Large time behavior in perfect incompressible flows

Dragos Iftimie

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Dragoş IFTIMIE

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Introduction

These lecture notes correspond to an eight hours mini-course that the author taught at the CIMPA summer school in Lanzhou (China) during July 2004.

The equation of motion of a perfect incompressible fluid were deduced by Euler [18] by assuming that there is no friction between the molecules of the fluid. In the modern theory of existence and uniqueness of solutions, the case of the dimension two is by far the richest. Global existence and uniqueness of bidimensional solutions was first proved by Wolibner [49] for smooth initial data and by Yudovich [51] for data with bounded vorticity. There are also some global existence results (no uniqueness yet) when the vorticity is in L^p or is a nonnegative compactly supported H^{-1} Radon measure. As far as the dimension three is concerned, only some local in time results are known, except in some very particular cases.

After obtaining this global existence theory in dimension two under more or less satisfactory hypothesis, a natural question arises: what is the large time behavior of these solutions? Unfortunately, the answer to this question is still largely unknown. The few results that are known give some information on the vorticity rather than the velocity itself. This 8 hours mini-course is intended to present the latest developments on the subject together with a introduction to the equations and a review of the main global existence of solutions results.

The structure of these notes is the following. In Chapter 1 we start by giving a very short presentation of the equations, we introduce the main quantities and list without proof the conservations laws that will be used in the sequel. Next, we review the most important global existence and uniqueness of solutions results; the main ideas of the proofs are also highlighted. After this introductory part, we discuss in Chapter 2 some relevant examples of solutions for the Euler equations and the vortex model; the behavior observed here will be precious in the sequel. Chapter 3 deals with the confinement properties of nonnegative vorticity. We end this work with the most general case, the case of unsigned vorticity. Here we will find another point of view for the large time behavior: we will try to describe the weak limits of different rescalings of the vorticity.

Chapter 1 is given only to make these lecture notes self-contained. For these reasons, the write-up is rather sketchy with very few details given. The main part of this work consists in Chapters 2, 3 and 4 which are more complete and carefully written.

Chapter 1

Presentation of the equations and existence of solutions

1.1 Presentation of the equations, Biot-Savart law and conserved quantities

Let u be the velocity of a perfect incompressible fluid filling $\mathbb{R}^n$ and p the pressure. Assuming that the density is constant equal to 1, then the vector field u and the scalar function p must satisfy the following Euler equation

$$\partial_t u + u \cdot \nabla u = -\nabla p, \quad \operatorname{div} u = 0, \quad u\big|_{t=0} = u_0,$$

where $\operatorname{div} u = \sum_i \partial_i u_i$ and $u \cdot \nabla = \sum_i u_i \partial_i$. If we place ourselves on a bounded domain, then we must also assume the so-called slip boundary conditions which say that the velocity is tangent to the boundary and express the fact that the boundary is not permeable. We define the vorticity to be the following antisymmetric matrix

$$\Omega = (\partial_j u_i - \partial_i u_j)_{i,j}.$$

In dimension 2 we identify Ω to a scalar function,

$$\Omega \equiv \omega = \partial_1 u_2 - \partial_2 u_1$$

while in dimension 3 we identify it with the following vector field.

$$\Omega \equiv \omega = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}.$$

From the divergence free condition on u, one can check that

$$\triangle u = \operatorname{div} \Omega = \Big(\sum_j \partial_j \Omega_{ij} \Big)_i$$

Using the formula for the fundamental solution of the Laplacian in $\mathbb{R}^n$ we deduce the following formula expressing the velocity in terms of the vorticity.

$$u = C_n \int_{\mathbb{R}^n} \Omega(y) \frac{x-y}{|x-y|^n} \,\mathrm{d}y.$$

The above relation is called the Biot-Savart law. In dimension 2, the Biot-Savart law can be expressed as follows:

$$u = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{2\pi|x-y|^2} \omega(y) \,\mathrm{d}y = \frac{x^\perp}{2\pi|x|^2} * \omega,$$

where $x^\perp = (-x_2, x_1)$.

It is a simple calculation to check that the vorticity equation is

$$\partial_t \Omega + u \cdot \nabla \Omega + (\nabla u)\Omega + \Omega(\nabla u)^t = 0$$

while in dimension 2 it can be expressed as a simple transport equation:

$$\partial_t \omega + u \cdot \nabla \omega = 0. \tag{1.1}$$

From this transport equation it is not difficult to deduce that the following quantities are conserved in dimension 2:

- $\int_{\mathbb{R}^2} u$;
- the energy $\|u\|_{L^2}$ and the generalized energy $\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y|\omega(x)\omega(y) \,\mathrm{d}x \,\mathrm{d}y$;
- $\int_{\mathbb{R}^2} \omega$ and all L^p norms of ω, $1 \le p \le \infty$;
- center of mass $\int_{\mathbb{R}^2} x\omega(x) \,\mathrm{d}x$;
- moment of inertia $\int_{\mathbb{R}^2} |x|^2\omega(x) \,\mathrm{d}x$;
- circulation on a material curve $\int_\Gamma u \cdot \mathrm{d}s$ (Γ is a curve transported by the flow).

1.2 Existence and uniqueness results

The aim of this section is to give a review of the most important global existence (and sometimes uniqueness) of bidimensional solutions to the Euler equations and also to give a very short sketch of the proof with the main ingredients. We start with the case of classical solutions in Subsection 1.2.1, we continue with L^p vorticities in Subsection 1.2.2 and we end with the very interesting case of vortex sheets in Subsection 1.2.3.

1.2.1 Strong solutions and the blow-up criterion of Beale-Kato-Majda

We first deal with strong solutions that belong to the Sobolev space $H^m(\mathbb{R}^n)$, $m > \frac{n}{2}+1$. By Sobolev embeddings, such a solution is C^1 so it verifies the equation in the classical sense. Their existence is in general only local in time, but the Beale, Kato and Majda [3] blow-up criterion ensures that the existence is global in dimension 2. More precisely, we have the following result.

Theorem 1.2.1 *Suppose that the initial velocity u_0 is divergence free and belongs to the Sobolev space $H^m(\mathbb{R}^n)$ where $m > \frac{n}{2}+1$. There exists a unique local solution $u \in C^0\big([0,T);H^m\big)$ with $T \geq \frac{C}{\|u_0\|_{H^m}}$. Moreover, the following blow-up criterion due to Beale, Kato and Majda holds: if T^*, the maximal time existence of this local solution, is finite, then $\int_0^{T^*}\|\Omega\|_{L^\infty} = \infty$.*

Corollary 1.2.2 *In dimension 2 the above solution is global.*

Proof of the corollary. The proof is trivial from the Beale, Kato and Majda blow-up criterion since the L^∞ norm of the vorticity is conserved. □

Sketch of proof of Theorem 1.2.1. The *a priori* estimates

$$\partial_t\|u\|_{H^m}^2 \leq C\|u\|_{H^m}^2\|\nabla u\|_{L^\infty}$$

follow from the following Gagliardo-Nirenberg inequality

$$\|D^\ell u\|_{L^{\frac{2k}{\ell}}} \leq C\|u\|_{L^\infty}^{1-\frac{\ell}{k}}\|D^k u\|_{L^2}^{\frac{\ell}{k}}, \qquad 0 \leq \ell \leq k,$$

and from the cancellation $\int u\cdot\nabla D^m u D^m u = 0$. The first part of the theorem follows from the Sobolev embedding $H^{m-1}\subset L^\infty$ used to estimate $\|\nabla u\|_{L^\infty} \leq C\|u\|_{H^m}$.

