z = X - iY\ ,\ \text{ where }\ X := \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial u},\ Y := \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial u}\ .$$

Not quite incidentally, Z is a left–invariant operator on a 2–step nilpotent Lie group, the Heisenberg group $\mathbb{H}_1$.

This example gave rise to an intensive study of so-called principal type operators, which eventually led, most notably through the work of Hörmander, Maslov, Egorov, Nirenberg–Trèves and Beals–Fefferman, to a complete solution of the problem of local solvability of such operators (see [7]).

Let us recall some notation. Denote by

$$p_k(x,\xi) := \sum_{|\alpha|=k} c_\alpha(x)\xi^\alpha$$

the *principal symbol* of L. We shall consider p_k as an invariantly defined function on the reduced cotangent bundle $\mathcal{C} := T^*\Omega \setminus 0 = \Omega \times (\mathbb{R}^n \setminus \{0\})$ of Ω.

Let us denotes by π_1 the base projection $\pi_1 : T^*\Omega \to \Omega$, $(x, \xi) \mapsto x$. $T^*\Omega$ carries a canonical 1-form, which, in the usual coordinates, is given by $\alpha = \sum_{j=1}^n \xi_j dx_j$, so that $T^*\Omega$ has a canonical symplectic structure, given by the 2-form $d\alpha = \sum_{j=1}^n d\xi_j \wedge dx_j$. In particular, for any smooth real function a on Ω, its corresponding *Hamiltonian vector field* H_a is well-defined, and explicitly given by

$$H_a := \sum_{j=1}^n \left(\frac{\partial a}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial a}{\partial x_j} \frac{\partial}{\partial \xi_j} \right).$$

If γ is an integral curve of H_a, i.e., if $\frac{d}{dt}\gamma(t) = H_a(\gamma(t))$, then a is constant along γ, and γ is called a *null bicharacteristic* of a, if a vanishes along γ. Finally, the (canonical) Poisson bracket of two smooth functions a and b on $T^*(\Omega)$ is given by

$$\{a, b\} := d\alpha(H_a, H_b) = H_a b = \sum_{j=1}^n \left(\frac{\partial a}{\partial \xi_j} \frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j} \frac{\partial b}{\partial \xi_j} \right).$$

Let

$$\Sigma = \{p_k = 0\} \subset \mathcal{C}$$

denote the *characteristic variety* of L. L is said to be of *principal type*, if $D_\xi p_2$ does not vanish on Σ (or, more generally, if for every $\zeta \in \Sigma$ there is a real number θ such that $d(\mathrm{Re}\,(e^{i\theta} p_k))(\zeta)$ and $\alpha(\zeta)$ are non–proportional).

In 1960, Hörmander proved the following fundamental result on non–existence of solutions (see [9]):

Theorem 1.1 (Hörmander) *Suppose there is some $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ such that*

$$f(x_0, \xi_0) = g(x_0, \xi_0) = 0 \quad \textit{and } \{f, g\}(x_0, \xi_0) \neq 0,$$

where $f := \mathrm{Re}\, p_k$ and $g := \mathrm{Im}\, p_k$. Then L is not locally solvable at x_0.

For example, if $L = Z$ is the Lewy operator, and if we choose coordinates $((x, y, u), (\xi, \eta, \mu))$ for the cotangent bundle of $\mathbb{R}^3$, then

$$f = i(\xi - \tfrac{1}{2}\mu y), \quad g = i(\eta + \tfrac{1}{2}\mu x) \quad \text{and} \quad \{f, g\} = -\mu.$$

Given (x_0, y_0, u_0), choosing e.g. $\mu_0 = 1$, we can find ξ_0, η_0 such that Hörmander's condition is satisfied, so that Z is nowhere locally solvable. Hörmander's theorem thus represents a wide extension of Lewy's result.

Remark 1.2 Notice that commutation relations play a crucial role here. In view of this, it appears natural to study local solvability of invariant differential operators on Lie groups, where commutation relations assume a particularly clear cut form. The results obtained in these settings can then give at least some guidance as to what may rule local solvability in more general situations. Moreover, on Lie groups, further tools are available and have proved to be useful, such as representation theory. We shall see instances of such ideas in the course of these lectures.

A complete answer to the question of local solvability of principal type operators L was eventually given in terms of the following condition $(\mathcal{P})$ of Nirenberg and Trèves:

$(\mathcal{P})$. The function $\mathrm{Im}\,(e^{i\theta}p_k)$ does not take both positive and negative values along a null–bicharacteristic $\gamma_\theta(t)$ of $\mathrm{Re}\,(e^{i\theta}p_k)$, for any $\theta \in \mathbb{R}$.

In fact, L of principal type is locally solvable at x_0 if and only if $(\mathcal{P})$ holds over some neighborhood of x_0. Notice that this is a condition solely on the principal symbol of L.

In these lectures, which are largely based on the joint article [21] with F. Ricci, and on [18], [19], we shall consider second order differential operators L with double characteristics. Let

$$\Sigma_2 := \{(x,\xi) \in \mathcal{C} : \quad D_\xi p_2(x,\xi) = 0\}\,.$$

By Euler's identity, Σ_2 is contained in the characteristic variety Σ, and so Σ_2 consists of the doubly characteristic cotangent vectors of L.

If p_2 is real, and if we assume that Σ_2 is a submanifold of codimension $m < n$ in $\mathcal{C}$, such that rank $D^2_\xi p_2 = m$ for every $\xi \in \Sigma_2$ (notice that for our class, $D^2_\xi p_2$ is in fact independent of ξ), then the following has been shown in [29] :

Given any $x_0 \in \pi_1(\Sigma_2)$, there exist suitable linear coordinates near x_0 such that, in those new coordinates, p_2 can be written in the form

$$p_2(x,\xi) = {}^t\!(\xi' + E(x)\,\xi'')\mathcal{A}(x)(\xi' + E(x)\,\xi'') \tag{1.1}$$

with respect to some splitting of coordinates $\xi = (\xi', \xi'')$, $\xi' \in \mathbb{R}^m$, $\xi'' \in \mathbb{R}^{n-m}$. Here, $\mathcal{A}(x) \in \mathrm{Sym}\,(m,\mathbb{R})$ is non-degenerate and $E(x) \in M^{m\times(n-m)}(\mathbb{R})$, and both matrices vary smoothly in x (as usually, $\mathrm{Sym}\,(n,\mathbb{K})$ denotes the space of all symmetric $n \times n$ matrices over the field $\mathbb{K}$, where $\mathbb{K}$ will be either $\mathbb{R}$ or $\mathbb{C}$.)

Motivated by this result, let us assume here that L is a complex coefficient differential operator with smooth coefficients, whose principal symbol is given by (1.1) where, however, now $\mathcal{A}(x) \in \mathrm{Sym}\,(m,\mathbb{C})$ is a complex matrix. We then write

$$\mathcal{A}(x) = A(x) + iB(x), \quad x \in \Omega\,,$$

with $A(x), B(x) \in \mathrm{Sym}\,(m,\mathbb{R})$. Notice that (1.1) means that, up to a factor $(2\pi)^{-2}$, L can be written as

$$L = \sum_{j,k=1}^{m} \alpha_{jk}(x)V_jV_k + \text{ lower order terms}\,, \tag{1.2}$$

where V_j is the real vector field

$$V_j = \frac{\partial}{\partial x_j} + \sum_{l=m+1}^{n} E_{jl}(x)\frac{\partial}{\partial x_l}\,, \quad j = 1,\ldots,m\,, \tag{1.3}$$

with $\mathcal{A}(x) = \{\alpha_{jk}(x)\}_{j,k}$, $E(x) = \{E_{jl}(x)\}_{j,l}$.

It may be worth while mentioning that every second order differential operator L, whose principal part is of the form

$$\sum_{j,k=1}^{m} \beta_{jk}(x)Y_jY_k, \tag{1.4}$$

with smooth real vector fields Y_j which are linearly independent at x_0, can be put into this form, locally near x_0. This can easily be seen by means of a suitable change of coordinates and choice of a suitable new basis $V_1, \ldots, V_m$ of the C^∞-module spanned by $Y_1, \ldots, Y_m$.

Moreover, by (1.1),

$$\Sigma_2 \supset \{(x,\xi) \in \mathcal{C} : \xi' + E(x)\,\xi'' = 0\} \subset \{(x,\xi) \in \mathcal{C} : dp_2(x,\xi) = 0\},$$

so that in particular $\pi_1(\Sigma_2) = \Omega$, and equality holds in these relations, if, e.g., $\mathcal{A}(x)$ is non-degenerate. Finally, denote by q_j the symbol of V_j (up to a factor 2π), i.e.,

$$q_j(x,\xi) := \xi_j + \sum_{l=m+1}^{n} E_{jl}(x)\xi_l\,, \quad j = 1, \ldots, m\,,$$

and define the skew-symmetric matrix

$$J_{(x,\xi'')} := \big(\{q_j, q_k\}(x,\xi)\big)_{j,k=1,\ldots,m}$$

(compare [29]). Here, $\{\cdot,\cdot\}$ denotes again the canonical Poisson bracket on $T^*\Omega$. Observe that $J_{(x,\xi'')}$ depends indeed only on x and the ξ''-component of ξ. Notice also that if $\mathcal{A}(x)$ is non-degenerate, then $J_{(x_0,\xi_0'')}$ is non-degenerate if and only if Σ_2 is symplectic in a neigborhood of (x_0,ξ_0), where ξ_0' is chosen so that $(x_0,\xi_0) \in \Sigma_2$ (see, e.g. [37], Proposition 3.1, Ch. VII).

Assume that $J_{(x,\xi'')}$ is non-degenerate. Then we can associate to $J_{(x,\xi'')}$ the skew form

$$\sigma_{(x,\xi'')}(v,w) := {}^t v\; {}^t(J_{(x,\xi'')})^{-1}\, w, \quad v, w \in \mathbb{R}^m,$$

which defines a symplectic structure on $\mathbb{R}^m$. In particular, $m = 2d$ is even.

Recall that if σ is an arbitrary symplectic form on $\mathbb{R}^{2d}$, then we can associate to any smooth function a on $\mathbb{R}^{2d}$ the Hamiltonian vector field H_a^σ such that $\sigma(H_a^\sigma, Y) = da(Y)$ for all vector fields Y on $\mathbb{R}^{2d}$, and define the associated Poisson bracket accordingly by

$$\{a,b\}_\sigma := \sigma(H_a^\sigma, H_b^\sigma)\,.$$

We can thus define a Poisson structure $\{\cdot,\cdot\}_{(x,\xi'')}$ on $\mathbb{R}^m$ (depending on the point (x,ξ'')) by putting $\{\cdot,\cdot\}_{(x,\xi'')} := \{\cdot,\cdot\}_{\sigma_{(x,\xi'')}}$.

In order to formulate our first main theorem, we need to introduce some further notation concerning quadratic forms.

If $A \in \mathrm{Sym}\,(n,\mathbb{K})$, we shall denote by Q_A the associated quadratic form

$$Q_A(z) := {}^t z A z, \quad z \in \mathbb{K}^n,$$

on $\mathbb{K}^n$. For any non-empty subset M of a $\mathbb{K}$- vector space V, $\mathrm{span}_{\mathbb{K}} M$ will denote its linear span over $\mathbb{K}$ in V.

Let $A, B \in \mathrm{Sym}\,(m,\mathbb{R})$. We say that A, B form a *non-dissipative pair,* if 0 is the only positive-semidefinite element in $\mathrm{span}_{\mathbb{R}}\{A,B\}$. Moreover, we put

$$\begin{aligned} \mathrm{maxrank}\,\{A,B\} &:= \max\{\mathrm{rank}\,F : F \in \mathrm{span}_{\mathbb{R}}\{A,B\}\} \\ \mathrm{minrank}\,\{A,B\} &:= \min\{\mathrm{rank}\,F : F \in \mathrm{span}_{\mathbb{R}}\{A,B\}\,, F \neq 0\}. \end{aligned}$$

If A, B form a non-dissipative pair, then one can show ([18]) that there is some $Q \in GL(m,\mathbb{R})$ such that

$$\mathrm{tr}\ {}^t\!QAQ = \mathrm{tr}\ {}^t\!QBQ = 0,$$

so that $\mathrm{minrank}\,\{A,B\} \geq 2$.

Observe finally that if Q_A and Q_B are quadratic forms on $\mathbb{R}^m$, then $\{Q_A, Q_B\}_{(x,\xi'')}$ is again a quadratic form. We then have

Theorem 1.3 ([18]) *Let L be given by* (1.2), *and let $x_0 \in \Omega$. Assume that*

(a) $A(x_0), B(x_0)$ *forms a non-dissipative pair.*

(b) There exists some $\xi_0'' \in \mathbb{R}^{n-m} \setminus \{0\}$ such that $J_{(x_0,\xi_0'')}$ is non-degenerate, and the matrices $A(x_0), B(x_0)$ and $C(x_0,\xi_0'')$ are linearly independent over $\mathbb{R}$, where $C(x_0,\xi_0'') \in$ $\mathrm{Sym}\,(m,\mathbb{R})$ *is defined by*

$$Q_{C(x_0,\xi_0'')} := \{Q_{A(x_0)}, Q_{B(x_0)}\}_{(x_0,\xi_0'')}\,.$$

(c) Either

(i) $\mathrm{minrank}\,\{A(x_0), B(x_0)\} \geq 3$ *and* $\mathrm{maxrank}\,\{A(x_0), B(x_0)\} \geq 17$, *or*

(ii) $\mathrm{minrank}\,\{A(x_0), B(x_0)\} = 2$, $\mathrm{maxrank}\,\{A(x_0), B(x_0)\} \geq 9$, *and the joint kernel* $\ker A(x_0) \cap \ker B(x_0)$ *of $A(x_0)$ and $B(x_0)$ is either trivial, or a symplectic subspace with respect to the symplectic form $\sigma_{(x_0,\xi_0'')}$.*

Then L is not locally solvable at x_0.

Notice that, like condition $(\mathcal{P})$, the condition in (a) is again a sign condition on the principal symbol of L.

We also remark that the conditions in (c) are automatically satisfied, if $\mathcal{A}$ is non-degenerate and $m \geq 18$.

Theorem 1.3 shows that a "generic" operator L of the form (1.2) can be locally solvable at x_0 only, if there is some $\theta \in \mathbb{R}$ such that $\mathrm{Re}\,(e^{i\theta}\mathcal{A}(x_0)) \geq 0$. A major task which remains is thus to study local solvability of L under the assumption that $\mathrm{Re}\,\mathcal{A}(x) \geq 0$ for every $x \in \Omega$. A stronger condition is the condition

(1.5) $$|\mathrm{Im}\,\mathcal{A}(x)| \leq C\mathrm{Re}\,\mathcal{A}(x), \quad x \in \Omega,$$

for some constant $C \geq 0$. This condition is equivalent to Sjöstrand's *cone condition* [36] (see also [10]). It implies hypoellipticity with loss of one derivative of the transposed operator ${}^tL$, for "generic" first order terms in (1.2), and thus local solvability of L at x_0 (see [7], Ch. 22.4, for details and further references).

Since, however, local solvability of L is in general a much weaker condition than hypoellipticity of ${}^tL$, we are still rather far from understanding what rules local solvability in general, even when the cone-condition is satisfied.

We should like to mention that, even if the cone-condition is satisfied, for instance small perturbations of the coefficients of the first order terms preserving the values at x_0, may influence local solvability and lead to local solvability in situations where the unperturbed operator is not locally solvable at x_0 (see, e.g., [2]). Moreover, if, e.g., maxrank $\{A(x_0), B(x_0)\} = 4$ or 6, then the conclusion in Theorem 1.3 may not be true (see [12],[20]).

All these results indicate that there is rather little hope for a complete characterization of local solvability for doubly characteristic operators in general, but that Theorem 1.3 in combination with the above mentioned results on hypoellipticity give at least rather satisfactory answers in the "generic" case.

Nevertheless, for the case of homogeneous, left-invariant second order differential operators on the Heisenberg group $\mathbb{H}_n$, a fairly comprehensive answer can be given.