We now prove the blow-up condition. Assume, by absurd, that $\int_0^{T^*}\|\Omega\|_{L^\infty} < \infty$. From the vorticity equation and using that $\|\nabla u\|_{L^2}\simeq\|\Omega\|_{L^2}$, one can easily deduce that $\Omega\in L^\infty(0,T^*;L^2)$. We now use the following standard logarithmic inequality

$$\|\nabla u\|_{L^\infty} \leq C[1+\|\Omega\|_{L^2}+\|\Omega\|_{L^\infty}(1+\log_+\|u\|_{H^m})]$$

to deduce that

$$\|\nabla u\|_{L^\infty} \leq C(1+\|\Omega\|_{L^\infty}\int_0^t\|\nabla u\|_{L^\infty}).$$

Gronwall's inequality therefore implies that $\int_0^{T^*}\|\nabla u\|_{L^\infty} < \infty$ which in turn gives that $u\in L^\infty(0,T^*;H^m)$ which obviously contradicts the maximality of T^*. □

1.2.2 Solutions with compactly supported L^p vorticity

Let L^p_c denote the space of compactly supported L^p functions. If $p > 1$ and $\omega_0 \in L^p_c$ then $\omega \in L^\infty(\mathbb{R}_+; L^p)$ and therefore $u \in L^\infty(\mathbb{R}_+; W^{1,p}_{loc})$. Global existence of solutions follows with a standard approximation procedure and basically from the compact embedding $W^{1,p}_{loc} \hookrightarrow L^2_{loc}$, see [15]. Uniqueness of these solutions is not known unless $p = \infty$ when the following uniqueness result due to Yudovich [51] holds.

Theorem 1.2.3 (Yudovich) *Suppose that $\omega_0 \in L^\infty_c$. There exists a unique global solution such that $\omega \in L^\infty(\mathbb{R}_+; L^\infty_c)$.*

Sketch of proof of uniqueness. The proof relies on the following singular integral estimate:

$$\|\nabla u\|_{L^p} \leq Cp\|\omega\|_{L^p} \qquad \forall 2 \leq p < \infty.$$

Let u and v be two solutions and set $w = u - v$. Then

$$\partial_t w + u \cdot \nabla w + w \cdot \nabla v = \nabla p'.$$

We now make L^2 energy estimates on this equation by multiplying with w to obtain

$$\partial_t \|w\|^2_{L^2} = -2\int w \cdot \nabla v w \leq 2\|w\|_{L^2}\|\nabla v\|_{L^p}\|w\|_{L^{\frac{2p}{p-2}}} \leq Cp\|w\|^{2-\frac{2}{p}}_{L^2}.$$

After integration we get $\|w(t)\|_{L^2} \leq (Ct)^p$. Sending $p \to \infty$ yields $w\big|_{[0,\frac{1}{C}]} = 0$. Global uniqueness follows by repeating this argument. □

1.2.3 Vortex sheets and the Delort theorem

The vortex sheet problem appears when the velocity has a jump over an interface. In this case, the vorticity is no longer a function but a measure since it must contain the Dirac mass of the interface. Previous global existence results do not apply. Nevertheless, we have the following very important global existence result due to Delort [12].

Theorem 1.2.4 (Delort) *Suppose that $u_0 \in L^2_{loc}(\mathbb{R}^2)$ such that the initial vorticity ω_0 is a nonnegative compactly supported Radon measure. Then there exists a global solution $u \in L^\infty_{loc}(\mathbb{R}_+; L^2_{loc})$.*

Sketch of proof. We give here the main ideas of the proof in the version of Schochet [46]. First of all, it is very easy to see by standard energy estimates that *a priori* $u \in L^\infty_{loc}(\mathbb{R}_+; L^2_{loc})$ which implies that $\omega \in L^\infty_{loc}(\mathbb{R}_+; H^{-1}_{loc})$.

The first main ingredient is the following weak definition of the nonlinear term from the vorticity equation:

$$\langle \operatorname{div}(u\omega), \varphi\rangle = -\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{(x-y)^\perp}{2\pi|x-y|^2}[\nabla\varphi(x) - \nabla\varphi(y)]\omega(x)\omega(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Since the kernel above is bounded and smooth outside the diagonal, the double integral above makes sense if the measure $\omega \otimes \omega$ doesn't charge the diagonal which is the case since $\omega \in H^{-1}_{loc}$ and Dirac masses are not in H^{-1}_{loc}.

The second main ingredient is to control the way how the vorticity doesn't charge the points. This control is contained in the following non-concentration lemma.

Lemma 1.2.5 *For all $T > 0$, there exists $C = C(\|u\|_{L^\infty(0,T;L^2_{loc})})$ such that*

$$\int_{B(x_0,r)} \omega(t,x)\,dx \le \frac{C}{\sqrt{|\log r|}} \qquad \textit{for all } t \in [0,T],\ r \in (0,1),\ x_0 \in \mathbb{R}^2.$$

Proof of lemma. Let

$$h_r(x) = \begin{cases} 1, & |x| < r; \\ \frac{\log|x|}{\log\sqrt{r}} - 1, & r \le |x| \le \sqrt{r}; \\ 0, & |x| \ge r. \end{cases}$$

Then h_r is a continuous and nonnegative function such that $\|\nabla h_r\|_{L^2} \le \frac{C}{\sqrt{|\log r|}}$. The desired bound follows from an integration by parts and a simple estimate.

$$\begin{aligned} \int_{B(x_0,r)} \omega(t,x)\,\mathrm{d}x &\le \int h_r(x-x_0)\omega(t,x)\,\mathrm{d}x = \int h_r(x-x_0)\,\mathrm{curl}\,u(t,x)\,\mathrm{d}x \\ &= \int u(t,x)\cdot\nabla^\perp h_r(x-x_0)\,\mathrm{d}x \le \|u\|_{L^2(B(x_0,1))}\|\nabla h_r\|_{L^2} \le \frac{C}{\sqrt{|\log r|}}\|u\|_{L^\infty(0,T;L^2_{loc})}. \end{aligned}$$

□

The passing to the limit with a standard approximation scheme is now easy since what is not on the diagonal passes to the limit immediately and what is on the diagonal gives no contribution because of the above lemma. □

Chapter 2

Some examples of solutions

In order to understand the large time behavior of solutions, a good starting point is to look at the available examples. However, the smooth examples are not so numerous and rather difficult to examine. On the other hand, there exists an approximation of the Euler equations called the vortex model which is a system of ordinary differential equations much more tractable from the point of view of examples.

The aim of this chapter is to examine all types of large time behavior that can be observed in examples of solutions of the vortex model and of the Euler equation. We start with the richer case of the vortex model and end with the more complicated case of smooth solutions of the Euler equations.

2.1 Discrete examples, the vortex model

The vortex model corresponds to vorticity that is a sum of Dirac measures of some points:

$$\omega(t,x) = \sum_{i=1}^{k} a_i \delta_{z_i(t)}.$$

Accordingly, for $x \notin \{z_1, z_2, \ldots, z_k\}$, the associated velocity is

$$u(x) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{2\pi|x-y|^2}\omega(y)\,\mathrm{d}y = \sum_{j=1}^{k} a_j \frac{(x-z_j)^\perp}{2\pi|x-z_j|^2}.$$

The problem is of course how to define the velocity of each of the points $z_1, z_2, \ldots, z_k$ since the velocity is clearly not defined on each of these points. The vortex model consists simply in ignoring the undefined terms and therefore reads

$$z_i' = \sum_{j\in\{1,\ldots,k\}\setminus\{i\}} a_j \frac{(z_i-z_j)^\perp}{2\pi|z_i-z_j|^2}, \qquad i \in \{1,\ldots,k\}. \tag{2.1}$$

This system of ordinary differential equations holds similar conservations as the Euler equations, namely:

- center of mass $\sum a_i z_i$;
- moment of inertia $\sum a_i |z_i|^2$;
- generalized energy $\sum\limits_{i\neq j} a_i a_j \log|z_i - z_j|$.

Global existence of solutions for the vortex model holds for almost every initial data (meaning that the set of initial data leading to blow-up is of vanishing Lebesgue measure) but not for every data. An example of collapse will be given in subsection 2.1.5. We refer to the excellent book by Marchioro and Pulvirenti [37] for a nice presentation and results on the vortex model, and more generally on perfect incompressible flow.

2.1.1 Justification of the model

First we note that the solution of the vortex model is not a solution of the Euler equation in the sense of distributions. The reason is that the velocity is not locally square integrable as it would be required in order to define the terms $u_i u_j$ that appear in the Euler equation. Nevertheless, it can be considered as a good discrete approximation for the Euler system.

Formally, this can be justified in the following way. The vortex approximation consists in ignoring the term $\frac{(x-z_i)^\perp}{2\pi|x-z_i|^2}$ when it comes to define the velocity of the point z_i. But this contribution is just rotation about z_i (faster and faster as x approaches z_i) so it shouldn't affect z_i itself.

Rigorously, the first complete justification is due to Marchioro and Pulvirenti [36] and was later improved by Marchioro [34] and Serfati [47]. It consists in proving that if the initial vorticity is localized and converges to a sum of Dirac masses in a certain way not too restrictive, then at later times it will stay localized and converge to a sum of Dirac masses that are the solutions to the vortex system. More precisely, we have the following theorem.

Theorem 2.1.1 (Serfati) *Suppose that* $\omega_\varepsilon(0) = \sum\limits_{j=1}^{k} \omega_\varepsilon^j(0)$ *and* $z_1(0), \dots, z_k(0)$ *are distinct points such that*

- $\omega_\varepsilon^j(0)$ *has definite sign;*
- $\operatorname{supp} \omega_\varepsilon^j(0) \subset D\big(z_j(0), \varepsilon\big)$*;*
- $\|\omega_\varepsilon^j(0)\|_{L^1} = a_j$*;*
- $|\omega_\varepsilon(0)| \leq \frac{C}{\varepsilon^k}$ *for some arbitrary* $k \in \mathbb{N}$*.*

Let $\omega_\varepsilon^j(t)$ *denote the time evolution of* $\omega_\varepsilon^j(0)$ *and* $\sum\limits_{j=1}^{k} a_j \delta_{z_j(t)}$ *the solution of the vortex model with initial data* $\sum\limits_{j=1}^{k} a_j \delta_{z_j(0)}$*. Then for any* $T > 0$ *and* $\mu < \frac{1}{2}$ *there exists a constant*

$C_1 = C_1(T, \mu)$ *such that*

$$\operatorname{supp} \omega_\varepsilon^j(T) \subset D\big(z_j(T), C_1 \varepsilon^\mu\big).$$

Moreover, for any $T \geq 0$, *we have the following weak convergence in the sense of measures:*

$$\omega_\varepsilon(T, \cdot) \rightharpoonup \sum_{j=1}^k a_j \delta_{z_j(T)} \qquad as \quad \varepsilon \to 0.$$

2.1.2 The case when all masses are positive

If all masses a_i are positive, then the conservation of the moment of inertia implies that the trajectories $z_j(t)$ stay bounded. Moreover, the conservation of the generalized energy also shows that collapse cannot occur as this would require blow-up of the generalized energy. We infer that the right-hand side of (2.1) stays bounded and therefore global existence of solutions of the vortex model holds in the case of positive masses and *no spreading* of the vortices is observed.

2.1.3 Discrete vortex pairs

We call discrete vortex pairs a couple of two vortices with vanishing sum of masses. The motion in this case is translation with constant velocity parallel to the perpendicular bisector of the segment formed by the vortices. More precisely, suppose that $z_1(0) = (0, \alpha)$, $z_2(0) = (0, -\alpha)$, $a_1 = a > 0$ and $a_2 = -a$. The vortex system then reads

$$z_1(t) = \big(x(t), \alpha\big), \qquad z_2(t) = \big(x(t), -\alpha\big), \qquad x'(t) = \frac{a}{4\pi\alpha}.$$

The important thing to note that, in contrast to the positive masses case, the vortices move linearly to infinity.

2.1.4 Vortices with diameter growing linearly

The previous example shows a couple of vortices moving fast to infinity. However, the distance between the two vortices stays bounded. Is there any configuration showing linear growth of the distance between the two vortices too? The answer is yes and here is an example. Consider $z = (x, y)$ a point vortex of mass $a > 0$ situated in the first quadrant and extend it by symmetry with respect to the axis of coordinates and the masses by antisymmetry. In other words,

$$\omega = a\delta_{(x,y)} - a\delta_{(x,-y)} + a\delta_{(-x,-y)} - a\delta_{(-x,y)}, \qquad a > 0.$$

This special symmetry is preserved by the flow and the vortex model simply reads

$$x' = \frac{ax^2}{4\pi y(x^2 + y^2)}, \qquad y' = -\frac{ay^2}{4\pi x(x^2 + y^2)}.$$

Therefore, x increases and y decreases. From the conservation of the generalized energy we see that the quantity $\frac{1}{x^2}+\frac{1}{y^2}$ is conserved, so the minimum distance between vortices has a positive lower bound. We infer that $\lim\limits_{t\to\infty} y(t)>0$ and, since x has a limit at infinity too, it follows that $x'=\frac{ax^2}{4\pi y(x^2+y^2)}$ has a finite limit. This shows that $x(t)\simeq O(t)$ and so does the diameter of this configuration since it equals $2\sqrt{x^2+y^2}$.

2.1.5 Collapse and special growth

We end this sequence of discrete examples with a configuration that can be found in [37] and that leads on one hand to collapse and on the other hand to a peculiar kind of growth. We consider an initial configuration of three point vortices

$$\omega=a_1\delta_{z_1}+a_2\delta_{z_2}+a_3\delta_{z_3}$$

such that

$$a_1a_2+a_2a_3+a_3a_1=0$$

and

$$a_1a_2|z_1-z_2|^2+a_2a_3|z_2-z_3|^2+a_3a_1|z_3-z_1|^2=0.$$

According to the known conservation laws, the above quantity is conserved and therefore it will vanish for all times. Under this assumption it is not difficult to check that

$$\frac{d}{dt}\Big(\frac{|z_1-z_2|^2}{|z_1-z_3|^2}\Big)=\frac{d}{dt}\Big(\frac{|z_1-z_2|^2}{|z_2-z_3|^2}\Big)=\frac{d}{dt}\Big(\frac{|z_1-z_3|^2}{|z_2-z_3|^2}\Big)=0.$$

This means that the triangle formed by these vortices changes only in size by similitude. We infer from this observation that

$$\frac{d}{dt}|z_1-z_2|^2=\frac{2Aa_3}{\pi}\Big[\frac{1}{|z_2-z_3|^2}-\frac{1}{|z_1-z_3|^2}\Big]=\text{ constant in time},$$

where A is the area of the triangle formed by the three vortices. Setting

$$M=\frac{2A(0)a_3}{\pi}\Big[\frac{1}{|z_2(0)-z_3(0)|^2}-\frac{1}{|z_1(0)-z_3(0)|^2}\Big]$$

we get

$$|z_1-z_2|^2=|z_1(0)-z_2(0)|^2+Mt.$$

Depending on the sign of M, that is on the one of a_3, we get one of the following two peculiar situations:

- either $M<0$ which implies that the three vortices collapse at time $t=-\frac{|z_1(0)-z_2(0)|^2}{M}$;

- or $M > 0$ which shows growth of the distance between the vortices as $O(t^{\frac{1}{2}})$.

An example of such an initial configuration is given by $a_1 = a_2 = 2$, $a_3 = -1$, $z_1(0) = (-1, 0)$, $z_2(0) = (1, 0)$, $z_3(0) = (1, \sqrt{2})$. Even though the growth is of only $O(t^{\frac{1}{2}})$ instead of $O(t)$ as observed in the previous subsection, the interest of this example stems from the fact that the total mass is non-zero. The significance of this will be obvious in section 4.1, see Remark 4.1.2.