In the sequel, we shall therefore entirely concentrate on the case of left-invariant operators on the Heisenberg group.

Recall that the *Heisenberg group* $\mathbb{H}_d$ of dimension $n = 2d+1$ is $\mathbb{R}^{2d} \times \mathbb{R}$ (as a manifold), with the group law

$$(v, u) \cdot (v', u') = (v + v', u + u' + \tfrac{1}{2}\, {}^t v J v'\,),$$

for $v, v' \in \mathbb{R}^{2d}, u, u' \in \mathbb{R}$, where J is the standard skew matrix

$$J := \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}.$$

Here, $m = 2d$ and $n = 2d + 1$.

A basis of the Lie algebra $\mathfrak{h}_d$ of left-invariant vector fields is then given by

$$\begin{aligned} V_j &:= \frac{\partial}{\partial v_j} + \tfrac{1}{2}({}^t v \cdot J)_j \frac{\partial}{\partial u}, \qquad j = 1, \dots, m\,, \\ U &:= \frac{\partial}{\partial u}. \end{aligned}$$

Consider an operator L on $\mathbb{H}_d$ of the form

$$L = \sum_{j,k=1}^{m} \alpha_{jk} V_j V_k + \text{ lower order terms}\,, \tag{1.6}$$

where the coefficient matrix $\mathcal{A} = (\alpha_{jk})_{j,k} \in \mathrm{Sym}\,(m, \mathbb{C})$ is symmetric. We put $A := \mathrm{Re}\,\mathcal{A}$, $B := \mathrm{Im}\,\mathcal{A}$. The matrix $\{E_{jl}(x)\}_{jl}$ in (1.3) is then given by a vector, with components

$$E_j(v, u) = \frac{1}{2} \sum_{k=1}^{m} J_{kj} v_k\,,$$

and, if we split a cotangent vector $\xi = (\nu, \mu) \in \mathbb{R}^m \times \mathbb{R}$ according to the coordinates $x = (v, u) \in \mathfrak{h}_d$, we have

$$q_j((v, u), (\nu, \mu)) = \nu_j + \tfrac{1}{2} \sum_s \mu J_{sj}\, v_s,$$

so that one easily computes that

$$J_{((v,u),\mu)} = \Big(\{q_j, q_k\}((v, u), (\nu, \mu))\Big)_{j,k=1,\dots,m} = \mu J\,,$$

which is non-degenerate whenever $\mu \neq 0$, and then the associated Poisson bracket on $\mathbb{R}^{2d}$ is given by

$$\{a, b\}_{((v,u),\mu)} = -\mu\,(\nabla a) J\ {}^t\!(\nabla b)\,.$$

Notice that this is the skew form associated to the matrix $-\mu J$. This implies in particular that

$$\{Q_A, Q_B\}_{((v,u),\mu)} = -2\mu(AJB - BJA).$$

Putting

$$C := 2(AJB - BJA)\,,$$

we thus obtain the following corollary to Theorem 1.3.

Corollary 1.4 *Assume that A, B forms a non-dissipative pair, and that A, B and C are linearly independent. Moreover, suppose that either*

(i) $\operatorname{minrank}\{A, B\} \geq 3$ *and* $\operatorname{maxrank}\{A, B\} \geq 17$, *or*

(ii) $\operatorname{minrank}\{A, B\} = 2$, $\operatorname{maxrank}\{A, B\} \geq 9$, *and that the joint kernel* $\ker A \cap \ker B$ *of Q_A and Q_B is either trivial, or a symplectic subspace with respect to the canonical symplectic form on $\mathbb{R}^{2d}$ (associated to $-J$).*

Then the operator L in (1.6) *on $\mathbb{H}_d$ is nowhere locally solvable.*

Before we turn to sufficient conditions for local solvability of the operators L, let me briefly comment on the proof of Corollary 1.4.

Denote by f and g the real and imaginary parts of the principal symbol of L. Writing $q := (q_1, \ldots, q_{2d})$, we have

$$f = Q_A(q(\cdot)) \quad \text{and} \quad g = Q_B(q(\cdot)),$$

and one easily computes that

$$\{f, g\}((v, u), (\nu, \mu)) = 4 \sum_{j,k} (A J_{((v,u),\mu)} B)_{jk}\, q_j((v, u), (\nu, \mu))\, q_k((v, u), (\nu, \mu)),$$

hence

$$\{f, g\} = Q_C(q(\cdot)).$$

Since

$$q((v, u), (\nu, \mu)) = \nu - \tfrac{1}{2}\mu J v,$$

so that the mapping $(\nu, \mu) \mapsto q((v, u), (\nu, \mu))$ is surjective for every (v, u), Corollary 1.4 can thus obviously be reduced by means of Hörmander's Theorem 1.1 to the following result concerning real quadrics, which represents the core of the work in [18] (the proof of Theorem 1.3 is based on similar ideas, but more involved):

Theorem 1.5 *Assume that $\mathbb{R}^n = \mathbb{R}^{2d}$ is endowed with the canonical symplectic form. Let $A, B \in \operatorname{Sym}(n, \mathbb{R})$ be linearly independent, and assume that A, B forms a non-dissipative pair. Define $Q_C := \{Q_A, Q_B\}$ as the Poisson bracket of Q_A and Q_B, and assume that A, B and C are linearly independent.*

Then there exists a point $x \in \mathbb{R}^n$ such that

$$Q_A(x) = Q_B(x) = 0 \text{ and } Q_C(x) \neq 0,$$

provided one of the following conditions are satisfied:

(i) minrank $\{A, B\} \geq 3$ *and* maxrank $\{A, B\} \geq 17$;

(ii) minrank $\{A, B\} = 2$, maxrank $\{A, B\} \geq 9$, *and the joint radical* $\ker A \cap \ker B$ *of* Q_A *and* Q_B *is either trivial, i.e.,* $\ker A \cap \ker B = \{0\}$, *or a symplectic subspace of* $\mathbb{R}^n$.

Put another way: If A and B are linearly independent and form a non-dissipative pair, then under the rank conditions above, if the quadratic form Q_C vanishes on the set of joint zeros of the forms Q_A and Q_B, there are $\alpha, \beta \in \mathbb{R}$ such that $C = \alpha A + \beta B$.

Corollary 1.4 shows that local solvability of L on Heisenberg groups can essentially only arize if the operator $e^{i\theta}L$ is dissipative, for some $\theta \in \mathbb{R}$. This statement is true in the strict sense, if, e.g., the matrix A, or if $A+iB$, is non-degenerate so that maxrank $\{A, B\} = 2d$, and $d \geq 9$.

That the conclusion of the theorem cannot hold without any rank condition is demonstrated by the following examples (compare also [18]).

Example 1.6 a) On $\mathbb{R}^2$, take

$$Q_A := x_1 x_2, \;\; Q_B := x_1^2 - x_2^2.$$

b) A more sophisticated example on $\mathbb{R}^4$, with coordinates $(x, y) = (x_1, x_2, y_1, y_2)$, is given as follows:

$$Q_A := x_1 y_2 + x_2 y_1, \;\; Q_B := x_1 y_1 - x_2 y_2.$$

Here, one computes that A and B are non-degenerate, have vanishing traces (so that they form a non-dissipative pair) and that

$$Q_C = \{Q_A, Q_B\} = 2(x_1 y_2 - x_2 y_1)$$

vanishes on the joint zeros of Q_A and Q_B. Nevertheless, A, B and C are linearly independent.

In order to describe our sufficient conditions for local solvability, it is convenient to slightly modify the notation. Let us first split coordinates $z = (x, y)$, with $x, y \in \mathbb{R}^d$. Then the previous basis $V_1, \ldots, V_{2d}, U$ of the Lie algebra of $\mathbb{H}_d$ can be written more explicitly as $X_1, \ldots, X_d, Y_1, \ldots, Y_d, U$ (in this order), where

$$\begin{aligned} X_j &= \partial_{x_j} - \tfrac{1}{2} y_j \partial_u \;, \qquad & j = 1, \cdots, d \;, \\ Y_j &= \partial_{y_j} + \tfrac{1}{2} x_j \partial_u \;, \qquad & j = 1, \cdots, d \;, \\ U &= \partial_u. \end{aligned} \tag{1.7}$$

Given a $2d \times 2d$ complex symmetric matrix $A = (a_{jk})$, set

$$\mathcal{L}_A = \sum_{j,k=1}^{2d} a_{jk} V_j V_k \;, \tag{1.8}$$

and, for $\alpha \in \mathbb{C}$,

$$\mathcal{L}_{A,\alpha} = \mathcal{L}_A + i\alpha U \;. \tag{1.9}$$

(notice that what used to be $\mathcal{A}$ has become A now).

These operators can be characterized as the second order left-invariant differential operators on $\mathbb{H}_d$ which are homogeneous of degree 2 under the automorphic dilations $D_r : (v, u) \mapsto (rv, r^2u), \quad r > 0$.

Recall also that the real symplectic group $\mathrm{Sp}\,(d, \mathbb{R})$, i.e., the group of all $2d \times 2d$ real matrices g such that ${}^t g J g = J$, acts by automorphisms on $\mathbb{H}_d$ fixing the center, namely by $(z, u) \mapsto (gz, u)$, if $g \in \mathrm{Sp}\,(d, \mathbb{R})$.

Since we now assume that $\mathcal{L}_{A,\alpha}$ is essentially dissipative, after multiplying it with a suitable complex number of modulus 1, we may assume that $\mathcal{L}_{A,\alpha}$ is dissipative, or, equivalently, that

$$\mathrm{Re}\, A \geq 0. \tag{1.10}$$

This condition is considerably weaker than Sjöstrand's cone condition (1.5).

The operators $\mathcal{L}_{A,\alpha}$ have double characteristics, and for such operators it is known that it is not only the principal symbol that governs local solvability, but that also the subprincipal symbol in combination with the Hamiltonian mappings associated with doubly characteristic points plays an important role. Due to the translation invariance of our operators and the symplectic structure that is inherent in the Heisenberg group law, these Hamiltonians are essentially encoded in the *Hamiltonian* $S \in \mathfrak{sp}(n, \mathbb{C})$, associated to the coefficient matrix A by the relation

$$S := -AJ$$

(see e.g. [29]).

In order to emphasize the central role played by S, we shall therefore also denote $\mathcal{L}_{A,\alpha}$ by $L_{S,\alpha}$.

One of the main results in [19] (Theorem 2.2), states that, under a further, natural condition, the question of local solvability of the operators $L_{S,\alpha}$ can essentially be reduced to the case where the Hamiltonian S has only real eigenvalues.

This is achieved by showing that an integration by parts technic, which had been introduced by R. Beals and P.C. Greiner in [1], and since then been applied in modified ways in various subsequent articles, e.g. in [21], when viewed in the right way, ultimately allows to show that $L_{S,\alpha}$ is locally solvable, provided that $L_{S_r,\beta}$ is locally solvable for particular values of β. Here, S_r is the "part" of S comprising all Jordan blocks associated with real eigenvalues.

For the case of real eigenvalues, one can prove partial results which, in combination with Theorem 2.2 of [19], allow to widely extend all the results known to date for operators $\mathcal{L}_{A,\alpha}$ with non-real coefficient matrices A (see Theorems 2.6, 2.7 in [19]).

However, these results are fairly involved and their proofs tend to be very technical. Therefore we shall restrict ourselves for these lectures to the special case where A satisfies the cone condition (1.5), and refer the interested reader for more general results to [19].

We should also like to mention that there is an abundance of literature on the question of local solvability of various classes of invariant operators on Lie groups. It would be impossible to list all the relevant articles, so that we restrict ourselves to pointing out just a few references, and apologizing to all whom we haven't done justice in doing so:

The case of bi-invariant differential operators of arbitrary order has been studied in particular by M. Rais [34],[35] (see also Helgason's work [6] for a related context). Some necessary and various sufficient representation theoretic conditions for local solvability of left-invariant differential operators of arbitrary order on nilpotent Lie groups can be found in work of P. Lévy-Bruhl [17],[16], [15], [14], [13] (see also B. Helffer's survey [5]). Finally, a complete answer for a particular class of second order operators on nilpotent Lie groups of step 3 and higher has been given in [30].

2 Sufficient conditions: Statement of the main results

In order to emphasize the symplectic structure on $\mathbb{R}^{2d}$ which is implicit in (1.7), and at the same time to provide a coordinate-free approach, we shall work within the setting of an arbitrary $2d$-dimensional real vector space V, endowed with a symplectic form σ. The extension of σ to a complex symplectic form on $V^{\mathbb{C}}$, the complexification of V, will also be denoted by σ.

If Q is a complex-valued quadratic form on V, we shall often view it as a symmetric bilinear form on $V^{\mathbb{C}}$, and shall denote by $Q(v)$ the quadratic form $Q(v,v)$. Q and σ determine a linear endomorphism S of $V^{\mathbb{C}}$ by imposing that

$$\sigma(v, Sw) = Q(v, w).$$

Then, $S \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma)$, i.e.,

$$\sigma(Sv, w) + \sigma(v, Sw) = 0. \tag{2.1}$$

S is called the *Hamilton map* of Q. We shall then also write $Q = Q_S$. Clearly, S is real, i.e., $S \in \mathfrak{sp}(V, \sigma)$, if Q is real.

Let us endow V with the Poisson bracket associated to σ, and denote by $\mathcal{Q}(V)$ the space of all complex symmetric quadratic forms on V. One easily computes that

$$\{Q_{S_1}, Q_{S_2}\} = Q_{-2[S_1,S_2]}, \quad S_1, S_2 \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma), \tag{2.2}$$

which proves the well-known fact that $\Big(\mathcal{Q}(V), \{\cdot,\cdot\}\Big)$ is a Lie algebra, isomorphic to $\mathfrak{sp}(V^{\mathbb{C}}, \sigma)$ under the isomorphism $Q_S \mapsto -2S$.

If $T : U \to W$ is a linear homomorphism of real or complex vector spaces, we shall denote by ${}^tT : W^* \to U^*$ the transposed homomorphisms between the dual spaces W^* and U^* of W and U, respectively, i.e.,

$$({}^tTw^*)(u) = w^*(Tu), \ u \in U, w^* \in W^*.$$

As usually, we shall identify the bi-dual W^{**} with W.

If Q is any bilinear form on V (respectively $V^{\mathbb{C}}$), there is a unique linear map $\mathcal{Q} : V \to V^*$ (respectively $\mathcal{Q} : V^{\mathbb{C}} \to (V^{\mathbb{C}})^*$) such that

$$(\mathcal{Q}v)(w) = Q(v, w), \tag{2.3}$$

and $\mathcal{Q}$ is a linear isomorphism if and only if Q is non-degenerate. In particular, the map $\mathcal{J} : V^{\mathbb{C}} \to (V^{\mathbb{C}})^*$, given by

$$(\mathcal{J}v)(w) = \sigma(w, v), \tag{2.4}$$

is a linear isomorphism, which restricts to a linear isomorphisms from V to V^*, also denoted by $\mathcal{J}$. We have ${}^t\mathcal{J} = -\mathcal{J}$, so that (2.1) can be read as

$$\mathcal{J}S + {}^tS\mathcal{J} = 0. \tag{2.5}$$

Moreover, if the form Q in (2.3) is symmetric, then ${}^t\mathcal{Q} = \mathcal{Q}$ and $\mathcal{Q} = \mathcal{J}S$.

By composition with $\mathcal{J}$, bilinear forms on V can be transported to V^*, e.g. we put

$$\sigma^*(\mathcal{J}v, \mathcal{J}w) := \sigma(v, w), \quad Q^*(\mathcal{J}v, \mathcal{J}w) := Q(v, w).$$

In analogy with (2.3) and (2.4), we obtain maps from V^* to V (respectively from $(V^{\mathbb{C}})^*$ to $V^{\mathbb{C}}$) which satisfy the following identities:

$$\mathcal{J}^* = -\mathcal{J}^{-1},\ \mathcal{Q} = {}^t\mathcal{J}\mathcal{Q}^*\mathcal{J} = -\mathcal{J}\mathcal{Q}^*\mathcal{J},\ S^* = \mathcal{J}S\mathcal{J}^{-1} = -{}^tS. \tag{2.6}$$

The canonical model of a symplectic vector space is $\mathbb{R}^{2d}$, with symplectic form $\sigma(v, w) = {}^tvJw$, where

$$J := \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}.$$

Identifying also the dual space with $\mathbb{R}^{2d}$ (via the canonical inner product on $\mathbb{R}^{2d}$), we have $\mathcal{J}v = Jv$. Moreover, of course $(\mathbb{R}^{2d})^{\mathbb{C}} = \mathbb{C}^{2d}$. As usually, in this case we also write $\mathfrak{sp}(d, \mathbb{C})$ in place of $\mathfrak{sp}(V^{\mathbb{C}}, \sigma)$.