2.2 Smooth examples

Smooth examples are much more difficult to obtain. To exhibit similar large time behavior as in the previous section is not always possible and when it is possible it requires a nontrivial proof, not just simple observations and calculations. For instance, we cannot prove that a smooth nonnegative vorticity has support bounded in time; for more details we refer to section 3.1. What we can do, is to prove that the smooth versions of the examples from subsections 2.1.3 and 2.1.4 retain some of the properties of their discrete counterparts and this is our aim for the rest of this chapter.

2.2.1 Vortex pairs and nonnegative vorticity in the half plane

The initial-boundary value problem for the incompressible 2D Euler equations in the half-plane (1.1) with bounded initial vorticity ω_0 is globally well-posed since it is equivalent, through the method of images, to an initial-value problem in the full-plane, with bounded, compactly supported initial vorticity (shown to be well-posed by Yudovich in [51]). The method of images consists of the observation that the Euler equations are covariant with respect to mirror-symmetry. Thus an initial vorticity which is odd with respect to reflection about the horizontal axis will remain so, and give rise to flow under which the half-plane is invariant. Conversely, the odd extension, with respect to $x_2 = 0$, of vorticity in half-plane flow gives rise to full-plane flow. This observation is especially useful in order to deduce the Biot-Savart law for half-plane flow, to recover velocity from vorticity.

Steady vortex pairs are a remarkable example of exact smooth solutions whose motion is just translation at constant speed without deformation (*i.e.* traveling waves). The initial vorticity is antisymmetric with respect to some axis of symmetry and has definite sign on each side of the axis. An explicit example can be found in [2] p.534, while some mathematical studies can be found in [6, 24, 39]. The sign and antisymmetry hypothesis given above are of course not sufficient to define a steady vortex pair; we call it just vortex pair. In fact it is equivalent to the motion of nonnegative vorticity in the half-plane. However, it can be proved that for any vortex pair, the center of mass behaves like the one of a steady vortex pair, meaning that it is exactly like $O(t)$. More precisely, it is proved in [23] the following theorem.

Theorem 2.2.1 *Consider the Euler equation in the half-plane $x_2 > 0$. Suppose that the initial vorticity is nonnegative and compactly supported, $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then* *the center of mass $P(t) = \int x\omega(t,x)\,dx$ is moving parallel to the boundary with a velocity bounded from below by a positive constant. In other words, there exists a constant $C > 0$ such that $P_2 = cst.$ and $P_1(t) \geq Ct$ for t sufficiently large.*

Proof. In the following, $C, C_1, \ldots$ denote some constants which may depend on ω_0 and may change from one line to another. The set D denotes the half-plane $x_2 > 0$.

The following lemma will be very useful in the sequel.

Lemma 2.2.2 *Let $a \in (0,2)$, $S \subset \mathbb{R}^2$ and $h : S \to \mathbb{R}_+$ be a function belonging to $L^1(S) \cap L^p(S)$, $p > \frac{2}{2-a}$. Then*

$$\int\limits_S \frac{h(y)}{|x-y|^a}\,dy \leq C\|h\|_{L^1(S)}^{\frac{2-a-2/p}{2-2/p}} \|h\|_{L^p(S)}^{\frac{a}{2-2/p}}.$$

Proof of lemma. Let $k \in \mathbb{N}$ be arbitrary. We can bound by Hölder's inequality

$$\begin{aligned}
\int\limits_S \frac{h(y)}{|x-y|^a}\,dy &= \int\limits_{S\cap\{|x-y|>k\}} \frac{h(y)}{|x-y|^a}\,dy + \int\limits_{S\cap\{|x-y|<k\}} \frac{h(y)}{|x-y|^a}\,dy \\
&\leq \frac{\|h\|_{L^1(S)}}{k^a} + \|h\|_{L^p(S)} \left\| \frac{1}{|x|^a} \right\|_{L^{\frac{p}{p-1}}(|x|\leq k)} \\
&= \frac{\|h\|_{L^1(S)}}{k^a} + C\|h\|_{L^p(S)} k^{2-a-2/p}.
\end{aligned}$$

The choice $k = \left(\|h\|_{L^1(S)} \|h\|_{L^p(S)}^{-1} \right)^{\frac{1}{2-2/p}}$ completes the proof of the lemma. □

Let us return to the proof of Theorem 2.2.1. First remark that the conservations of the center of mass and moment of inertia are no longer true in D. We assume for simplicity that $\int\limits_D \omega(x)\,dx = 1$. It is not difficult to see that if we extend ω by antisymmetry with respect to the axis $x_2 = 0$, then the resulting vorticity obeys the Euler equations in $\mathbb{R}^2$. The Biot-Savart law in the full plane therefore gives the Biot-Savart law in $x_2 > 0$:

$$v(x) = \frac{1}{2\pi} \int\limits_D \left(\frac{(x-y)^\perp}{|x-y|^2} - \frac{(x-\overline{y})^\perp}{|x-\overline{y}|^2} \right) \omega(y)\,dy,$$

where $\overline{y} = (y_1, -y_2)$ denotes the complex conjugate of y. A very simple calculation now shows that

$$v_1(x) = \frac{1}{2\pi} \int\limits_D \left(-\frac{x_2 - y_2}{|x-y|^2} + \frac{x_2 + y_2}{|x-\overline{y}|^2} \right) \omega(y)\,dy \tag{2.2}$$

$$v_2(x) = \frac{2}{\pi} \int\limits_D \frac{(x_1 - y_1)x_2 y_2}{|x-y|^2 |x-\overline{y}|^2} \omega(y)\,dy. \tag{2.3}$$

Let

$$P(t) = \int\limits_D x\omega(t,x)\,dx,$$

be the center of mass of the vorticity. We get from (1.1) and after an integration by parts that

$$P'(t) = \int\limits_D x\partial_t\omega(t,x)\,dx = -\int\limits_D xv(x)\cdot\nabla\omega(x)\,dx = \int\limits_D v(x)\omega(x)\,dx. \tag{2.4}$$

The Biot-Savart law (2.2)–(2.3) now implies that

$$\begin{aligned} P_1'(t) &= \frac{1}{2\pi}\iint\limits_{D^2} \frac{x_2+y_2}{|x-\overline{y}|^2}\omega(x)\omega(y)\,dx\,dy, \\ P_2'(t) &= 0, \end{aligned}$$

where we have used the antisymmetry of the expressions $\frac{x_2-y_2}{|x-y|^2}\omega(x)\omega(y)$ and $\frac{(x_1-y_1)x_2y_2}{|x-y|^2|x-\overline{y}|^2}\omega(x)\omega(y)$ with respect to the change of variables $(x,y) \longleftrightarrow (y,x)$. We immediately obtain a new conservation law.

$$\int\limits_D x_2\omega(x)\,dx = cst.$$

Let us now prove that there exists a constant $C>0$ such that $P_1' \geq C$. For notational convenience, we denote by ω the extension of the vorticity by antisymmetry with respect to the axis $x_2 = 0$. Since this new vorticity verifies the Euler equations in $\mathbb{R}^2$, the following (generalized) energy is conserved.

$$\begin{aligned} E_0 &= -\frac{1}{2\pi}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y|\omega(x)\omega(y)\,dx\,dy \\ &= \frac{1}{2\pi}\iint\limits_{D^2} \log\frac{|x-\overline{y}|^2}{|x-y|^2}\omega(x)\omega(y)\,dx\,dy \\ &= \frac{1}{2\pi}\iint\limits_{D^2} \log\left(1+\frac{4x_2y_2}{|x-y|^2}\right)\omega(x)\omega(y)\,dx\,dy. \end{aligned}$$