If a general symmetric form Q is given by $Q(v, w) = {}^tvAw$, where A is a symmetric matrix, we have the following formulas:

$$\mathcal{Q}v = Av,\ Sv = -JAv,\ S^*v = -AJv. \tag{2.7}$$

These formulas apply whenever we introduce coordinates on V adapted to a *symplectic basis* of V, i.e., to a basis $X_1, \dots, X_d, Y_1, \dots, Y_d$ such that

$$\sigma(X_j, X_k) = \sigma(Y_j, Y_k) = 0,\ \sigma(X_j, Y_k) = \delta_{jk}$$

for every j, k. Observe that in V^*, the dual of a symplectic basis is symplectic with respect to σ^*. The Heisenberg group $\mathbb{H}_V$ built on V is $V \times \mathbb{R}$, endowed with the product

$$(v, u)(v', u') := (v + v', u + u' + \frac{1}{2}\sigma(v, v')).$$

Its Lie algebra $\mathfrak{h}_V$ is generated by the left-invariant vector fields

$$X_v = \partial_v + \tfrac{1}{2}\sigma(\cdot, v)\partial_u, \quad v \in V.$$

The Lie brackets are given by $[X_v, X_w] = \sigma(v, w)U$, with $U := \partial_u$.

We regard the formal expression (1.8) defining the operator $\mathcal{L}_A$ as an element of the symmetric tensor product $\mathfrak{S}^2(V^{\mathbb{C}})$ (with $V^{\mathbb{C}} = \mathbb{C}^{2d}$), hence as a complex symmetric bilinear form Q^* on $(V^{\mathbb{C}})^*$. With this notation, the Hamilton map S^* of Q^* is

$$S^* = -JA, \tag{2.8}$$

and the Hamilton map of the corresponding form Q on $V^{\mathbb{C}}$ is

$$S = -AJ. \tag{2.9}$$

Since the solvability of $\mathcal{L}_{A,\alpha}$ is closely connected to the spectral properties of the associated Hamilton map S, we shall sometimes also write

$$\mathcal{L}_A =: L_S, \quad \mathcal{L}_{A,\alpha} =: L_{S,\alpha},$$

and shall call S the *Hamiltonian* associated to $\mathcal{L}_A$. We remark (compare (2.2)) that

$$[L_{S_1}, L_{S_2}] = -2\ L_{[S_1,S_2]}U. \tag{2.10}$$

Remark 2.1 Notice that the notation used here to parametrize the operators L slightly deviates from that used in the introduction. Since in this part of my lectures, the separation between the real- and the imaginary part of the complex matrix A will only play a minor role, we have decided to use roman letters for A here. Moreover, what had been called Q_A before is now called Q_S.

If $Q(v,w) = Q_1(v,w) + iQ_2(v,w)$ is a complex-valued symmetric bilinear form on V, we say that Q satisfies the *cone condition* if

$$|Q_2(v)| \leq CQ_1(v) \tag{2.11}$$

for some $C > 0$ and every $v \in V$. If $Q = Q_S$, then this is equivalent to the cone condition (1.5) for the associated matrix A (possibly with a different constant C).

Denote by

$$\Omega = \Omega_Q := \mathrm{conv}\,\{Q(v) : v \in V\}$$

the convex hull of the set $\{Q(v) : v \in V\}$ in the complex plane. Clearly Ω is contained in the proper angle $\{\zeta \in \mathbb{C} : |\mathrm{Im}\,\zeta| \leq C\mathrm{Re}\,\zeta\}$.

The cone condition is obviously satisfied if Q_1 is positive definite. It is also evident from (2.6) that Q and Q^* satisfy the cone condition for the same values of C.

As we shall see in Section 3, the cone condition implies that the eigenvalues of S are in $i\Omega \cup -i\Omega$. Also, the non-zero eigenvalues come in pairs $\pm\lambda$ with the same multiplicity.

Theorem 2.2 *Assume that* $\mathrm{Re}\,A > 0$, *and let* $\lambda_1, \ldots, \lambda_d$ *be the eigenvalues of* $S = -AJ$ *contained in* $i\Omega$, *counted with multiplicities. Then* $\mathcal{L}_{A,\alpha}$ *is locally solvable if and only if*

$$\alpha \notin \mathcal{E} := \left\{ \pm i \sum_{j=1}^{d} (2k_j + 1)\lambda_j : k_j \in \mathbb{N} \right\}.$$

If $\mathrm{Re}\,A$ is only semi-definite and the cone condition is satisfied, then A itself is clearly degenerate, so that 0 is an eigenvalue of $S = -AJ$. Let $\lambda_1, \ldots, \lambda_m$ (with $m < d$) be the non-zero eigenvalues of S contained in $i\Omega$, counted with multiplicities.

We recall that a subspace V' of V (or a complex subspace of $V^{\mathbb{C}}$) is called *symplectic* if the restriction of σ to $V' \times V'$ is non-degenerate.

Theorem 2.3 *Assume that A satisfies the cone condition (so that in particular* $\operatorname{Re} A \geq 0$*), and that A is degenerate. If* $\ker S \cap V$ *is symplectic (or, equivalently, if* $\ker \operatorname{Re} S$ *is symplectic in V), then $\mathcal{L}_{A,\alpha}$ is locally solvable if and only if*

$$\alpha \notin \mathcal{E} := \left\{ \pm i \sum_{j=1}^{m} (2k_j+1)\lambda_j : k_j \in \mathbb{N} \right\} .$$

If $\ker S \cap V$ *is not symplectic, then $\mathcal{L}_{A,\alpha}$ is locally solvable for every α.*

For the sake of completeness, let me finally state a theorem which gives the answer in the rather opposite case where $C = 0$. For further results complementing these theorems see also [19], Theorem 2.8.

Theorem 2.4 ([19]) *Let $S = S_1 + iS_2 \in \mathfrak{sp}(d, \mathbb{C})$, and assume that $[S_1, S_2] = 0$. Then L is locally solvable, unless both S_1 and S_2 have purely imaginary spectrum and are semisimple. In the latter case, one can find a suitable automorphism of $\mathbb{H}_d$ leaving the center fixed such that, in the new coordinates given by this automorphism, L takes on the form*

$$L = \sum_{j=1}^{n_1} i\lambda_j (X_j^2 + Y_j^2) + i\alpha U, \tag{2.12}$$

with $n_1 \leq d$ and $\lambda_1, \ldots, \lambda_{n_1} \in \mathbb{C} \setminus \{0\}$. Then L is locally solvable if and only if there are constants $C, N > 0$, such that

$$\left| i \sum_{j=1}^{n_1} (2k_j+1)\lambda_j \pm \alpha \right| \geq C \ (1 + k_1 + \cdots + k_{n_1})^{-N} \tag{2.13}$$

for all $k_1, \ldots, k_{n_1} \in \mathbb{N}$.

In particular cases, condition (2.13) can be considered as a problem of diophantine approximation, see [32], Proposition 3.9.

Observe that in view of (2.2), $\{Q_{S_1}, Q_{S_2}\} = 0$ in this theorem, whereas we had been assuming in Corollary 1.4 that Q_{S_1}, Q_{S_2} and $\{Q_{S_1}, Q_{S_2}\}$ are linear independent. An important special case of Theorem 2.4 is the case where $S_1 = 0$, (or $S_2 = 0$; these two cases are of course equivalent), i.e., the case of real coefficient operators. This case had been dealt with in a complete way in [33], [32], and these results have been extended to arbitrary two-step nilpotent Lie groups in [28]. For yet further results, we refer to the References.

The proofs of the previous theorems will be based on the following formal identity for "the inverse" to $\mathcal{L}_{A,\alpha}$:

$$(\mathcal{L}_{A,\alpha})^{-1} = -\int_0^\infty e^{t\mathcal{L}_{A,\alpha}} \, dt.$$

This formula is not completely unreasonable, since $\operatorname{Re} A \geq 0$, which just means that the operator $\mathcal{L}_{A,\alpha}$ is dissipative on $L^2(\mathbb{H}_d)$, so that it generates a contraction semigroup $\{e^{t\mathcal{L}_{A,\alpha}}\}_{t\geq 0}$ on $L^2(\mathbb{H}_d)$, by the Phillips-Lumer theorem. However, the operators $e^{t\mathcal{L}_{A,\alpha}}$ will in general have norm 1, so that the integral above will not make sense on $L^2(\mathbb{H}_d)$. Indeed, it turns out that a somewhat better approach is based on the related formal identity

$$
\begin{aligned}
(\mathcal{L}_{A,\alpha})^{-1} &= -|U|^{-1}\int_0^\infty e^{t|U|^{-1}\mathcal{L}_{A,\alpha}}\,dt \\
&= -|U|^{-1}\int_0^\infty e^{t|U|^{-1}\mathcal{L}_A+it\alpha U/|U|}\,dt.
\end{aligned}
\tag{2.14}
$$

It turns out that, after suitable modifications, the integral expression in (2.14) can be given meaning, in the distributional sense.

A first step will consist in taking the partial Fourier transform with respect to the central variable u, say at the Fourier parameter μ. This will turn the operator $\mathcal{L}_A$ into the so-called μ-twisted operator $\mathcal{L}_A^\mu$.

In a second step, we shall have to identify the one-parameter semigroups generated by the $\mathcal{L}_A^\mu$. This step involves a careful discussion of the spectral properties of S: location of its eigenvalues in the complex plane, symplectic properties of its generalized eigenspaces (as subspaces of $\mathbb{C}^{2n}$ with the symplectic form induced by J). In this analysis, which will be carried out in the next Section, we shall follow [8], [10],[21] and [19] (see also [36]).

3 On the algebraic structure of S

Complex conjugation in $V^{\mathbb{C}}$ is meant with respect to the real form V, i.e., for $z = v+iw \in V^{\mathbb{C}}$, we set $\bar{z} = v - iw$.

Recall that the *radical* $\operatorname{rad} B$ of a bilinear form B on a (real or complex) space V is the space of the $v \in V$ such that $B(v, v') = 0$ for all $v' \in V$.

Also recall that a subspace V' of a (real or complex) symplectic space is called *isotropic* if the restriction of the symplectic form to $V' \times V'$ is trivial. A maximal isotropic subspace of V is called *Lagrangian.* If V' is Lagrangian, then $\dim V' = \frac{1}{2}\dim V$.

The following structure theory for elements $S \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma)$ will be important. Let $\operatorname{spec} S \subset \mathbb{C}$ denote the spectrum of S, and for $\lambda \in \operatorname{spec} S$ by V_λ the generalized eigenspace of S belonging to the eigenvalue λ.

Lemma 3.1 *If $\lambda+\mu \neq 0$, then $\sigma(V_\lambda, V_\mu) = 0$. Also, if $\lambda \neq 0$, then $\dim V_\lambda = \dim V_{-\lambda}$ and $V_\lambda \oplus V_{-\lambda}$ is a symplectic subspace in which each summand is Lagrangian.*

Proof. The map $S + \mu I$ is a bijection of V_λ onto itself, so every $z \in V_\lambda$ can be written as $(S + \mu I)^n z_n$, for arbitrary n, with $z_n \in V_\lambda$.

If we now take $z' \in V_\mu$, we have, for n large,

$$\sigma(z, z') = (-1)^n \sigma\big(z_n, (S - \mu I)^n z'\big) = 0\ .$$

Assume that $v \in V_{-\lambda}$ is such that $\sigma(v, V_\lambda) = 0$, where $\lambda \neq 0$. Then $v = 0$, since σ would be degenerate otherwise. Hence $V_\lambda \oplus V_{-\lambda}$ is symplectic. As both V_λ and $V_{-\lambda}$ are isotropic, they must necessarily be Lagrangian. As such, they have the same dimension.

Q.E.D.

The fact that $\dim V_\lambda = \dim V_{-\lambda}$ also follows from the fact that S and $-\,{}^tS$ are conjugate under $\mathcal{J}$.

In particular, V_λ and $V_{-\lambda}$ are isotropic subspaces with respect to the symplectic form σ, and $V_\lambda \oplus V_{-\lambda}$ is a symplectic subspace of $V^{\mathbb{C}}$, if $\lambda \neq 0$, while V_0 is symplectic too. We thus obtain a decomposition of $V^{\mathbb{C}}$ as a direct sum of symplectic subspaces which are σ-orthogonal:

$$V^{\mathbb{C}} = V_0 \oplus \sum_{\lambda\neq 0}^{\oplus} V_\lambda \oplus V_{-\lambda}. \tag{3.1}$$

Here, the summation takes place over a suitable subset of $\operatorname{spec} S$. Notice that the decomposition above is also orthogonal with respect to the symmetric form $Q(v,w) = \sigma(v, Sw)$, since the spaces V_λ are S-invariant. (3.1) induces an orthogonal decomposition

$$V^{\mathbb{C}} = V_r \oplus V_i, \tag{3.2}$$

where $V_r := \sum_{\lambda\in\mathbb{R}\cap\operatorname{spec} S} V_\lambda$ and $V_i := \sum_{\mu\in(\mathbb{C}\setminus\mathbb{R})\cap\operatorname{spec} S} V_\mu$. Correspondingly, S decomposes as

$$S = S_r + S_i, \tag{3.3}$$

where we have put $S_r(u+w) := S(u)$, $S_i(u+w) := S(w)$, if $u \in V_r$ and $w \in V_i$. Then also S_r and S_i are in $\mathfrak{sp}(V^{\mathbb{C}}, \sigma)$, and S_r respectively S_i corresponds to the Jordan blocks of S associated with real eigenvalues respectively non-real eigenvalues.

Lemma 3.2 *Assume that Q satisfies the cone condition. Denote by S_1 and S_2 the real and imaginary part of the Hamilton map S of Q, respectively, and let $K_1 = \operatorname{rad} Q_1 = \ker S_1 \subset V$. Then, as subspaces of $V^{\mathbb{C}}$, $\operatorname{rad} Q = \ker S = K_1^{\mathbb{C}}$. Also, $z \in \operatorname{rad} Q$ if and only if $Q(z,\bar z) = 0$.*

Proof. Let $z = v + iw \in \operatorname{rad} Q$. Then

$$Q(z,\bar z) = Q(v) + Q(w) = 0\ .$$

As Q_1 is positive semi-definite, this implies that $Q_1(v) = Q_1(w) = 0$, hence $v, w \in K_1$. This shows that $\operatorname{rad} Q \subset K_1^{\mathbb{C}}$.

To prove the converse, take $v \in K_1$, $w \in V$ and $t \in \mathbb{R}$. By the cone condition, $Q_2(v) = 0$ and

$$|2tQ_2(v,w) + Q_2(w)| = |Q_2(tv+w)| \leq CQ_1(tv+w) = CQ_1(w) \quad \forall t \in \mathbb{R}.$$

This implies that $Q_2(v,w) = 0$, i.e., $v \in \operatorname{rad} Q_2 = \ker S_2$. The rest of the proof is trivial.

Q.E.D.

Lemma 3.3 *If λ is an eigenvalue of S, then $\lambda \in (i\Omega) \cup (-i\Omega)$, where $\Omega = \Omega_Q$.*

Proof. If $\lambda = 0$, there is nothing to say. Let $\lambda \neq 0$ be an eigenvalue of S and let $z = v + iw \in V^{\mathbb{C}}$ be an associated eigenvector. Then

$$Q(z, \bar{z}) = Q(v) + Q(w) \in \Omega \ ,$$

and, by Lemma 3.2, $Q(z, \bar{z}) \neq 0$.

On the other hand,

$$\begin{aligned} Q(z, \bar{z}) &= -\sigma(Sz, \bar{z}) \\ &= -\lambda\sigma(v + iw, v - iw) \\ &= 2i\lambda\sigma(v, w) \ . \end{aligned}$$

This implies that $\lambda \in \pm i\Omega$, depending on the signum of $\sigma(v, w)$.

Q.E.D.

Lemma 3.4 *Assume that Q_1 is positive definite, and let*

$$V^{\pm} := \sum_{\lambda \in \pm i\Omega} V_{\lambda}.$$

Then V^+ and V^- are S-invariant (complex) Lagrangian subspaces of $V^{\mathbb{C}}$,

$$V^+ \oplus V^- = V^{\mathbb{C}}, \qquad \text{and} \quad V^{\pm} \cap V = \{0\}.$$

Furthermore, the Hermitean form

$$B(z, z') := -i\sigma(z, \bar{z}')$$

is positive definite on V^+ and negative definite on V^-.

In particular, if $Q = Q_1$ is real, we can find a symplectic basis $X_1, \dots, X_d, Y_1, \dots, Y_d$ of V and $\rho_j > 0, j = 1, \dots, d$, such that in the coordinates $x_1, \dots, x_d, y_1, \dots, y_d$ associated to this basis, Q takes on the normal form

$$Q = \sum_{j=1}^{d} \rho_j (x_j^2 + y_j^2). \tag{3.4}$$

Proof. That $V^{\pm}$ are Lagrangian subspaces follows immediately from Lemma 3.1, since Ω is a proper cone, and since clearly $V^+ \oplus V^- = V^{\mathbb{C}}$ (see Lemma 3.3).

Next, if $v \in V^{\pm} \cap V$, then $Q(v) = \sigma(v, Sv) = 0$, since Sv is also in $V^{\pm}$. Hence $v = 0$.

In order to prove the statement concerning the Hermitean form B, assume first that $Q_2 = 0$. Then $S = S_1$ is real,and by Lemma 3.3, its eigenvalues are purely imaginary.

Consider an eigenvalue $i\nu$ with associated eigenvector $z = v + iw$, i.e., $Sz = i\nu z$; then, as S is real, $S\bar{z} = -i\nu\bar{z}$. Let V' be the orthogonal complement in V of $\text{span}_{\mathbb{R}}\{v, w\}$, relative to the positive definite form $Q = Q_1$. If $u \in V'$, then

$$\begin{aligned} Q(Su, v \pm iw) &= \sigma\big(Su, S(v \pm iw)\big) \\ &= \pm i\nu\sigma(Su, v \pm iw) \\ &= \mp i\nu Q(u, v \pm iw) = 0 \ . \end{aligned}$$

Hence S maps V' into itself, which implies that S is diagonalizable on $V^{\mathbb{C}}$.

So $V^\pm$ is the linear span of the eigenvectors of S relative to the eigenvalues $\pm i\nu$ with $\nu > 0$. In order to see that B is positive (resp. negative) definite on V^+ (resp. V^-), by Lemma 3.1 it is sufficient to test it on eigenvectors. But, if $Sz = i\nu z$ and $z = v + iw$, then we have

$$\begin{aligned} \nu B(z,z) &= -i\nu\sigma(z,\bar z) \\ &= -\sigma(Sz,\bar z) \\ &= Q(\bar z, z) \\ &= Q(v) + Q(w) > 0 \ . \end{aligned}$$

Notice also that $-i\nu\sigma(z,\bar z) = -2\nu\sigma(v,w)$, so that the above inequality further implies $\sigma(v,w) < 0$. After scaling, we may thus assume that $\sigma(v,w) = -1$. If we then put $\rho_1 := \nu$, $X_1 := v$ and $Y_1 := -w$, we obtain $S(X_1 - iY_1) = i\rho_1(X_1 - iY_1)$, i.e., $SX_1 = \rho_1 Y_1, SY_1 = -\rho_1 X_1$, so that X_1, Y_1, forms a symplectic basis of an S-invariant subspace of V. And, on this space, we have

$$Q(x_1X_1 + y_1Y_1) = \sigma\Big((x_1X_1 + y_1Y_1, S((x_1X_1 + y_1Y_1)\Big) = \rho_1(x_1^2 + y_1^2).$$

Applying the same reasoning to V', and proceeding by induction, we obtain (3.4).

Consider finally the case $Q_2 \neq 0$, and assume, by contradiction, that B is not positive definite on V^+. As the forms $Q_1 + itQ_2$ and the corresponding Lagrangian subspaces V_t^+ depend continuously on t, there will be $t_0 > 0$ such that B is positive semi-definite and degenerate on $V_{t_0}^+$. Without loss of generality, we can assume that $t_0 = 1$.

Hence there is some $z \in V^+ \setminus \{0\}$ such that $\sigma(z', \bar z) = iB(z', z) = 0$ for every $z' \in V^+$. As V^+ is Lagrangian, this implies that also $\bar z \in V^+$. But this is in contradiction with the fact that V^+ does not contain real vectors other than zero.

The same argument applies to V^-.

Q.E.D.

Recall that if V' is a subspace of a symplectic space V, then its *σ-orthogonal* $V'^\perp$ consists of the elements $v \in V$ such that $\sigma(v, v') = 0$ for every $v' \in V'$. The subspace V' is symplectic if and only if $V' \cap V'^\perp$ is trivial (in which case $V = V' \oplus V'^\perp$), it is isotropic if and only if $V' \subseteq V'^\perp$, and it is Lagrangian if and only if $V' = V'^\perp$. Any subspace V' decomposes as the direct sum of $V' \cap V'^\perp$ (which is isotropic) and a symplectic subspace. In fact, any complementary subspace of $V' \cap V'^\perp$ in V' is symplectic.

Assuming now that Q satisfies the cone condition, but Q_1 is only positive semi-definite, we discuss the structure of the generalized eigenspace V_0 of S, relative to the eigenvalue 0. It follows from Lemma 3.1 that V_0 is a symplectic subspace of $V^{\mathbb{C}}$. In the following, σ-orthogonality is referred to V_0 as the ambient space.

It is important at this point to remark that the space V_0 will in general not be invariant under complex conjugation, so that we cannot reduce considerations to $V_0 \cap V$ in place of V!

Let $K := \ker S \subseteq V_0$, and decompose K as the σ-orthogonal direct sum

$$K = K_0 \oplus W_0$$

of the isotropic subspace $K_0 := K \cap K^\perp$ and a symplectic subspace W_0. Let $W_1 := W_0^\perp$. Then W_1 is also symplectic, and $K^\perp \subset W_1$, since $W_0 \subset K$. Moreover,

$$V_0 = W_1 \oplus W_0.$$

Observe that, since $K = \overline{K}$ (by Lemma 3.2), also $K^\perp \cap K$ is complex conjugate, i.e. $\overline{K_0} = K_0$, and so we can choose W_0 so that $W_0 = \overline{W_0}$. In general, V_0 is not self-conjugate, so we cannot assume that also W_1 is self-conjugate.

The following results complete the picture of the Jordan structure of S.

Lemma 3.5 *Assume that Q satisfies the cone condition. Then $K^\perp = K_0$, and K_0 is a Lagrangian subspace of W_1. Moreover, $SV_0 = SW_1 = K_0$, so that $S^2 = 0$ on V_0, so that $S_r^2 = 0$. Finally, $S = 0$ on V_0, i.e., $S_r = 0$, if and only if $K = \ker S$ is symplectic.*

Proof. Notice first that, since $S|_{V_0} \in \mathfrak{sp}(V_0, \sigma)$, the σ-orthogonal of an S-invariant subspace of V_0 is also S-invariant. This implies that

$$SV_0 = K^\perp,$$

because clearly $K = (SV_0)^\perp$. Hence all the spaces K_0, W_0 and W_1 are S-invariant.

So all conclusions will follow if we prove that $K^\perp = K_0$ (the fact that K_0 is Lagrangian in W_1 then follows from $(K^\perp)^\perp \cap W_1 = K \cap W_1 \subset K_0$).

To this end, consider the quotient space $K^\perp/K_0$. As $K^\perp$ and K_0 are S-invariant, S projects to a linear map $\tilde{S}$ of $K^\perp/K_0$ into itself. The form σ also projects to a symplectic form $\tilde{\sigma}$ on $K^\perp/K_0$, since $K_0 = K^\perp \cap (K^\perp)^\perp$ is just the radical of σ in $K^\perp$. Also, $\tilde{S} \in \mathfrak{sp}(K^\perp/K_0, \tilde{\sigma})$.

Consider therefore the symmetric bilinear form $\tilde{Q}(\xi, \eta) := \tilde{\sigma}(\xi, \tilde{S}\eta)$ on the complex space $K^\perp/K_0$. If $\xi = z + K_0$ and $\eta = w + K_0$, we have

$$\tilde{Q}(\xi, \eta) = \sigma(z, Sw) = Q(z, w) \ .$$

We show by means of Lemma 3.2 that $\tilde{Q}$ is non-degenerate. A technical problem is that $K^\perp$ may not be self-conjugate. Therefore, we consider $K^\perp$ is a subspace of the σ-orthogonal complement $U = K^\perp + V_i$ of K in $V^\mathbb{C}$, which is self-conjugate.

Let $\xi \in \operatorname{rad} \tilde{Q}$. Then $Q(z, w) = \sigma(Sz, w) = 0$ for every $w \in K^\perp$, and the same is true for every $w \in V_i$ (since V_0 and V_i are σ-orthogonal), hence for every $w \in U$. Since $\overline{z} \in U$, we thus find that $Q(z, \overline{z}) = 0$, hence $Sz = 0$, by Lemma 3.2, and thus $\xi = 0$.

Since $\tilde{Q}$ is non-degenerate, $\tilde{S}$ is non-degenerate too. But $\tilde{S}$ is nilpotent, and hence $K^\perp = K_0$.

Q.E.D.

Observe finally that $\overline{SV_0} \subset \overline{K} = K \subset V_0$. It is not difficult to show (see Lemma 3.3 in [19]) that this implies $\operatorname{Re} Q_{S_r} \geq 0$, so that under the hypotheses of Lemma 3.5, we can apply the following result from [19] (Proposition 7.2) to S_r.

Proposition 3.6 *Assume that $S^2 = 0$ and $\operatorname{Re} Q_S \geq 0$. Then we can select a symplectic basis $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ of V such that $L_S = \sum_{j,k=1}^m b_{jk} Y_j Y_k$ for some $m \leq n$ and $b_{jk} \in \mathbb{C}$. Since the vector fields Y_j all commute among themselves, we see in particular that L_S is a constant coefficient operator, when written in suitable coordinates.*

The proof is rather involved. If we assume, however, that V_0 in Lemma 3.5 is self-conjugate, we can give a short argument for S_r : If $\overline{V_0} = V_0$, we can choose a self-conjugate Lagrangian subspace K_1 of W_1 complementary to K_0, i.e. $W_1 = K_1 \oplus K_0$. Then $SK_1 \subset K_0$. Moreover, if we put $W_2 := W_0 \oplus V_i$, then W_2 is a self-conjugate symplectic subspace complementary to W_1, which is annihilated by S_r. We decompose W_2 as the direct sum $W_2 = H_1 \oplus H_0$ of self-conjugate Lagrangian subspaces, so that

$$V^{\mathbb{C}} = (K_1 \oplus K_0) \oplus (H_1 \oplus H_0).$$

In suitable blocks of symplectic coordinates adapted to this decomposition, we then can represent S_r and J by block matrices of the form

$$S_r = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad J = \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \end{array}\right),$$

hence Q_{S_r} by the symmetric matrix $A_r := S_r J = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

This shows that L_{S_r} assumes the following form in these coordinates:

$$L_{S_r} = \sum_{j,k=1}^{m} b_{jk} Y_j Y_k, \tag{3.5}$$

with $b_{jk} \in \mathbb{C}$.

4 Non-solvability.

In this section we sketch how to prove the negative part of Theorem 2.2.

We shall make use of the *Schrödinger representations* of the Heisenberg group in the following form: let $\{X_j, Y_j\}_{j=1,\ldots,d}$ be a real symplectic basis of V, and denote by the same letters also the corresponding elements of the Lie algebra $\mathfrak{h}_V$, regarded as left-invariant vector fields on $\mathbb{H}_V$.

Given $\mu \in \mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, we denote by π_μ the irreducible unitary representation of $\mathbb{H}_V$ on $L^2(\mathbb{R}^d)$ such that

$$d\pi_\mu(X_j) = \frac{\partial}{\partial \xi_j}\,, \quad d\pi_\mu(Y_j) = 2\pi i \mu \xi_j, \quad j = 1, \ldots, d.$$

Notice that then

$$d\pi_\mu(U) = 2\pi i \mu.$$

Different choices of the symplectic basis produce equivalent representations for the same value of μ. If $f \in L^1(\mathbb{H}_V)$, we shall define

$$\hat{f}(\pi_\mu) := \pi_\mu(\check{f}) = \int_{\mathbb{H}_V} f(g) \pi_\mu(g)^* \, dg,$$

and correspondingly also $\hat{P}(\pi_\mu) := \widehat{P\delta_0}(\pi_\mu)$ for any left-invariant differential operator $P \in \mathfrak{U}(\mathfrak{h}_V)$ (compare Appendix 8.) Notice that then

$$\widehat{X_j}(\pi_\mu) = -\frac{\partial}{\partial \xi_j}\,, \quad \widehat{Y_j}(\pi_\mu) = -2\pi i \mu \xi_j, \quad j = 1, \ldots, d,$$

and $\widehat{U}(\pi_\mu) = -2\pi i\mu$.

We shall make use of the following representation-theoretic condition for local solvability, due to Corwin and Rothschild:

Theorem 4.1 *If L is a homogeneous left-invariant differential operator on $\mathbb{H}_d$, and if $\widehat{{}^tL}(\pi_{\mu_0})$ annihilates some non-trivial Schwartz function for some $\mu_0 \neq 0$, then L is not locally solvable.*

An analogue of this theorem holds indeed true on arbitrary homogeneous Lie groups (see [3]). A wide extension based on a simplified proof has been given in [31]. For a direct proof in case of the Heisenberg group, see Appendix 8 (and also my ICMS-Lecture notes [25] for more details and further information).

Theorem 4.2 *Assume that* $\operatorname{Re} A > 0$*, and let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of S contained in $i\Omega$. If $\alpha = \pm i \sum_{j=1}^d (2k_j + 1)\lambda_j$, with $k_j \in \mathbb{N}$, then $\mathcal{L}_{A,\alpha}$ is not locally solvable.*

Proof in an important special case. Let us assume for simplicity that $A > 0$. Then, by Lemma 3.4 (more precisely (3.4)), one can find a suitable symplectic automorphism of $\mathbb{H}_d$ such that, in the new coordinates given by this automorphism, $L := \mathcal{L}_{A,\alpha}$ takes on the form

$$L = \sum_{j=1}^d \rho_j (X_j^2 + Y_j^2) + i\alpha U, \tag{4.1}$$

with $\rho_1, \ldots, \rho_d > 0$. One easily computes that the eigenvalues of the associated Hamilton map $S = -AJ$ contained in $i\Omega$ are then given by

$$\lambda_j = i\rho_j.$$

For the image of ${}^tL$ under the Schrödinger representation $\pi_{\pm 1}$ we then obtain

$$\widehat{{}^tL}(\pi_{(\pm\frac{1}{2\pi})}) = \sum_{j=1}^d \rho_j \left(\partial_{\xi_j}^2 - \xi_j^2\right) \mp \alpha.$$

Now, $\partial_x^2 - x^2$ is just the *Hermite operator*, whose eigenfunctions are of the form

$$h_n(x) := H_n(x) e^{-x^2/2},$$

where H_n is a polynomial of degree n, namely the n-th order *Hermite polynomial*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad n = 0, 1, 2, \ldots.$$

The associated eigenvalue is given by

$$(\partial_x^2 - x^2) h_n = -(2n+1) h_n.$$

Consequently, if we put $h_k(\xi) := h_{k_1}(\xi_1) \cdots h_{k_d}(\xi_d)$, for $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, then $h_k \in \mathcal{S}(\mathbb{R}^d)$, and

$$\widehat{{}^tL}(\pi_{(\pm\frac{1}{2\pi})}) h_k = \left(-\sum_{j=1}^d \rho_j (2k_j + 1) \mp \alpha \right) h_k = 0, \tag{4.2}$$

if $\alpha = \pm i \sum_{j=1}^{d}(2k_j+1)\lambda_j$. This proves the claim.