Note that the kernel above is nonnegative, in contrast to what happens for a nonnegative vorticity in $\mathbb{R}^2$. An application of Hölder's inequality gives

$$E_0 \leq C(P_1')^{1-1/q}\left[\iint\limits_{D^2}\left(\frac{|x-\overline{y}|^2}{x_2+y_2}\right)^{q-1}\left[\log\left(1+\frac{4x_2y_2}{|x-y|^2}\right)\right]^q\omega(x)\omega(y)\,dx\,dy\right]^{1/q},$$

with $q > 1$ to be chosen later. We now use the obvious inequality $\log(1+t) \le C\frac{t}{(1+t)^\alpha}$, $1 - 1/q \le \alpha < 1$, with $t = \frac{4x_2y_2}{|x-y|^2}$ which implies $1 + t = \frac{|x-\overline{y}|^2}{|x-y|^2}$. We therefore get

$$\iint_{D^2} \left(\frac{|x-\overline{y}|^2}{x_2+y_2}\right)^{q-1} \left[\log\left(1 + \frac{4x_2y_2}{|x-y|^2}\right)\right]^q \omega(x)\omega(y)\,dx\,dy$$

$$\le C \iint_{D^2} \frac{|x-\overline{y}|^{2q-2} x_2^q y_2^q}{(x_2+y_2)^{q-1}|x-y|^{2q-2\alpha q}|x-\overline{y}|^{2\alpha q}} \omega(x)\omega(y)\,dx\,dy$$

$$\le C \iint_{D^2} \frac{(x_2+y_2)^{3q-2\alpha q-1}}{|x-y|^{2q-2\alpha q}} \omega(x)\omega(y)\,dx\,dy$$

$$= C \iint_{D^2} \frac{x_2+y_2}{|x-y|^{2-q}} \omega(x)\omega(y)\,dx\,dy$$

$$= 2C \int_D x_2\omega(x)\Big(\int_D \frac{1}{|x-y|^{2-q}}\omega(y)\,dy\Big)\,dx$$

where we have chosen $\alpha = 3/2 - 1/q$ which is allowed if $q < 2$. Lemma 2.2.2 therefore yields

$$E_0 \le C(P_1')^{1-1/q}P_2^{1/q} = CP_2(0)^{1/q}(P_1')^{1-1/q},$$

from which we deduce that P_1' is bounded from below by a positive constant. Let us also note that the velocity v being bounded in space and time and relation (2.4) implies that P_1' is bounded by another constant. □

2.2.2 Smooth vorticity with diameter growing linearly

The aim of this subsection is to present an example of vorticity, with indefinite sign, whose support grows like $\mathcal{O}(t)$. This rate is optimal since the growth can be at most linear in time. The initial vorticity is not positive, rather it consists of four blobs, identical except for alternating sign, located symmetrically in the four quadrants. The initial configuration is inspired by two examples. First, the discrete analog of this set-up was investigated above in subsection 2.1.4 and the point vortices are seen to spread at a rate of $\mathcal{O}(t)$. Secondly, at the other extreme, Bahouri and Chemin [1] consider an example for which the initial vorticity is piecewise constant with alternating values ± 1 in the unit square of the four quadrants. There one finds rapid loss of Hölder regularity of the flow map. The motion in our example restricts to a solution of the Euler equations in the first quadrant with slip boundary conditions. The proof will show that the center of the mass located in the first quadrant moves at a rate of $\mathcal{O}(t)$. In this case, the conservation of the center of mass and moment of inertia are no longer useful since both quantities vanish. Instead, we shall use conservation of energy.

Let us denote the first quadrant by $\mathbf{Q}$. Let $\widetilde{\omega}_0$ be a nonnegative function, belonging to L^∞, compactly supported in $\mathbf{Q}$. We denote $m_0 = \int \widetilde{\omega}_0(x)\,dx$, $M_0 = \|\widetilde{\omega}_0\|_{L^\infty}$, and

$\mathbf{P}_0 = \int x\,\widetilde{\omega}_0(x)\,dx$. Our example of initial vorticity is a function antisymmetric with respect with both coordinate axes and equal to $\widetilde{\omega}_0$ in the first quadrant. In other words, using $\overline{x}$ for the complex conjugate of x, we define $\omega_0(x) = \widetilde{\omega}_0(x)$ for $x \in \mathbf{Q}$ and extend ω_0 to $\mathbb{R}^2$ so as to have $\omega_0(x) = -\omega_0(\overline{x}) = -\omega_0(-\overline{x}) = \omega_0(-x)$. We shall prove the following theorem from [22].

Theorem 2.2.3 *There exists a constant $C_0 = C_0(m_0, M_0, \mathbf{P}_0)$ such that, for every time t, the diameter, $d(t)$, of the support of the vorticity evolved from ω_0 satisfies $d(t) \geq C_0 t$.*

Proof. By uniqueness, the vorticity $\omega(t,x)$ preserves the antisymmetry of the initial data,

$$\omega(t,x) = -\omega(t,\overline{x}) = -\omega(t,-\overline{x}) = \omega(t,-x).$$

Moreover, the flow map is antisymmetric, and so it leaves each quadrant and both coordinate axes invariant. Consequently, we have

$$\int_{\mathbf{Q}} \omega(t,x)\,dx = \int_{\mathbf{Q}} \omega(0,x)\,dx = \int_{\mathbf{Q}} \widetilde{\omega}_0(x)\,dx = m_0. \tag{2.5}$$

We shall consider the evolution of the center of mass of $\omega(t,x)$ restricted to $\mathbf{Q}$ defined by

$$\mathbf{P}(t) = \frac{1}{m_0}\int_{\mathbf{Q}} x\,\omega(t,x)\,dx.$$

Let $\mathbf{P}(t) = (P_1(t), P_2(t))$. The support of ω has a non-empty intersection with the region $\{x_1 \geq P_1\}$. Therefore, the symmetry properties of $\omega(t,x)$ imply that the diameter of the support of the vorticity is bounded by below by $P_1(t)$. So, in order to prove Theorem 2.2.3, it is enough to prove that $P_1(t) \geq C_0(m_0, M_0, \mathbf{P}_0)t$. In the course of the proof, we shall also see that $P_1(t)$ is increasing and that $P_2(t)$ is decreasing.

From the Biot-Savart law (3.3) along with the obvious changes of coordinates, we deduce

$$\begin{aligned} v(x) &= \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)\,dy \\ &= \int_{\mathbf{Q}} \left(\frac{(x-y)^\perp}{|x-y|^2} + \frac{(x+y)^\perp}{|x+y|^2} - \frac{(x-\overline{y})^\perp}{|x-\overline{y}|^2} - \frac{(x+\overline{y})^\perp}{|x+\overline{y}|^2} \right)\omega(y)\,dy. \end{aligned}$$

Separating the components, we can further write

$$\begin{aligned} v_1(x) = \int_{\mathbf{Q}} \Bigg[&-(x_2-y_2)\left(\frac{1}{|x-y|^2} - \frac{1}{|x+\overline{y}|^2}\right) \\ &+ (x_2+y_2)\left(\frac{1}{|x-\overline{y}|^2} - \frac{1}{|x+y|^2}\right)\Bigg]\omega(y)\,dy \\ v_2(x) = \int_{\mathbf{Q}} \Bigg[&(x_1-y_1)\left(\frac{1}{|x-y|^2} - \frac{1}{|x-\overline{y}|^2}\right) \\ &+ (x_1+y_1)\left(\frac{1}{|x+y|^2} - \frac{1}{|x+\overline{y}|^2}\right)\Bigg]\omega(y)\,dy. \end{aligned} \tag{2.6}$$