The proof in the general case where $\operatorname{Re} A > 0$, as well as that of Theorem 2.3, is based on similar ideas (and the general case of Lemma 3.4), but more involved.

Q.E.D.

5 Twisted convolution and Gaussians generated by the operators $\tilde{L}_S^\mu$.

Assume that $S \in \mathfrak{sp}(d, \mathbb{C})$ is such that $\operatorname{Re} Q_S \geq 0$. It is our main goal in this section to determine the semigroup generated by the operator $|U|^{-1}L_S$. Our results are directly related to those in [8] by means of the Weyl transform. Instead of transferring the result from [8], Theorem 4.3, by means of the inverse Weyl transform, we prefer, however, to give a direct argument.

We shall work in the setting of an arbitrary real symplectic vector space (V, σ) of dimension $2d$. Given two suitable functions φ and ψ on V and $\mu \in \mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, we define the *$\mu$-twisted convolution* of φ and ψ as

$$\varphi \times_\mu \psi(v) := \int_V \varphi(v - v')\psi(v')e^{-\pi i \mu \sigma(v,v')}\, dv',$$

where dv' stands for the volume form $\sigma^{\wedge(d)}$.

If f is a suitable function on $\mathbb{H}_V$, we denote by

$$f^\mu(v) := \int_{-\infty}^{\infty} f(v,u)e^{-2\pi i \mu u} du$$

the partial Fourier transform of f in the central variable u at $\mu \in \mathbb{R}$.

For $\mu \neq 0$, we have

$$(f \star g)^\mu(v) = f^\mu \times_\mu g^\mu(v). \tag{5.1}$$

Moreover, if L is any left-invariant differential operator on $\mathbb{H}_V$, then there exists a differential operator $\tilde{L}^\mu$ on V such that

$$(Lf)^\mu = \tilde{L}^\mu f^\mu. \tag{5.2}$$

Explicitly, if $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ are coordinates on V associated with a symplectic basis $\{X_j, Y_j\}$, then

$$\begin{aligned} \tilde{X}_j^\mu \varphi &= (\partial_{x_j} - \pi i \mu y_j)\varphi = \varphi \times_\mu (\partial_{x_j}\delta_0), \\ \tilde{Y}_j^\mu \varphi &= (\partial_{y_j} + \pi i \mu x_j)\varphi = \varphi \times_\mu (\partial_{y_j}\delta_0) \\ \tilde{U}^\mu \varphi &= 2\pi i \mu \varphi, \end{aligned} \tag{5.3}$$

and consequently, if $Q^* = Q_S$,

$$\tilde{L}_S^\mu = \tilde{\mathcal{L}}_A^\mu = \sum_{j,k} a_{jk} \tilde{V}_j^\mu \tilde{V}_k^\mu = f^\mu \times_\mu \big(Q^*(\partial)\delta_0\big) \tag{5.4}$$

is obtained from $\mathcal{L}_A$ by replacing each V_j in (1.8) by $\tilde{V}_j^\mu$. On V, we define the *(adapted) Fourier transform* by

$$\hat{f}(w) := \int_V f(v) e^{-2\pi i \sigma(w,v)} dv, \quad w \in V.$$

Observe that then $\hat{\hat{f}} = f$ and $\int fg = \int \hat{f}\hat{g}$, for suitable functions f and g on V.

Consider an arbitrary quadratic form Q on $V^{\mathbb{C}}$, with associated Hamilton map $S \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma)$. Once we have fixed a symplectic basis $\{X_j, Y_j\}$ of V, we may identify S with a $2d \times 2d$-matrix. If $\pm\lambda_1, \ldots, \pm\lambda_m$ are the non-zero eigenvalues of S, then $\det(\cos S) = \prod_{j=1}^m \cos^2 \lambda_j$, so that the square root

$$\sqrt{\det(\cos S)} := \prod_{j=1}^m \cos \lambda_j \tag{5.5}$$

is well-defined. Observe that this expression is invariant under all permutations of the roots of the characteristic polynomial $\det(S - \lambda I)$, hence an entire function of the elementary symmetric functions, which are polynomials in (the coefficients of) S.

Thus, as already observed in [8], $\sqrt{\det(\cos S)}$, given by (5.5), is a well-defined analytic function of $S \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma)$.

We shall always consider $T_S := \tilde{L}_S^\mu$ as the maximal operator defined by the differential operator (5.4) on $L^2(V)$; its domain $\mathrm{dom}(T_S)$ consists of all functions $f \in L^2(V)$ such that $\tilde{L}_S^\mu f$, defined in the distributional sense, is in $L^2(V)$. One can show that $\tilde{L}_S^\mu$ is a closed operator, which is the closure of its restriction to $\mathcal{S}(V)$.

Moreover, its adjoint operator is obviously given by $(\tilde{L}_S^\mu)^* = \tilde{L}_{\overline{S}}^\mu$.

Lemma 5.1 *Assume that* $\mathrm{Re}\, Q_S \geq 0$. *Then the operator* $\tilde{\mathcal{L}}_A^\mu = \tilde{L}_S^\mu$ *and its adjoint are dissipative, so that it generates, by the Phillips-Lumer theorem, a contraction semigroup* $\{\exp(t\tilde{L}_S^\mu)\}_{t \geq 0}$ *on* $L^2(V)$.

Proof. Clearly, for $f \in \mathcal{S}(V)$, we have

$$\begin{aligned}
\mathrm{Re}\,(\tilde{\mathcal{L}}_A^\mu f, f) &= -\mathrm{Re} \sum_{j,k} a_{jk} (\tilde{V}_j^\mu f, \tilde{V}_k^\mu f) \\
&= -\mathrm{Re} \int_V a_{jk} g_j(v) \overline{g_k(v)}\, dv \\
&= -\int_V \mathrm{Re}\, Q_S(g_j(v), \overline{g_k(v)})\, dv \leq 0,
\end{aligned}$$

if we set $g_j := \tilde{V}_j^\mu f$. This inequality remains true for arbitrary $f \in \mathrm{dom}(\tilde{L}_A^\mu)$, since $\mathcal{S}(V)$ is a core for $\tilde{L}_S^\mu$, hence $\tilde{L}_S^\mu$ is dissipative, and the same is true of the adjoint operator $\tilde{L}_{\overline{S}}^\mu$, since $\mathrm{Re}\, Q_S = \mathrm{Re}\, Q_{\overline{S}}$.

Q.E.D.

For the case where $\mathrm{Re}\, Q_S > 0$, an explicit formula for the semigroup $\exp(\frac{t}{|\mu|}\tilde{L}_S^\mu)$ has been given in [21], Theorem 5.2 (for the correct determination of the square roots of the determinants arizing in the subsequent formulas, see [21]).

Theorem 5.2 *If* $\operatorname{Re} Q_S > 0$*, then for* $f \in L^2(V)$

$$\exp(\frac{t}{|\mu|}\tilde{L}_S^\mu)f = f \times_\mu \Gamma_{t,S}^\mu, \quad t \geq 0, \tag{5.6}$$

where, for $t > 0$*,* $\Gamma_{t,S}^\mu$ *is a Schwartz function given by*

$$\Gamma_{t,S}^\mu(v) = \frac{|\mu|^d}{\big(\det(2\sin 2\pi tS)\big)^{\frac{1}{2}}} e^{\frac{\pi}{2}|\mu|\sigma\big(v,(\cot 2\pi tS)v\big)} . \tag{5.7}$$

Moreover, the Fourier transform of $\Gamma_{t,S}^\mu$ *is given by*

$$\widehat{\Gamma_{t,S}^\mu}(w) = \frac{1}{\sqrt{\det(\cos 2\pi tS)}} e^{-\frac{2\pi}{|\mu|}\sigma(w,\tan(2\pi tS)w)}. \tag{5.8}$$

We say that $S \in \mathfrak{sp}_+(V^{\mathbb{C}}, \sigma)$ if Q_S has positive definite real part. To every $S \in \mathfrak{sp}_+(V^{\mathbb{C}}, \sigma)$ we associate the Gaussian function $e^{-\pi Q_S} \in \mathcal{S}(V)$. Observe that Theorem 5.2 gives an explicit expression for the one-parameter semigroups contained in Howe's *oscillator semigroup* [11] (which, by definition, is just the semigroup of all Gaussians $e^{-\pi Q}$ with $\operatorname{Re} Q > 0$ and twisted convolution as product).

The proof is based on the following lemma on the twisted convolution of Gaussian functions which can easily be verified by direct computations (compare also formulas (8.1) - (8.3) in [11]). Notice the formal resemblance of these formulas with the addition theorem

$$\cot(\alpha+\beta) + i = \frac{(\cot\alpha + i)(\cot\beta + i)}{\cot\alpha + \cot\beta}$$

for the cotangent (if $\mu = 2$), which strongly suggest the exponent of the exponential factor in (5.7).

Lemma 5.3 *Let* $S_1, S_2 \in \mathfrak{sp}_+(V^{\mathbb{C}}, \sigma)$*. Then*

$$e^{-\pi Q_{S_1}} \times_\mu e^{-\pi Q_{S_2}} = \det(S_1 + S_2)^{-\frac{1}{2}} e^{-\pi Q_{S_3}} , \tag{5.9}$$

and

$$S_3 + \frac{i}{2}\mu = \left(S_2 + \frac{i}{2}\mu\right)(S_1 + S_2)^{-1}\left(S_1 + \frac{i}{2}\mu\right) , \tag{5.10}$$

or, equivalently,

$$S_3 - \frac{i}{2}\mu = \left(S_1 - \frac{i}{2}\mu\right)(S_1 + S_2)^{-1}\left(S_2 - \frac{i}{2}\mu\right) , \tag{5.11}$$

In particular, since the left-hand side of (5.9) *is in* $\mathcal{S}(V)$*, we have* $S_3 \in \mathfrak{sp}_+(V^{\mathbb{C}}, \sigma)$*.*

Proof of Theorem 5.2. Choosing a symplectic basis of V, we may assume that $V = \mathbb{R}^{2d}$, and then also write $\mathfrak{sp}_+(d, \mathbb{C})$ in place of $\mathfrak{sp}_+(V^{\mathbb{C}}, \sigma)$.

Observe that if a symmetric matrix A has positive definite real part, the same is true for A^{-1}, and thus the matrix $(JS)^{-1} = -S^{-1}J$ has a positive definite real part. Hence the same is true for its conjugate $J(-S^{-1})J\,{}^tJ = -JS^{-1}$. So $-S^{-1} \in \mathfrak{sp}_+(d,\mathbb{C})$. Moreover, as $t \to 0$, $-\cot tS = -(tS)^{-1} + O(t)$, so that there exists a $t_0 > 0$ such that

$$-\cot 2\pi tS \in \mathfrak{sp}_+(d,\mathbb{C}) \quad \forall t \in]0,t_0[.$$

Let Q_t denote the quadratic form

$$Q_t(v) := -\frac{\pi}{2}|\mu|\sigma\big(v,(\cot 2\pi tS)v\big) = -\frac{\pi}{2}|\mu|\,{}^tvJ(\cot 2\pi tS)v.$$

Clearly, $\Gamma^\mu_{t,S} \in \mathcal{S}$ if and only if $\operatorname{Re} Q_t > 0$. We claim that, for every $t,t' > 0$,

$$\operatorname{Re} Q_t > 0 \tag{5.12}$$

and

$$\Gamma^\mu_{t,S} \times_\mu \Gamma^\mu_{t',S} = \Gamma^\mu_{t+t',S}. \tag{5.13}$$

Indeed, it follows from Lemma 5.3 by direct computation that the semigroup property (5.13) holds for every t,t' satisfying (5.12), and (5.12) holds at least on the interval $]0,t_0[$.

Thus, if we assume that (5.12) holds on an interval $]0,2^mt_0[$, for some $m \in \mathbb{N}$, then, by (5.13), $\Gamma^\mu_{t+t',S}$ will be in $\mathcal{S}$ for every $t,t' \in]0,2^mt_0[$, and so (5.12) remains valid on the interval $]0,2^{m+1}t_0[$. Our claim thus follows by induction.

Next, according to [11] (15.1), the operators $f \mapsto f \times_{\pm 1} \big(\det(S+i/2)\big)^{\frac{1}{2}} e^{-\pi\sigma(v,Sv)}$ are contractions on $L^2(V)$ when $S \in \mathfrak{sp}_+(d,\mathbb{C})$. By scaling, the same is true for $f \mapsto f \times_\mu |\mu|^n\big(\det(S+i/2)\big)^{\frac{1}{2}} e^{-\pi|\mu|\sigma(v,Sv)}$, with $\mu \neq 0$. Then, in order to see that T_t is a contraction, we just have to observe that

$$-\frac{1}{2}\cot 2\pi tS + \frac{i}{2} = -e^{-2\pi itS}(2\sin 2\pi tS)^{-1}\ ,$$

and that $\det e^{-2\pi itS} = 1$ as $\operatorname{tr} S = 0$.

The proof will be completed if we show that, for $f \in \mathcal{S}(\mathbb{R}^{2d})$,

$$\lim_{t\to 0}\langle \Gamma^\mu_{t,S}, f\rangle = f(0)\ ,\quad \frac{d}{dt}_{|t=0}\langle \Gamma^\mu_{t,S}, f\rangle = \frac{1}{|\mu|}Q^*(\partial)f(0)\ , \tag{5.14}$$

where $Q^*(\xi,\eta) = {}^t\xi A\eta$.

Now, from (5.7), one easily computes that the Fourier transform of $\Gamma^\mu_{t,S}$ is given by formula (5.8). In particular, since $\Gamma^\mu_{t,S} \in \mathcal{S}$, we see that

$$\tan tS \in \mathfrak{sp}_+(d,\mathbb{C})$$

for every $t > 0$. By dominated convergence, it is now easy to check that, for a test function g,

$$\begin{aligned}\lim_{t\to 0}\langle \widehat{\Gamma^\mu_{t,S}}, g\rangle &= \int g(\xi)\,d\xi = \hat{g}(0)\ ,\\ \frac{d}{dt}_{|t=0}\langle \widehat{\Gamma^\mu_{t,S}}, g\rangle &= \frac{1}{|\mu|}\int {}^t(2\pi i\xi)JS(2\pi i\xi)g(\xi)\,d\xi = \frac{1}{|\mu|}Q^*(\partial)\hat{g}(0)\ .\end{aligned}$$

This gives (5.14). Q.E.D.

This result can be extended to the semi-definite case by means of a continuity argument, since $\mathfrak{sp}_+(V^{\mathbb{C}},\sigma)$ is dense in the cone $\overline{\mathfrak{sp}}_+(V^{\mathbb{C}},\sigma)$ of all elements $S\in\mathfrak{sp}(V^{\mathbb{C}},\sigma)$ such that $\operatorname{Re}Q_S\geq 0$ (see [19]). We remark that in the limiting case where $\operatorname{Re}Q_S=0$, one finds the one-parameter (semi)-groups of the metaplectic representation of $\mathfrak{sp}(V,\sigma)$ (compare [11]).

Theorem 5.4 *The mapping* $S\mapsto\exp(\tilde{L}^\mu_S)f$ *is continuous from* $\overline{\mathfrak{sp}}_+(V^{\mathbb{C}},\sigma)$ *to* $L^2(V)$ *(respectively to* $\mathcal{S}(V)$*), if* $f\in L^2(V)$ *(respectively if* $f\in\mathcal{S}(V)$*), and, for* $S\in\overline{\mathfrak{sp}}_+(V^{\mathbb{C}},\sigma)$ *the mapping* $t\mapsto\exp(t\tilde{L}^\mu_S)f$ *is smooth from* $\mathbb{R}_+$ *to* $\mathcal{S}(V)$*, for every* $f\in\mathcal{S}(V)$*. Moreover, for* $t\geq 0$*, the operator* $\exp(\frac{t}{|\mu|}\tilde{L}^\mu_S)$ *is given by* (5.6)*, where* $\Gamma^\mu_{t,S}$ *is a tempered distribution depending continuously on* S*, whose Fourier transform is given by* (5.8) *whenever* $\det(\cos(2\pi tS)\neq 0$.

Observe that Theorem 5.4 implies that

$$\operatorname{Re}\sigma(w,\tan(2\pi tS)w)\geq 0\quad\forall w\in V,\ t\geq 0,\tag{5.15}$$

whenever $\det(\cos 2\pi tS)\neq 0$.

For *arbitrary* $S\in\mathfrak{sp}(V^{\mathbb{C}},\sigma)$, and complex $t\in\mathbb{C}$, $w\in V^{\mathbb{C}}$, let us *define* $\widehat{\Gamma^\mu_{t,S}}(w)$ by formula (5.8), whenever $\det(\cos 2\pi tS)\neq 0$. Observe that $\widehat{\Gamma^\mu_{t,S}}$ may not be tempered, if $S\notin\mathfrak{sp}_+(V^{\mathbb{C}},\sigma)$ or $t\notin\mathbb{R}_+$.

For $S\in\mathfrak{sp}(V^{\mathbb{C}},\sigma)$, consider again the decomposition $S=S_r+S_i$ given by (3.3). Notice, however, that in general we do not have that $S_i\in\mathfrak{sp}_+(V^{\mathbb{C}},\sigma)$, even if $S\in\mathfrak{sp}_+(V^{\mathbb{C}},\sigma)$. Nevertheless, since S_i and S_r act on different blocks of complex coordinates, one can prove

Proposition 5.5 *Assume that* $\det\cos(2\pi tS)\neq 0$*. Then*

$$\widehat{\Gamma^\mu_{t,S}}(w)=\widehat{\Gamma^\mu_{t,S_r}}(w)\widehat{\Gamma^\mu_{t,S_i}}(w),\tag{5.16}$$

and

$$\hat{L}^\mu_{S_r}\widehat{\Gamma^\mu_{t,S}}(w)=(\hat{L}^\mu_{S_r}\widehat{\Gamma^\mu_{t,S_r}})(w)\widehat{\Gamma^\mu_{t,S_i}}(w)=|\mu|(\partial_t\widehat{\Gamma^\mu_{t,S_r}})(w)\widehat{\Gamma^\mu_{t,S_i}}(w),\tag{5.17}$$

if $t\in\mathbb{C},\ w\in V^{\mathbb{C}}$*, where* $\hat{L}^\mu_{S_r}$ *is defined by* $\hat{L}^\mu_{S_r}\hat{f}=\widehat{L^\mu_{S_r}f}$.

6 Solvability for $\operatorname{Re}A>0$.

Let $\mathcal{L}_{A,\alpha}=\mathcal{L}_A+i\alpha U$ be as in (1.9), and let $S=-AJ$. Our starting point is the formal identity (compare (2.14))

$$\begin{aligned}(\mathcal{L}_A+i\alpha U)^{-1}\delta_0&=\int_{-\infty}^{+\infty}(\tilde{\mathcal{L}}^\mu_A-2\pi\alpha\mu)^{-1}\delta_0\,e^{2\pi iu\mu}\,d\mu\\&=-\int_{-\infty}^{+\infty}\int_0^\infty e^{t(\tilde{\mathcal{L}}^\mu_A-2\pi\alpha\mu)}\delta_0e^{2\pi iu\mu}\,dt\,d\mu\\&=-\int_{-\infty}^{+\infty}\int_0^\infty e^{\frac{t}{|\mu|}\tilde{\mathcal{L}}^\mu_A-2\pi\alpha t\operatorname{sgn}\mu}\delta_0e^{2\pi iu\mu}\,dt\,\frac{d\mu}{|\mu|}\ ,\end{aligned}$$

where we must remember that, according to Theorem 5.2, $e^{\frac{t}{|\mu|}\tilde{\mathcal{L}}_A^\mu}$ is the μ-twisted convolution operator with kernel $\Gamma^\mu_{t,S}$, so that $e^{\frac{t}{|\mu|}\tilde{\mathcal{L}}_A^\mu}\delta_0 = \Gamma^\mu_{t,S}$.

Let $\lambda_1, \cdots, \lambda_d$ be the eigenvalues of S in $i\Omega$ (i.e., with positive imaginary parts). For $1 \le j \le d$, we set $\nu_j := \operatorname{Im}\lambda_j > 0$, and $\nu := \sum_j \nu_j$.

Theorem 6.1 *Let* $\operatorname{Re} A > 0$. *For very* $f \in \mathcal{S}(\mathbb{H}_V)$, *the integral*

$$\langle K_\alpha, f\rangle := -\int_{-\infty}^{+\infty} \frac{1}{|\mu|}\, d\mu \int_0^\infty dt\, e^{-2\pi\alpha t\, sgn\,\mu} \int_V \Gamma^\mu_{t,S}(v) f^{-\mu}(v)\, dv \tag{6.1}$$

converges absolutely for $|\operatorname{Re}\alpha| < \nu$ *and it extends analytically to a meromorphic function of* α *in the complex plane with poles at the points*

$$\alpha \in \mathcal{E} = \left\{ \pm i \sum_{j=1}^d (2k_j + 1)\lambda_j : k_j \in \mathbb{N} \right\} .$$

Also, K_α *is a tempered fundamental solution of* $\mathcal{L}_{A,\alpha}$ *for* $\alpha \notin \mathcal{E}$.

From this we immediately derive the following conclusion.

Corollary 6.2 $\mathcal{L}_{A,\alpha}$ *is locally solvable for* $\alpha \notin \mathcal{E}$.

The proof of Theorem 6.1 will be preceded by a lemma, which we emphasize for future reference.

Lemma 6.3 *There are quadratic forms* Q_{jk} *on* V *such that*

$$\sigma\big(v, (\cot tS)v\big) = \sum_{j=1}^d \sum_{k=0}^m t^k \cot^{(k)}(\lambda_j t) Q_{jk}(v) \ ,$$

where $\cot^{(k)}$ *denotes the* k*-th derivative of the cotangent function and* $m+1$ *is the dimension of the largest Jordan block of* S. *Moreover* $\lim_{t\to\infty} \cot tS$ *exists and its real part is an invertible operator.*

Proof. If λ is an eigenvalue of S, let V_λ be its generalized eigenspace in $V^{\mathbb{C}}$. Then $S = \lambda I + N_\lambda$ on V_λ, with $N_\lambda^{m+1} = 0$. Denoting by γ a small circle around λ, we have

$$\begin{aligned}
(\cot tS)|_{V_\lambda} &= \frac{1}{2\pi i}\int_\gamma (\zeta I - S)^{-1} \cot t\zeta\, d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma \big((\zeta - \lambda)I - N_\lambda\big)^{-1} \cot t\zeta\, d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma \sum_{k=0}^m (\zeta-\lambda)^{-k-1} N_\lambda^k \cot t\zeta\, d\zeta \\
&= \sum_{k=0}^m \frac{1}{k!}\frac{d^k}{d\zeta^k}|_{\zeta=\lambda} (\cot t\zeta) N_\lambda^k \\
&= \sum_{k=0}^m \frac{1}{k!} t^k \cot^{(k)}(\lambda t) N_\lambda^k \ .
\end{aligned}$$

Therefore, extending to all of $V^{\mathbb{C}}$ and summing over the various eigenvalues $\pm\lambda_j$,

$$\cot tS = \sum_{j=1}^{d}\sum_{k=0}^{m}\frac{1}{k!}t^k\cot^{(k)}(\lambda_j t)M_{jk}\ ,$$

for appropriate operators M_{jk}. This gives the first part of the statement. Observe now that

$$\cot\lambda t = -i - 2i\sum_{m=0}^{\infty}e^{2im\lambda t} \tag{6.2}$$

if $\operatorname{Im}\lambda > 0$ and

$$\cot\lambda t = i + 2i\sum_{m=0}^{\infty}e^{-2im\lambda t}$$

if $\operatorname{Im}\lambda < 0$, where the series are uniformly convergent for $t \geq 1$ together with all their derivatives. Therefore $\lim_{t\to\infty} t^k\cot^{(k)}(\lambda_j t) = 0$ if $k \geq 1$, and

$$\lim_{t\to\infty}(\cot tS)|_{V^\pm} = \mp iI\ ,$$

where $V^\pm$ are as in Lemma 3.4.

Call $T = \lim_{t\to\infty}\cot tS$. Given $v \in V$, decompose it as $v = v^+ + v^-$, with $v^\pm \in V^\pm$. Then $\operatorname{Im} v^- = -\operatorname{Im} v^+$, and $Tv = -iv^+ + iv^-$, so that $\operatorname{Re} Tv = 2\operatorname{Im} v^+$. If $\operatorname{Re} Tv = 0$, then $\operatorname{Im} v^+ = \operatorname{Im} v^- = 0$. By Lemma 3.4, $v^\pm = 0$, hence $v = 0$.

Q.E.D.

Proof of Theorem 6.1. In (6.1) we change $2\pi t$ into t, so that it becomes, apart from a constant factor,

$$\int_{-\infty}^{+\infty}|\mu|^{d-1}\,d\mu\int_0^{+\infty}dt\,\frac{e^{-\alpha t\operatorname{sgn}\mu}}{\big(\det(\sin tS)\big)^{\frac12}}\int_V e^{\frac{\pi}{2}|\mu|\sigma\left(v,(\cot tS)v\right)}f^{-\mu}(v)\,dv \tag{6.3}$$

We split the integral (6.3) into the sum of three terms, according to the following limitations in the integrals in $d\mu$ and in dt:

$$\int_{-\infty}^{+\infty}d\mu\int_0^1 dt\ ,\qquad \int_0^{+\infty}d\mu\int_1^{+\infty}dt\ ,\qquad \int_{-\infty}^{0}d\mu\int_1^{+\infty}dt\ . \tag{6.4}$$

The first integral in (6.4) is absolutely convergent for every α. In order to see this, observe that, applying Hölder's inequality and making use of well-known formulas for the integral of a Gaussian, we have

$$\left|\int_V e^{\frac{\pi}{2}|\mu|\sigma\left(v,(\cot tS)v\right)}f^{-\mu}(v)\,dv\right| \leq C_{\delta,N}(1+|\mu|)^{-N}|\mu|^{-\delta d}\big(\det(-\operatorname{Re}(\cot tS))\big)^{-\frac{\delta}{2}}\ , \tag{6.5}$$

for every $\delta \leq 1$ and every N.

For t small, $\cot tS \sim (tS)^{-1}$, so that $\det(-\operatorname{Re}(\cot tS)) \sim t^{-2d}$; similarly, $\big|\det(\sin tS)\big|^{\frac12} \sim t^d$. Matters are so reduced to discussing the convergence of

$$\int_{-\infty}^{+\infty}\frac{|\mu|^{(1-\delta)d-1}}{(1+|\mu|)^N}\,d\mu\int_0^1\frac{e^{|\operatorname{Re}\alpha|t}}{t^{(1-\delta)d}}\,dt\ ,$$

for some δ; for every α it suffices to take $0 < 1-\delta < 1/d$. It is also clear that the first integral in (6.4) is analytic in α on the whole plane.

We pass now to the second integral in (6.4). We apply again (6.5), observing that $\det(-\mathrm{Re}\,(\cot tS)) \sim 1$ by Lemma 6.3 and that $\left|\det(\sin tS)\right|^{\frac{1}{2}} \sim e^{\nu t}$. By (6.5), we are so led to discuss

$$\int_0^{+\infty} \frac{\mu^{(1-\delta)d-1}}{(1+\mu)^N}\,d\mu \int_1^{+\infty} e^{-(\mathrm{Re}\,\alpha+\nu)t}\,dt\ ,$$

which converges for $\mathrm{Re}\,\alpha > -\nu$, provided $\delta > 0$. As before, the second integral in (6.4) depends analytically on α for $\mathrm{Re}\,\alpha > -\nu$.

If we set

$$\varphi(t) := \frac{e^{-\alpha t}}{\big(\det(\sin tS)\big)^{\frac{1}{2}}} = \frac{i^d e^{-\alpha t}}{\prod_{j=1}^d \sin\lambda_j t}\ ,$$

we must discuss the analytic continuation of

$$\int_0^{+\infty} dt\,\mu^{n-1}\,d\mu \int_1^{\infty} \varphi(t) \int_V e^{\frac{\pi}{2}\mu\sigma\left(v,(\cot tS)v\right)} f^{-\mu}(v)\,dv\ . \tag{6.6}$$

Writing

$$\frac{1}{\sin\lambda_j t} = \frac{-2ie^{i\lambda_j t}}{1-e^{2i\lambda_j t}} = -2i\sum_{m=0}^{\infty} e^{(2m+1)i\lambda_j t}\ ,$$

it follows that

$$\varphi(t) = 2^d e^{-\alpha t} \sum_{m\in\mathbb{N}^d} e^{it\sum_j(2m_j+1)\lambda_j} \tag{6.7}$$

where the sum is absolutely convergent for $t \geq 1$.

We fix $r > \nu$, and we truncate the sum to those m for which $\sum_j(2m_j+1)\nu_j < r$, so that

$$\varphi(t) = 2^d e^{-\alpha t} \sum_{\sum_j(2m_j+1)\nu_j<r} e^{it\sum_j(2m_j+1)\lambda_j} + O\big(e^{-(\mathrm{Re}\,\alpha+r)t}\big)\ .$$

Accordingly, we split (6.6) into the sum of a finite number of integrals. Repeating the argument given above, the integral containing the remainder term is absolutely convergent for $\mathrm{Re}\,\alpha > -r$ and it depends analytically on α in this region.

We then discuss the other terms, each of them having the form

$$\int_0^{+\infty} \mu^{d-1}\,d\mu \int_1^{\infty} dt\, e^{-(\alpha+\beta)t} \int_V e^{\frac{\pi}{2}\mu\sigma\left(v,(\cot tS)v\right)} f^{-\mu}(v)\,dv \tag{6.8}$$

with $\beta = -i\sum_j(2m_j+1)\lambda_j$ and $\mathrm{Re}\,\beta = \sum_j(2m_j+1)\nu_j < r$. We set

$$Q_t(v) := -\sigma\big(v,(\cot tS)v\big)\ .$$

When $\mathrm{Re}\,\alpha > -\mathrm{Re}\,\beta$, the integral (6.8) converges absolutely, so we can integrate by parts in t first, to obtain two terms:

$$\frac{\pi}{2}(\alpha+\beta)^{-1} \int_0^{+\infty} \mu^d\,d\mu \int_1^{\infty} dt\, e^{-(\alpha+\beta)t} \int_V e^{-\frac{\pi}{2}\mu Q_t(v)} \frac{dQ_t}{dt}(v) f^{-\mu}(v)\,dv \tag{6.9}$$

and the boundary term

$$(\alpha+\beta)^{-1}e^{-\alpha-\beta}\int_0^{+\infty}\mu^{d-1}\,d\mu\int_V e^{-\frac{\pi}{2}\mu Q_1(w)}f^{-\mu}(v)\,dv\ . \tag{6.10}$$

By (6.5), the boundary term (6.10) extends analytically to all values of $\alpha\neq-\beta$.

By Lemma 6.3, we have

$$\frac{dQ_t}{dt}(v)=\sum_{j=1}^{d}\sum_{k=0}^{m}t^k\cot^{(k+1)}(\lambda_j t)Q_{jk}(v)\ ,$$

for some other quadratic forms $Q_{jk}(v)$.