Differentiating $\mathbf{P}(t)$, using the vorticity equation (3.2), and integrating by parts implies

$$\mathbf{P}'(t) = \frac{1}{m_0}\int_{\mathbf{Q}} x\,\partial_t\omega(t,x)\,dx = \frac{1}{m_0}\int_{\mathbf{Q}} v(t,x)\,\omega(t,x)\,dx.$$

Furthermore, according to the modified Biot-Savart law (2.6), we obtain

$$\begin{aligned}
P_1' = \frac{1}{m_0}\iint\limits_{\mathbf{Q}^2}\Big[&-(x_2-y_2)\Big(\frac{1}{|x-y|^2}-\frac{1}{|x+\overline{y}|^2}\Big)\\
&+(x_2+y_2)\Big(\frac{1}{|x-\overline{y}|^2}-\frac{1}{|x+y|^2}\Big)\Big]\omega(x)\omega(y)\,dx\,dy\\
P_2' = \frac{1}{m_0}\iint\limits_{\mathbf{Q}^2}\Big[&(x_1-y_1)\Big(\frac{1}{|x-y|^2}-\frac{1}{|x-\overline{y}|^2}\Big)\\
&+(x_1+y_1)\Big(\frac{1}{|x+y|^2}-\frac{1}{|x+\overline{y}|^2}\Big)\Big]\omega(x)\omega(y)\,dx\,dy.
\end{aligned} \tag{2.7}$$

Interchanging the coordinates, $x \leftrightarrow y$, yields

$$\begin{aligned}
&\iint\limits_{\mathbf{Q}^2}(x_2-y_2)\Big(\frac{1}{|x-y|^2}-\frac{1}{|x+\overline{y}|^2}\Big)\omega(x)\omega(y)\,dx\,dy\\
&\qquad = -\iint(x_2-y_2)\Big(\frac{1}{|x-y|^2}-\frac{1}{|x+\overline{y}|^2}\Big)\omega(x)\omega(y)\,dx\,dy,
\end{aligned}$$

so

$$\iint\limits_{\mathbf{Q}^2}(x_2-y_2)\Big(\frac{1}{|x-y|^2}-\frac{1}{|x+\overline{y}|^2}\Big)\omega(x)\omega(y)\,dx\,dy = 0.$$

In a similar manner, we see that

$$\iint\limits_{\mathbf{Q}^2}(x_1-y_1)\Big(\frac{1}{|x-y|^2}-\frac{1}{|x-\overline{y}|^2}\Big)\omega(x)\omega(y)\,dx\,dy = 0.$$

We conclude that relation (2.7) can be now written as

$$\begin{aligned}
P_1' &= \frac{1}{m_0}\iint\limits_{\mathbf{Q}^2}\frac{4x_1y_1(x_2+y_2)}{|x-\overline{y}|^2|x+y|^2}\omega(x)\omega(y)\,dx\,dy\\
P_2' &= -\frac{1}{m_0}\iint\limits_{\mathbf{Q}^2}\frac{4x_2y_2(x_1+y_1)}{|x+y|^2|x+\overline{y}|^2}\omega(x)\omega(y)\,dx\,dy.
\end{aligned} \tag{2.8}$$

The first thing to remark is that P_1 is increasing and P_2 is decreasing.

The second main ingredient is conservation of energy. When the velocity lies in L^2, its norm is equivalent to the quantity

$$E_0 = -\frac{1}{2\pi} \iint\limits_{\mathbb{R}^2 \times \mathbb{R}^2} \log|x-y|\, \omega(x)\omega(y)\, dx\, dy.$$

However, it can be seen directly that the latter integral is a constant of the motion. Thanks to the symmetry, a few changes of coordinates reduce the integration to the first quadrant

$$E_0 = \frac{2}{\pi} \iint\limits_{\mathbf{Q}^2} \log \frac{|x-\overline{y}||x+\overline{y}|}{|x-y||x+y|}\, \omega(x)\omega(y)\, dx\, dy.$$

The kernel is nonnegative, since we can write

$$\begin{aligned} \log \frac{|x-\overline{y}||x+\overline{y}|}{|x-y||x+y|} &= \frac{1}{2} \log \frac{|x-\overline{y}|^2|x+\overline{y}|^2}{|x-y|^2|x+y|^2} \\ &= \frac{1}{2} \log \left(1 + \frac{|x-\overline{y}|^2|x+\overline{y}|^2 - |x-y|^2|x+y|^2}{|x-y|^2|x+y|^2} \right) \\ &= \frac{1}{2} \log \left(1 + \frac{16 x_1 y_1 x_2 y_2}{|x-y|^2|x+y|^2} \right). \end{aligned} \tag{2.9}$$

Taking $1/p + 1/q = 1$, with $1 < q < 2$, Hölder's inequality along with relation (2.8) imply

$$E_0^p \le C\, m_0\, P_1'\, I^{1/(q-1)}, \tag{2.10}$$

in which

$$I \equiv \iint\limits_{\mathbf{Q}^2} \left[\frac{|x-\overline{y}|^2|x+y|^2}{x_1 y_1 (x_2+y_2)} \right]^{q-1} \left[\log \frac{|x-\overline{y}||x+\overline{y}|}{|x-y||x+y|} \right]^q \omega(x)\omega(y)\, dx\, dy. \tag{2.11}$$

In the following, we will derive an upper bound for the integral I.

Since the logarithm grows more slowly than any power, given $0 < \alpha < 1$, there is a constant C_α such that $\log(1+z) \le C_\alpha z/(1+z)^\alpha$, for all $z > 0$. Therefore, using (2.9), the logarithm has the bound

$$\begin{aligned} \log \left(1 + \frac{16 x_1 y_1 x_2 y_2}{|x-y|^2|x+y|^2} \right) &\le C \frac{x_1 y_1 x_2 y_2}{|x-y|^2|x+y|^2} \left[\frac{|x-\overline{y}|^2|x+\overline{y}|^2}{|x-y|^2|x+y|^2} \right]^{-\alpha} \\ &= C \frac{x_1 y_1 x_2 y_2}{|x-y|^{2(1-\alpha)}|x+y|^{2(1-\alpha)}|x-\overline{y}|^{2\alpha}|x+\overline{y}|^{2\alpha}}. \end{aligned}$$

From (2.11), this leads to the upper bound

$$I \le C \iint\limits_{\mathbf{Q}^2} \frac{x_1 y_1 (x_2 y_2)^q |x+y|^{2\alpha q - 2}}{(x_2+y_2)^{q-1}|x-y|^{2q(1-\alpha)}|x+\overline{y}|^{2\alpha q}|x-\overline{y}|^{2-2q(1-\alpha)}} \times \omega(x)\omega(y)\, dx\, dy.$$

If we agree to take $\alpha = 1/q$, then this simplifies to

$$I \leq C \iint\limits_{\mathbf{Q}^2} \frac{x_1 y_1 (x_2 y_2)^q}{(x_2+y_2)^{q-1}|x-y|^{2(q-1)}|x+\overline{y}|^2|x-\overline{y}|^{2(2-q)}} \omega(x)\omega(y)\, dx\, dy.$$

Now the trivial inequalities

$$x_1 y_1 \leq (x_1+y_1)^2 \leq |x+\overline{y}|^2 \quad \text{and} \quad x_2 y_2 \leq (x_2+y_2)^2 \leq |x-\overline{y}|^2$$

ensure that

$$I \leq C \iint\limits_{\mathbf{Q}^2} \frac{(x_2+y_2)^{3(q-1)}}{|x-y|^{2(q-1)}} \omega(x)\omega(y)\, dx\, dy.$$

If $q \leq 6/5$, so that $5(q-1) \leq 1$, we can apply Hölder's inequality to get

$$I \leq C I_1^{1-5(q-1)} I_2^{2(q-1)} I_3^{3(q-1)},$$

with

$$I_1 = \iint\limits_{\mathbf{Q}^2} \omega(x)\omega(y)\, dx\, dy, \quad I_2 = \iint\limits_{\mathbf{Q}^2} \frac{1}{|x-y|} \omega(x)\omega(y)\, dx\, dy,$$

$$I_3 = \iint\limits_{\mathbf{Q}^2} (x_2+y_2)\omega(x)\omega(y)\, dx\, dy.$$

From (2.5), we have that $I_1 = m_0^2$. Lemma 2.2.2 with $a = 1$ and $p = \infty$ tells us that $I_2 \leq C m_0^{3/2} M_0^{1/2}$. Also, the monotonicity of P_2 gives $I_3 \leq C m_0^2 P_2(0)$. Altogether, we have the bound

$$I \leq C(q) m_0^2 \left[\frac{M_0 P_2(0)^3}{m_0} \right]^{q-1}.$$

Going back to (2.10), we obtain

$$P_1' \geq C_0 \equiv C(q) \left[\frac{E_0}{m_0^2} \right]^{1/(q-1)} \frac{E_0}{M_0 P_2(0)^3},$$

so that

$$P_1(t) \geq P_1(0) + C_0 t.$$

This completes the proof of Theorem 2.2.3. □

Chapter 3

When the vorticity is nonnegative: growth of the support

The confinement results for the vorticity depend heavily on the (unbounded) domain. We first treat the most important case, the full plane, and we discuss at the end what can be proved for other domains.

3.1 The case of the full plane

The evolution of ideal incompressible fluid vorticity preserves compactness of support. We saw in Section 1.2 that the initial value problem for the 2d incompressible Euler equations is globally well-posed in variety of settings. The divergence-free fluid velocity vector field $v(t, x)$ generates a particle flow map $\Phi(t, p)$ through the system of ODE's

$$\frac{d}{dt}\Phi(t,p) = v(t, \Phi(t,p)), \quad \Phi(0,p) = p, \tag{3.1}$$

such that the map $p \mapsto \Phi(t, p)$ is a continuously varying family of area-preserving diffeomorphisms of the plane. Recall that the scalar vorticity $\omega = \partial_1 v_2 - \partial_2 v_1$ is transported by this flow

$$D_t\, \omega = \partial_t\, \omega + v \cdot \nabla\omega = 0, \quad \omega(0, x) = \omega_0(x), \tag{3.2}$$

and the velocity is coupled to the vorticity through the Biot-Savart law

$$v(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2}\, \omega(t,y)\, dy. \tag{3.3}$$

Despite the successful existence theory, little can be said about the large time behavior of solutions. This is not surprising since point vortex approximations, even using small numbers of particles, can generate complex dynamics. Given that the vorticity is transported by a area-preserving flow (3.2), it follows that its L^p norms are constant in time. In the case of smooth data, Hölder regularity of the flow map is preserved in time, but the

Hölder norm of the flow map is only known to be bounded by an expression of the form $\exp(\exp Ct)$. Clearly, any growth in the Hölder norm of the flow map would be related to the evolution of compact regions under the flow.

If the initial vorticity is supported in a compact set $\Omega \subset \mathbb{R}^2$, then equation (3.2) shows that at time $t > 0$ the vorticity is supported in $\Omega(t) = \Phi(t, \Omega)$. Nothing can be said about the geometry of $\Omega(t)$. However in the case where the vorticity equals the characteristic function of a set with smooth boundary, the so-called vortex patch, Chemin [8] proved that the regularity of the boundary is propagated, see also [5]. A simple estimate from (3.3), given in Lemma 2.2.2, provides a uniform bound for the velocity, and so the support of the vorticity can grow at most linearly in time. For nonnegative initial vorticity, Marchioro [31] demonstrated that the conservation of the *moment of inertia*, $\int_{\mathbb{R}^2} |x|^2 \omega(t,x) dx$, further acts to constrain the spreading of the support to a rate of $\mathcal{O}(t^{1/3})$. This result was generalized to include vorticity in L^p for $2 < p < \infty$, in [28].

We will present in this section a result from [22] (see Theorem 3.1.1 below) which shows that Marchioro's bound for the growth rate of the support of nonnegative vorticity can be improved to $\mathcal{O}[(t \log t)^{1/4}]$ by taking into account not only the conservation of the moment of inertia but also the conservation of the center of mass, $\int_{\mathbb{R}^2} x\, \omega(t,x)\, dx$. Bounds for the flow map will come from an estimate for the radial component of the velocity starting from (3.3). The heart of the matter is to measure the vorticity in L^1 outside of balls centered at the origin, Proposition 3.1.2. The approach taken here is to estimate higher momenta of the vorticity following the idea of Gamblin included in the Appendix of [22]. The analysis applies to weak solutions in L^p, $2 < p \le \infty$. We also note that Serfati [47] has independently obtained a result similar to Theorem 3.1.1 with the factor $t^{1/4} \log \circ \cdots \circ \log t$ replacing $(t \log t)^{1/4}$.

There are a few examples of nonnegative explicit solutions, but none of these exhibit any growth of support. Spherically symmetric initial vorticity gives rise to a stationary solution whose velocity vector field induces flow lines which follow circles about the origin. The support of the Kirchoff elliptical vortex patch rotates with constant angular velocity, although the velocity vector field has a nontrivial structure exterior to the support, (see [25], p.232). We also note that numerical simulations starting with a pair of positively charged vortex patches show homogenization of the patches simultaneous with the formation of long filaments [7]. On the other hand, when the vorticity is not signed, we saw in subsection 2.2.2 that it is useless to look for confinements results.

We will make use of several quantities that are conserved by the time evolution, namely the total mass

$$\int \omega(t,x)\, dx = \int \omega_0(x)\, dx = m_0,$$

the maximum norm

$$\|\omega(t)\|_{L^\infty} = \|\omega_0\|_{L^\infty} = M_0,$$

the center of mass

$$\int x\, \omega(t,x)\, dx = \int x\, \omega_0(x)\, dx = c_0,$$

and the moment of inertia

$$\int |x|^2\omega(t,x)\,dx = \int |x|^2\omega_0(x)\,dx = i_0.$$

Assume that the support of ω_0 is contained in the ball centered at the origin of radius d_0. We are going to prove the following theorem.

Theorem 3.1.1 *Let $\omega(t,x)$ be the solution of the 2d incompressible Euler equations with a nonnegative compactly supported initial vorticity $\omega_0 \in L^\infty(\mathbb{R}^2)$. There exists a constant $C_0 = C_0(i_0, d_0, m_0, M_0)$ such that, for every time $t \geq 0$, the support of $\omega(t,\cdot)$ is contained in the ball $|x| < 4d_0 + C_0[t\log(2+t)]^{1/4}$.*

Proof. First, by making the change of variable $x \to x - \frac{c_0}{m_0}$, we may assume, without loss of generality, that the center of mass is located at the origin.

In the following estimates, constants will be independent of ω_0, unless otherwise indicated, and then the dependence will be only through the quantities i_0, d_0, m_0, and M_0. We will establish the theorem for classical solutions, and the general result, for weak solutions, follows immediately since these quantities are stable under passage to the weak limit. The time variable will often be suppressed since it plays no role in the estimation of the various convolution integrals.

We are going to show that the radial component of the velocity satisfies an estimate of the form

$$\left|\frac{x}{|x|}\cdot v(t,x)\right| \leq \frac{C_0}{|x|^3}, \quad \text{for all} \quad |x| \geq 4d_0 + C_0[t\log(2+t)]^{1/4}, \tag{3.4}$$

with $C_0 = C_0(i_0, d_0, m_0, M_0)$. The proof of the theorem concludes by noticing that the region

$$\{(t,x) : t \geq 0,\ |x| < 4d_0 + C_0[t\log(2+t)]^{1/4}\}$$

is invariant for the flow

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = v(t,x)$$

since the bound (3.4) implies that the vector field $(1, v(t,x))$ points inward along the boundary of this region.