Then (6.9) decomposes as a finite sum of terms of the form

$$\begin{aligned}\frac{\pi}{2}(\alpha+\beta)^{-1}\int_0^{+\infty}\mu^d\,d\mu\quad&\int_1^{\infty}dt\,t^k\cot^{(k+1)}(\lambda_j t)e^{-(\alpha+\beta)t}\\&\times\int_V e^{-\frac{\pi}{2}\mu Q_t(v)}Q_{jk}(v)f^{-\mu}(v)\,dv\ .\end{aligned}\tag{6.11}$$

We can at this point iterate the argument above. We expand $\cot^{(k+1)}(\lambda_j t)$ on the basis of (6.2) and truncate the sum to the values of m for which $2m\nu_j<r-\operatorname{Re}\beta$. The remainder term gives a contribution to the integral in (6.11) that is absolutely convergent for $\operatorname{Re}\alpha>-r$. From this term we hence obtain an expression that is analytic on the half-plane $\operatorname{Re}\alpha>-r$, except at $\alpha=-\beta$, if this point lies in the half-plane.

The other terms have a form similar to (6.8), precisely

$$\frac{\pi}{2}(\alpha+\beta)^{-1}\int_0^{+\infty}\mu^d\,d\mu\int_1^{\infty}dt\,t^k e^{-(\alpha+\beta')t}\int_V e^{-\frac{\pi}{2}\mu Q_t(v)}Q_{jk}(v)f^{-\mu}(v)\,dv\ , \tag{6.12}$$

with $\beta'=\beta-2im\lambda_j$. It is important to observe that, as $m\geq 1$, $\operatorname{Re}\beta'>\operatorname{Re}\beta$.

If $\operatorname{Re}\beta'\geq r$, then the integral (6.12) is absolutely convergent and analytic for $\operatorname{Re}\alpha>-r$, except for $\alpha=-\beta$. If, instead, $\operatorname{Re}\beta'<r$, we can perform a new integration by parts, choosing $t^k e^{-(\alpha+\beta')}$ as the differential factor. At each step we obtain new exponents $\alpha+\beta',\alpha+\beta''$ etc. (together with factors $(\alpha+\beta')^{-1}$ etc.), where β',β'' etc. are all in $\mathcal{E}$, with positive and strictly increasing real parts. As the strip $\operatorname{Re}\alpha>-r$ contains only finitely many of such points $-\beta$, with $\beta\in\mathcal{E}$ and $\operatorname{Re}\beta>0$, after a finite number of steps all the terms will be well-defined for $\operatorname{Re}\alpha>-r$, except for poles at $-\beta',-\beta''$ etc.

This takes care of the second integral in (6.4). The analysis of the third integral can be reduced to the previous one by replacing μ with $-\mu$ and α with $-\alpha$.

For $|\operatorname{Re}\alpha|<\nu$ we have

$$\begin{aligned}\langle K_\alpha,{}^t\mathcal{L}_{A,\alpha}f\rangle&=\langle K_\alpha,\mathcal{L}_{A,-\alpha}f\rangle\\&=-\int_{-\infty}^{+\infty}\frac{1}{|\mu|}\,d\mu\int_0^{\infty}dt\,e^{-2\pi\alpha t\operatorname{sgn}\mu}\int_V\Gamma^\mu_{t,S}(v)(\tilde L_A^{-\mu}-2\pi\alpha\mu)f^{-\mu}(v)\,dv\\&=-\int_{-\infty}^{+\infty}\frac{1}{|\mu|}\,d\mu\int_0^{\infty}dt\,e^{-2\pi\alpha t\operatorname{sgn}\mu}\int_V(\tilde L_A^{\mu}-2\pi\alpha\mu)\Gamma^\mu_{t,S}(v)f^{-\mu}(v)\,dv\\&=-\int_{-\infty}^{+\infty}\frac{1}{|\mu|}\,d\mu\int_0^{\infty}dt\,e^{-2\pi\alpha t\operatorname{sgn}\mu}\int_V(|\mu|\partial_t-2\pi\alpha\mu)\Gamma^\mu_{t,S}(v)f^{-\mu}(v)\,dv\ .\end{aligned}$$

Using the absolute convergence of the integral, we interchange the order of integration in dt and dv to obtain

$$\int_V f^{-\mu}(v)\,dv \int_0^\infty |\mu|\partial_t\Gamma^\mu_{t,S}(v)e^{-2\pi\alpha t\,\mathrm{sgn}\,\mu}\,dt$$
$$= -|\mu|f^{-\mu}(0) + 2\pi\alpha\mu\int_V f^{-\mu}(v)\,dv\int_0^\infty \Gamma^\mu_{t,S}(v)e^{-2\pi\alpha t\,\mathrm{sgn}\,\mu}\,dt\ .$$

Therefore

$$\langle K_\alpha,\,{}^t\mathcal{L}_{A,\alpha}f\rangle = \int_{-\infty}^{+\infty} d\mu f^{-\mu}(0) = f(0)\ , \tag{6.13}$$

i.e. $\mathcal{L}_{A,\alpha}K_\alpha = \delta_0$. By analytic continuation, (6.13) remains valid for $\alpha \notin \mathcal{E}$.

We finally remark that all constants arising in the estimates in the proof are majorized by some continuous norm on $\mathcal{S}(\mathbb{H}_V)$, which shows that $K_\alpha \in \mathcal{S}'(\mathbb{H}_V)$.

Q.E.D.

7 Local solvability when $\mathrm{Re}\,A$ is semi-definite.

Assume that A satisfies the cone condition, and that $\mathrm{Re}\,A$ is positive semi-definite and degenerate. Based on the factorization formula

$$\widehat{\Gamma^\mu_{t,S}}(w) = \widehat{\Gamma^\mu_{t,S_r}}(w)\widehat{\Gamma^\mu_{t,S_i}}(w) \tag{7.1}$$

in Proposition 5.5, we can then still apply a similar integration by parts method to the factor $\widehat{\Gamma^\mu_{t,S_i}}$ in place of $\widehat{\Gamma^\mu_{t,S}}$. In this way, we can then reduce the problem of local solvability of $\mathcal{L}_{A,\alpha}$ for α outside the exceptional set $\mathcal{E}$ in Theorem 2.3 to the problem of local solvability of the operators $L_{S_r,\beta}$ (for particular values of β.) And, we had seen in Lemma 3.6 (see also (3.5)) that the operators $L_{S_r,\beta}$ turn out to be constant coefficient operators, when written in suitable coordinates, so that all of them are indeed locally solvable, by the Malgrange-Ehrenpreis theorem.

To be a bit more precise, observe that similar considerations as before and Theorem 5.4 show that $\widehat{\Gamma^\mu_{t,S_i}}$ can be written as series of terms of the form $e^{-\beta t}e^{-\frac{2\pi}{\mu}\tilde{Q}_{i,t}}$, with

$$\tilde{Q}_{i,t}(w) := \sigma(w, (\tan tS_i)w)$$

(if, say, $\mu > 0$). Oversimplifying, let us then assume that

$$\widehat{\Gamma^\mu_{t,S_i}} = e^{-\beta t}e^{-\frac{2\pi}{\mu}\tilde{Q}_{i,t}}.$$

Then, by (7.1),

$$e^{-\alpha t}\,\widehat{\Gamma^\mu_{t,S}} = \Big[e^{-(\alpha+\beta)t}\,\widehat{\Gamma^\mu_{t,S_r}}\Big]e^{-\frac{2\pi}{\mu}\tilde{Q}_{i,t}}.$$

And,

$$\mu\partial_t\Big[e^{-(\alpha+\beta)t}\,\widehat{\Gamma^\mu_{t,S_r}}\Big] = \hat{L}^\mu_{S_r,\alpha+\beta}\Big[e^{-(\alpha+\beta)t}\widehat{\Gamma^\mu_{t,S_r}}\Big]$$

hence, at least formally,

$$e^{-\alpha t}\,\widehat{\Gamma^{\mu}_{t,S}} = \Big(\hat{L}^{\mu}_{S_r,\alpha+\beta}\Big)^{-1}\,\mu\partial_t\Big[e^{-(\alpha+\beta)t}\,\widehat{\Gamma^{\mu}_{t,S_r}}\Big]\,e^{-\frac{2\pi}{\mu}\tilde{Q}_{i,t}}. \tag{7.2}$$

If we then consider, e.g., the integral

$$\begin{aligned}\int_0^{+\infty}\frac{1}{\mu}\,d\mu\int_1^{\infty}dt\,e^{-\alpha t}\langle\Gamma^{\mu}_{t,S},f^{-\mu}\rangle &= \int_0^{+\infty}\frac{1}{\mu}\,d\mu\int_1^{\infty}dt\,\langle e^{-\alpha t}\widehat{\Gamma^{\mu}_{t,S}},\widehat{f^{-\mu}}\rangle\\ &= \int_0^{+\infty}\frac{1}{\mu}\,d\mu\int_1^{\infty}dt\,\Big\langle\Big[e^{-(\alpha+\beta)t}\,\widehat{\Gamma^{\mu}_{t,S_r}}\Big]e^{-\frac{2\pi}{\mu}\tilde{Q}_{i,t}},\widehat{f^{-\mu}}\Big\rangle,\end{aligned}$$

we may apply (7.2) to integrate by parts in t. Except for the boundary term, we are thus led to terms of the form

$$-2\pi\int_0^{+\infty}\frac{1}{\mu}\,d\mu\int_1^{\infty}dt\,\Big\langle\Big(\hat{L}^{\mu}_{S_r,\alpha+\beta}\Big)^{-1}\Big[e^{-(\alpha+\beta)t}\,\widehat{\Gamma^{\mu}_{t,S_r}}\Big]\,\partial_t\tilde{Q}_{i,t}\,e^{-\frac{2\pi}{\mu}\tilde{Q}_{i,t}},\widehat{f^{-\mu}}\Big\rangle.$$

And, arguing similarly as in the previous section, only with cotangent replaced by tangent, in view of the obvious analogue of (6.2), in the expression for $\partial_t\tilde{Q}_{i,t}$ we gain factors of the form $e^{-2im\lambda_j t}$, with $m\geq 1$, compared to $\tilde{Q}_{i,t}$.

Iterating this procedure, we can again extend in this way the family of distributions K_α analytically to $\alpha\in\mathbb{C}\setminus\mathcal{E}$, provided the operators $\Big(\hat{L}^{\mu}_{S_r,\alpha+\beta}\Big)^{-1}$, respectively $\Big(L_{S_r,\alpha+\beta}\Big)^{-1}$, that arize on the way can be given a meaning. However, this is the case, as we have seen before.

For details and extensions of these result, we refer the interested reader to [19].

8 Appendix: A representation theoretic necessary condition for local solvability

In this appendix, we provide a proof of the Corwin-Rothschild theorem 4.1 for the Heisenberg group. This proof is based on my approach in [31], which gives in fact a stronger result than the one in the original paper by Corwin and Rothschild. This improvement turned out to be crucial for the proof of results such as Theorem 2.4. I shall closely follow my survey lecture [25], which is also recommended for further information.

Let me first briefly recall some basics of the representation theory of the Heisenberg group. Consider for $\mu\in\mathbb{R}^{\times}:=\mathbb{R}\setminus\{0\}$ the *Schrödinger representation* π_μ of $\mathbb{H}_d$ acting on the Hilbert space $L^2(\mathbb{R}^d)$ as follows:

$$[\pi_\mu(x,y,u)f](\xi):=e^{2\pi i\mu(u+y\cdot\xi+\frac{1}{2}x\cdot y)}f(\xi+x),\quad f\in L^2(\mathbb{R}^d). \tag{8.1}$$

For $f\in L^1(\mathbb{H}_d)$, we define the *(group-) Fourier transform* $\hat{f}(\pi_\mu)$ of f at the representation π_μ as the bounded operator

$$\hat{f}(\pi_\mu):=\int_{\mathbb{H}_d}f(g)\pi_\mu(g)^*\,dg=\int_{\mathbb{H}_d}f(g)\pi_\mu(g^{-1})\,dg=\pi_\mu(\check{f})$$

on $L^2(\mathbb{R}^d)$. Observe that

$$(f_1 \star f_2)^\wedge(\pi_\mu) = \hat{f}_2(\pi_\mu) \circ \hat{f}_1(\pi_\mu). \tag{8.2}$$

Direct computations, based on formula (8.1), show that $\hat{f}(\pi_\mu)$ can be represented as a kernel operator

$$(\hat{f}(\pi_\mu)\varphi)(\xi) = \int_{\mathbb{R}^d} K_f^\mu(\xi,\eta)\varphi(\eta)\,d\eta, \qquad \varphi \in L^2(\mathbb{R}^d), \tag{8.3}$$

with integral kernel

$$\begin{aligned} K_f^\mu(\xi,\eta) \;&=\; \int\int f(\xi-\eta,y,u)e^{-2\pi i\mu(u+\frac{y}{2}(\xi+\eta))}\,dydu, \\ &=\; f(\xi-\eta,\widehat{\frac{\mu}{2}(\xi+\eta)},\widehat{\mu}). \end{aligned} \tag{8.4}$$

As usually, we shall identify the elements P of the universal envelopping algebra $\mathfrak{u}(\mathfrak{h}_d)$ with left-invariant differential operators by means of the formula

$$P\varphi = P(\varphi \star \delta_0) = \varphi \star (P\delta_0), \quad \varphi \in \mathcal{S},$$

i.e., P can be represented by convolution from the right with the compactly supported distribution $P\delta_0$. But from (8.4), one sees that K_f^μ is well-defined as a tempered distribution kernel $K_f^\mu \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ supported near the diagonal $\xi = \eta$, for every distribution $f \in \mathcal{E}'(\mathbb{H}_d)$ with compact support. This implies that the integral operator (8.4), defined in the Schwartz-sense of distributions, is well-defined on $\mathcal{S}(\mathbb{R}^d)$, and

$$\hat{f}(\pi_\mu) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$$

is continuous for every $f \in \mathcal{E}'(\mathbb{H}_d)$.

For $P \in \mathfrak{u}(\mathfrak{h}_d)$, we now define its Fourier transform by

$$\hat{P}(\pi_\mu) := \widehat{P\delta_0}(\pi_\mu) := \pi_\mu((P\delta_0)\check{}).$$

Approximating $P\delta_0$ by $P\delta_0 \star \varphi_\varepsilon \in \mathcal{D}$, where $\{\varphi_\varepsilon\}_{\varepsilon>0}$ denotes a Dirac sequence in $\mathcal{D}$, one finds from (8.2) that

$$\widehat{P\varphi}(\pi_\mu) = \hat{P}(\pi_\mu) \circ \hat{\varphi}(\pi_\mu), \qquad \varphi \in \mathcal{S}, \tag{8.5}$$

and

$$\widehat{AB}(\pi_\mu) = \hat{A}(\pi_\mu) \circ \hat{B}(\pi_\mu), \qquad \forall A, B \in \mathfrak{u}(\mathfrak{h}_d), \tag{8.6}$$

since $(AB)\delta_0 = A(B\delta_0 \star \delta_0) = B\delta_0 \star A\delta$.

Since $X_j\delta_0 = \frac{\partial}{\partial x_j}\delta_0$, $Y_j\delta = \frac{\partial}{\partial y_j}\delta_0$, $U\delta_0 = \frac{\partial}{\partial u}\delta_0$, we find from (8.4) that

$$\hat{X}_j(\pi_\mu) = \frac{\partial}{\partial \xi_j}, \quad \hat{Y}_j(\pi_\mu) = 2\pi i\mu\xi_j, \quad \hat{U}(\pi_\mu) = 2\pi i\mu. \tag{8.7}$$

Also, from (8.4), one sees that

$$K_{f\circ D_r}^{r^2\mu}(\xi,\eta) = r^{-d-2}K_f^\mu(r\xi,r\eta), \qquad r > 0. \tag{8.8}$$

If $P \in \mathfrak{u}(\mathfrak{h}_n)$ is homogeneous of degree q, i.e., if $P(f \circ D_r) = r^q (Pf) \circ D_r$ for every $r > 0$, then $f := P\delta_0$ satisfies $f \circ D_r = r^{-Q-q} f$, where $Q := 2d+2$ is the homogeneous dimension of $\mathbb{H}_d$, hence (8.8) implies

$$K^{r^2\mu}_{P\delta_0}(\xi, \eta) = r^{q+d} K^{\mu}_{P\delta_0}(r\xi, r\eta). \tag{8.9}$$

We can now turn to the problem of local solvability. The first, basic step consists in turning the qualitative statement of an operator L "to be locally solvable" into a quantitative statement. This is achieved by means of the following criterion, due to Hörmander.

Lemma 8.1 *Let L be a linear differential operator with smooth coefficients in an open subset Ω of $\mathbb{R}^n$. The equation $Lu = f$ can be solved for every $f \in \mathcal{D}(\Omega)$ with $u \in \mathcal{D}'(\Omega)$ if and only if the following holds true:*

For every relatively compact open subset $\Lambda \subset \Omega$ there exist constants C and $k \in \mathbb{N}$, such that for every $f, v \in C_0^\infty(\Lambda)$,

$$|\int f v \, dx| \leq C \sum_{|\alpha| \leq k} ||D^\alpha f||_2 \sum_{|\beta| \leq k} ||D^\beta \; {}^t L v||_2 \tag{8.10}$$

Here, ${}^tL$ denotes the formal transposed of L, defined by

$$\int v(Lu) \, dx = \int ({}^tLv) u \, dx.$$

Proof. The sufficiency of (8.10) follows by the Hahn-Banach theorem (exercise).

Conversely, if $Lu = f$ can be solved for every $f \in \mathcal{D}(\Omega)$ by some $u = u_f \in \mathcal{D}'(\Omega)$, then

$$\langle f, v \rangle = \int f v \, dx = \langle u_f, {}^tLv \rangle \quad \forall v \in \mathcal{D}(\Lambda). \tag{8.11}$$

Consider $\langle f, v \rangle$ as a bilinear form on $C_0^\infty(\overline{\Lambda}) \times C_0^\infty(\Lambda)$, where $C_0^\infty(\overline{\Lambda})$ is a Fréchet space with the topology induced by the semi-norms $||D^\alpha f||_2$, and where $C_0^\infty(\Lambda)$ is endowed with the metrizable topology induced by the semi-norms $||D^\beta \; {}^tPv||_2$.
Obviously, $f \mapsto \langle f, v \rangle$ is continuous for fixed v.
The continuity of $v \mapsto \langle f, v \rangle$, for fixed f, follows on the other hand by (8.11).
Thus, $(f, v) \mapsto \langle f, v \rangle$ is separately continuous, hence continuous, by the theorem of Banach-Steinhaus. This proves (8.10).

Q.E.D.

Remark 8.2 *Condition* (8.10) *is equivalent to*

$$||v||_{(-k)} \leq C || \, {}^tLv||_{(k)}, \tag{8.12}$$

where $||f||_{(\alpha)} = (\int (1 + |\xi|^2)^\alpha |\hat{f}(\xi)|^2 d\xi)^{1/2}$ denotes the Sobolev-norm of order α.

For homogeneous left-invariant differential operators on $\mathbb{H}_d$, the following necessary criterion for local solvability has proven extremely useful (analogues hold on general homogeneous groups).

Theorem 8.3 ([31]) *Let $P \in \mathfrak{u}(\mathfrak{h}_d)$ be homogeneous. If P is locally solvable, then there exist a Sobolev-norm $||\cdot||_{(k)}$ and a continuous "Schwartz-norm" $||\cdot||_{\mathcal{S}}$ on $\mathcal{S}(\mathbb{H}_d)$, such that*

$$|f(0)| \le ||f||_{\mathcal{S}}^{1/2}\, ||\,{}^tPf||_{(k)}^{1/2} \quad \forall f \in \mathcal{S}(\mathbb{H}_d). \tag{8.13}$$

Proof. Let Q be an elliptic, right-invariant Laplacian on $\mathbb{H}_d$, and let Ω be an open neighborhood of 0, and let $m > (d+1)/2$. Then, for $\varphi \in \mathcal{D}(\Omega)$, by Poincaré's inequality and standard elliptic regularity theory,

$$|\varphi(0)| \le C'||Q^m\varphi||_2 \le C||Q^{m+k/2}\varphi||_{(-k)},$$

provided Ω is chosen sufficiently small. We choose k is as in (8.12), and assume k to be even. Since $Q^{m+k/2}$ commutes with the left-invariant operator ${}^tP$, by (8.12) we have

$$\begin{aligned} ||Q^{m+k/2}\varphi||_{(-k)} &\le C||Q^{m+k/2}\ {}^tP\varphi||_{(k)} \\ &\le C'||\,{}^tP\varphi||_{(2m+2k)}, \end{aligned}$$

i.e., there exists a $K \in \mathbb{N},\ C \ge 0$, such that

$$|\varphi(0)| \le C\,||\,{}^tP\varphi||_{(K)} \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{8.14}$$

Let us us denote by

$$|(v,u)| := (|v|^4 + 16u^2)^{1/4}$$

the so-called Koranyi-norm on $\mathbb{H}_d$. Then $|\cdot|$ is a homogeneous norm, which means in particular that

$$|D_r g| = r|g|,\ |gh| \le |g| + |h| \text{ and } |g^{-1}| = |g| \quad \forall g,h \in \mathbb{H}_d, r > 0.$$

Let $B_r := \{g \in \mathbb{H}_d : |g| < r\}$ denote the corresponding ball of radius $r > 0$ centred at the origin. Re-scaling, we may assume that $\Omega = B_2$. Let ${}^tP$ be homogeneous of degree q. Choose $\chi \in \mathcal{D}(B_2)$ such that $\chi \equiv 1$ on B_1. Then, for $f \in \mathcal{S}$, by (8.14),

$$|f(0)| \le C\,||\,{}^tP(\chi(f \circ D_r))||_{(K)} \quad \forall r > 0. \tag{8.15}$$

But,

$$\begin{aligned} {}^tP(\chi(f \circ D_r)) &= \chi\ {}^tP(f \circ D_r) + R(f \circ D_r) \\ &= r^q\chi\,({}^tPf) \circ D_r + R(f \circ D_r), \end{aligned}$$

where $R = [{}^tP, \chi]$ is a PDO whose coefficients are supported in $\{1 \le |x| \le 2\}$. Thus, for $r \ge 1$,

$$\begin{aligned} &||\,{}^tP(\chi(f \circ D_r))||_{(K)} \\ &\quad \le Cr^A\{||\,{}^tPf||_{(K)} + \sum_{|\alpha| \le N}(\int_{1<|x|<2} |f^{(\alpha)}(D_r x)|^2 dx)^{1/2}\}, \end{aligned}$$

for some constants $A>0, N\geq 0$. Now,

$$\begin{aligned}\int_{1<|x|<2}|f^{(\alpha)}(D_r x)|^2 dx &\leq r^{-B}\int_{1<|x|<2}|D_r x|^B|f^{(\alpha)}(D_r x)|^2 dx\\ &\leq r^{-B-Q}\int |x|^B|f^{(\alpha)}(x)|^2 dx.\end{aligned}$$

Choosing B such that $A-\frac{1}{2}(B+Q)=-A$, we find a Schwartz-norm $||\cdot||_{\mathcal{S}}$ such that

$$||\,{}^tP(\chi(f\circ D_r))||_{(K)}\leq C(r^A||\,{}^tPf||_{(K)}+r^{-A}||f||_{\mathcal{S}}).$$

Combining this with (8.15) and optimizing in r we obtain (8.13) (if we assume without loss of generality that $||\,{}^tPf||_{(K)}\leq ||f||_{\mathcal{S}}$).

Q.E.D.

Corollary 8.4 ([3]) *Suppose there exists a non-trivial $f\in\mathcal{S}(\mathbb{H}_d)$ such that*

$$ {}^tPf=0. \tag{8.16}$$

Then P is not locally solvable.

Proof of Theorem 4.1: Assume that $\widehat{{}^tP}(\pi_{\mu_0})$ annihilates some non-trivial Schwartz function $\phi\in\mathcal{S}(\mathbb{R}^d)$ for some $\mu_0\neq 0$, say for instance $\mu^0>0$. For $\mu>0$, put

$$\phi^\mu(\xi):=\phi\left(\left(\frac{\mu}{\mu^0}\right)^{1/2}\xi\right).$$

Then, by (8.9), $\widehat{{}^tP}(\pi_\mu)\phi^\mu=0$ for every $\mu>0$. Let $\chi\in C_0^\infty(\mathbb{R}^+)$, and put

$$K^\mu(\xi,\eta):=\chi(\mu)\phi^\mu(\xi)\phi^\mu(\eta),\quad (\mu,\xi,\eta)\in\mathbb{R}^\times\times\mathbb{R}^d\times\mathbb{R}^d.$$

¿From (8.4), it follows that $K^\mu=K^\mu_f$ for some unique function $f\in\mathcal{S}(\mathbb{H}_d)$. And, by (8.5),

$$({}^tPf)^\wedge(\pi_\mu)=\widehat{{}^tP}(\pi_\mu)\hat{f}(\pi_\mu)=0,$$

since $\hat{f}(\pi_\mu)$ is represented by the kernel K^μ. Thus, by Fourier inversion on $\mathbb{H}_d$, we get ${}^tPf=0$. Since $f\neq 0$, the proof follows now from Corollary 8.4.

Q.E.D.

References

[1] Richard Beals and Peter Greiner. *Calculus on Heisenberg manifolds*, volume 119 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 1988.

[2] Michael Christ, G. E. Karadzhov, and Detlef Müller. Infinite-dimensional families of locally nonsolvable partial differential operators. *Math. Res. Lett.*, 3(4):511–526, 1996.

[3] Larry Corwin, Linda P. Rothschild. Necessary conditions for local solvability of homogeneous left invariant differential operators on nilpotent Lie groups. *Acta Math.*, 147:265–288, 1981.

[4] Filippo De Mari, Marco M. Peloso, and Fulvio Ricci. Analysis of second order differential operators with complex coefficients on the Heisenberg group. *J. Reine Angew. Math.*, 464:67–96, 1995.

[5] B. Helffer. Partial differential equations on nilpotent groups. In *Lie group representations, III (College Park, Md., 1982/1983)*, volume 1077 of *Lecture Notes in Math.*, pages 210–253. Springer, Berlin, 1984.

[6] S. Helgason. Fundamental solutions of invariant differential operators on symmetric spaces. *Bull. Amer. Math. Soc.*, 69:778–781, 1963.

[7] Lars Hrmander. *The analysis of linear partial differential operators. III*, volume 274 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1994. Pseudo-differential operators, Corrected reprint of the 1985 original.

[8] Lars Hrmander. Symplectic classification of quadratic forms, and general Mehler formulas. *Math. Z.*, 219(3):413–449, 1995.

[9] Lars Hrmander. *Linear partial differential operators.* Springer Verlag, Berlin, 1976.

[10] Lars Hrmander. A class of hypoelliptic pseudodifferential operators with double characteristics, *Math. Ann.*, 217:165–188, 1975.

[11] Roger Howe. The oscillator semigroup. *Proc. Symp. Pure Appl. Math.*, 48:61–132, 1988.

[12] Georgi E. Karadzhov and Detlef Müller. A remarkable class of second order differential operators on the Heisenberg group $\mathbb{H}_2$. *Math. Ann.*, 320(4):731–755, 2001.

[13] Pierre Lévy-Bruhl. Conditions suffisantes de résolubilité locale pour des opérateurs invariants à gauche sur des groupes nilpotents. II. *Trans. Amer. Math. Soc.*, 293(2):593–603, 1986.

[14] P. Lévy-Bruhl. Conditions suffisantes de résolubilité locale d'opérateurs invariants à gauche sur des groupes nilpotents. *Comm. Partial Differential Equations*, 9(9):839–888, 1984.

[15] P. Levy-Bruhl. Condition nécessaire de résolubilité locale d'opérateurs invariants sur des groupes nilpotents. *Comm. Partial Differential Equations*, 8(12):1325–1335, 1983.

[16] Pierre Lévy-Bruhl. Application de la formule de Plancherel à la résolubilité d'opérateurs invariants à gauche sur des groupes nilpotents d'ordre deux. *Bull. Sci. Math. (2)*, 106(2):171–191, 1982.

[17] Pierre Lévy-Bruhl. Résolubilité locale et globale d'opérateurs invariants du second ordre sur des groupes de Lie nilpotents. *Bull. Sci. Math. (2)*, 104(4):369–391, 1980.

[18] Detlef Müller. Local solvability of second order differential operators with double characteristics I: Necessary conditions. *submitted; download: Math. ArXiv: http://arxiv.org/abs/math.AP/0501452.*

[19] Detlef Müller. Local solvability of second order differential operators with double characteristics II: Sufficiant conditions for left-invariant operators on the Heisenberg group. *submitted; download: http://analysis.math.uni-kiel.de/mueller.*

[20] Detlef Müller and Marco M. Peloso. Non-solvability for a class of left-invariant second-order differential operators on the Heisenberg group. *Trans. Amer. Math. Soc.*, 355(5):2047–2064 (electronic), 2003.

[21] Detlef Müller and Fulvio Ricci. Solvability of second-order left-invariant differential operators on the Heisenberg group satisfying a cone condition. *J. Anal. Math.*, 89:169–197, 2003.

[22] Detlef Müller and Zhenqiu Zhang. A class of solvable non-homogeneous differential operators on the Heisenberg group. *Studia Math.*, 148(1):87–96, 2001.

[23] Detlef Müller and Zhenqiu Zhang. Local solvability for positive combinations of generalized sub-Laplacians on the Heisenberg group. *Proc. Amer. Math. Soc.*, 129(10):3101–3107 (electronic), 2001.

[24] Detlef Müller and Christoph Thiele. Normal forms of involutive complex Hamiltonian matrices under the real symplectic group. *J. Reine Angew. Math.*, 513:97–114, 1999.

[25] Detlef Müller. Analysis of invariant PDO's on the Heisenberg group, Lecture at ICMS-Instructional Conference, Edinburgh, 1999. *On webpage http://analysis.math.uni-kiel.de/mueller/*

[26] Detlef Müller, Marco M. Peloso, and Fulvio Ricci. On local solvability for complex coefficient differential operators on the Heisenberg group. *J. Reine Angew. Math.*, 513:181–234, 1999.

[27] Detlef Müller, Marco M. Peloso, and Fulvio Ricci. On the solvability of homogeneous left-invariant differential operators on the Heisenberg group. *J. Funct. Anal.*, 148(2):368–383, 1997.

[28] Detlef Müller and Fulvio Ricci. Solvability for a class of doubly characteristic differential operators on 2-step nilpotent groups. *Ann. of Math. (2)*, 143(1):1–49, 1996.

[29] Detlef Müller and Fulvio Ricci. Solvability of second-order PDOs on nilpotent groups—a survey of recent results. In *Fourier analysis and partial differential equations (Miraflores de la Sierra, 1992)*, Stud. Adv. Math., pages 77–97. CRC, Boca Raton, FL, 1995.

[30] Detlef Müller. Another example in the solvability theory of PDOs with double characteristics. *Comm. Partial Differential Equations*, 20(11-12):2165–2186, 1995.

[31] Detlef Müller. A new criterion for local nonsolvability of homogeneous left invariant differential operators on nilpotent Lie groups. *Journal für die Reine und Angewandte Mathematik.*, 416:207–219, 1991.

[32] Detlef Müller and Fulvio Ricci. Analysis of second order differential operators on Heisenberg groups. II. *J. Funct. Anal.*, 108(2):296–346, 1992.

[33] Detlef Müller and Fulvio Ricci. Analysis of second order differential operators on Heisenberg groups. I. *Invent. Math.*, 101(3):545–582, 1990.

[34] Mustapha Raïs. Solutions élémentaires des opérateurs différentiels bi-inbariants sur un groupe de Lie nilpotent. *C. R. Acad. Sci. Paris Sér. A-B*, 273:A495–A498, 1971.

[35] Mustapha Raïs. Opérateurs différentiels bi-invariants (d'après M. Duflo). In *Séminaire Bourbaki, 29e année (1976/77)*, volume 677 of *Lecture Notes in Math.*, pages Exp. No. 498, pp. 125–137. Springer, Berlin, 1978.

[36] Johannes Sjstrand. Parametrices for pseudodifferential operators with multiple characteristics. *Ark. Mat.*, 12:85–130, 1974.

[37] Franois Trèves. *Introduction to pseudodifferential and Fourier integral operators. Vol. 1.* Plenum Press, New York, 1980. Pseudodifferential operators, The University Series in Mathematics.

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