We now turn to the verification of (3.4). The radial part of the velocity is

$$\frac{x}{|x|}\cdot v(x) = \frac{1}{2\pi}\int \frac{x}{|x|}\cdot\frac{(x-y)^\perp}{|x-y|^2}\omega(y)\,dy.$$

The last integral will be divided into two pieces.

The portion of the integral over the region $|x-y| < |x|/2$ is immediately seen to be bounded by

$$C\int_{|x-y|<|x|/2} \frac{\omega(y)}{|x-y|}\,dy.$$

Using that $x \cdot (x-y)^\perp = -x \cdot y^\perp$ and the fact that the center of mass is at the origin, we can express the other portion as

$$\begin{aligned}
\int\limits_{|x-y|>|x|/2} \frac{x}{|x|} \cdot \frac{(x-y)^\perp}{|x-y|^2}\,\omega(y)\,dy &= - \int\limits_{|x-y|>|x|/2} \frac{x \cdot y^\perp}{|x||x-y|^2}\,\omega(y)\,dy \\
&= - \int\limits_{|x-y|>|x|/2} \frac{x \cdot y^\perp}{|x|} \left(\frac{1}{|x-y|^2} - \frac{1}{|x|^2} \right) \omega(y)\,dy \\
&\quad + \int\limits_{|x-y|<|x|/2} \frac{x \cdot y^\perp}{|x|^3}\,\omega(y)\,dy \\
&= - \int\limits_{|x-y|>|x|/2} \frac{x \cdot y^\perp}{|x|} \, \frac{\langle y, 2x-y \rangle}{|x-y|^2|x|^2}\,\omega(y)\,dy \\
&\quad + \int\limits_{|x-y|<|x|/2} \frac{x \cdot y^\perp}{|x|^3}\,\omega(y)\,dy.
\end{aligned}$$

Next, we note that $|x-y| > |x|/2$ implies

$$|2x - y| \le |x-y| + |x| < 3|x-y|,$$

and so the first of these integrals is bounded as follows

$$\begin{aligned}
\Bigg| \int\limits_{|x-y|>|x|/2} \frac{x \cdot y^\perp}{|x|} \, \frac{\langle y, 2x-y \rangle}{|x-y|^2|x|^2}\,\omega(y)\,dy \Bigg| &\le \int\limits_{|x-y|>|x|/2} \frac{|y|^2|2x-y|}{|x|^2|x-y|^2}\,\omega(y)\,dy \\
&\le \frac{C}{|x|^3} \int\limits_{|x-y|>|x|/2} |y|^2\omega(y)\,dy \le \frac{Ci_0}{|x|^3}.
\end{aligned}$$

On the grounds of simple homogeneity, it is difficult to see how to improve this estimate using only the conserved quantities at hand.

As for the second piece, we use that $|x-y| < |x|/2$ gives $|y| \le 3|x|/2$ to write

$$\Bigg| \int\limits_{|x-y|<|x|/2} \frac{x \cdot y^\perp}{|x|^3} \omega(y)\,dy \Bigg| \le C \int\limits_{|x-y|<|x|/2} \frac{\omega(y)}{|x-y|}\,dy.$$

We have deduced the following estimate for the radial component of velocity

$$\left| \frac{x}{|x|} \cdot v(x) \right| \le \frac{Ci_0}{|x|^3} + C \int\limits_{|x-y|<|x|/2} \frac{\omega(y)}{|x-y|}\,dy. \tag{3.5}$$

The rest of the proof consists in showing that the last integral is negligible for large $|x|$.

From (3.5), Lemma 2.2.2 with $a = 1$, $p = \infty$, and the fact that

$$\{y : |x - y| < |x|/2\} \subset \{y : |y| > |x|/2\},$$

the estimate for the radial component of the velocity is

$$\left| \frac{x}{|x|} \cdot v(x) \right| \leq \frac{Ci_0}{|x|^3} + CM_0^{1/2} \Big(\int_{|y|>|x|/2} \omega(y)\, dy \Big)^{1/2}.$$

Given the following proposition, with $k = 6$, the last integral is also $\mathcal{O}(|x|^{-3})$ for $|x|$ large so that inequality (3.4) holds, and hence Theorem 3.1.1 is valid. □

Proposition 3.1.2 *There exists a constant $C_0 = C_0(i_0, d_0, m_0, M_0, k)$ such that for any $k > 0$*

$$\int_{|y|>|x|/2} \omega(t, y)\, dy \leq \frac{C_0}{|x|^k},$$

for all $|x| > 4d_0 + C_0[t \log(2 + t)]^{1/4}$.

Proof of the Proposition. In order to estimate the decay of the mass of vorticity far from the center of mass, we introduce the higher momenta:

$$m_n(t) = \int |x|^{4n} \omega(t, x)\, dx.$$

Although these are not conserved quantities, a recursive estimate holds for their derivatives leading to the following result.

Lemma 3.1.3 *There exists a constant C_0 such that for any $n \geq 1$*

$$m_n(t) \leq m_0(d_0^4 + C_0 i_0 n t)^n. \tag{3.6}$$

Assume, for the moment, that Lemma 3.1.3 is true and let us use it to prove Proposition 3.1.2. Fix $k \geq 1$, and suppose that

$$r^4 \geq 2 \left[d_0^4 + C_0 i_0 k t \log(2 + t) \right]. \tag{3.7}$$

Choose $n \geq k/4$ in such a way that

$$k \log(2 + t) - 1 < n \leq k \log(2 + t). \tag{3.8}$$

Recalling that the vorticity remains nonnegative during the motion, we have using (3.6), (3.7), and (3.8)

$$\begin{aligned} \int_{|x|\geq r} \omega(t, x)\, dx &\leq \frac{m_n(t)}{r^{4n}} \leq \frac{m_0}{r^k}\ \frac{(d_0^4 + C_0 i_0 n t)^n}{r^{4n-k}} \\ &\leq \frac{m_0}{r^k}\ 2^{k/4-n}\ \left[d_0^4 + C_0 i_0 k t \log(2 + t) \right]^{k/4}. \end{aligned}$$

Note that by (3.8), we have that $2^{n+1} \geq (2+t)^{k\log 2}$. This means that the right-hand side can be bounded above by $C(i_0, d_0, m_0, k)/r^k$ when (3.7) holds, and so Proposition 3.1.2 follows. $\square$

Proof of Lemma 3.1.3. Using the vorticity equation (2) and the Biot-Savart law (3.3), we have after some integrations by parts

$$m_n'(t) = \frac{2n}{\pi} \iint \frac{\langle x, (x-y)^\perp \rangle}{|x-y|^2} |x|^{4n-2} \omega(t,x)\omega(t,y)\,dx\,dy.$$

We define

$$K(x,y) = \langle x, (x-y)^\perp \rangle \left(\frac{1}{|x-y|^2} - \frac{1}{|x|^2} \right).$$

Since the center of mass is at the origin, we can write

$$m_n'(t) = \frac{2n}{\pi} \iint K(x,y) |x|^{4n-2} \omega(t,x)\omega(t,y)\,dx\,dy.$$

Let us consider the following partition of the plane:

$$A_1 = \left\{ (x,y) : |y| \leq \left(1 - \frac{1}{2n}\right) |x| \right\},$$
$$A_2 = \left\{ (x,y) : \left(1 - \frac{1}{2n}\right)|x| < |y| < \left(1 - \frac{1}{2n}\right)^{-1} |x| \right\},$$
$$A_3 = \left\{ (x,y) : |x| \leq \left(1 - \frac{1}{2n}\right) |y| \right\}.$$

Then, we have

$$m_n'(t) = \alpha_1(t) + \alpha_2(t) + \alpha_3(t)$$

with

$$\alpha_i = \frac{2n}{\pi} \iint_{A_i} K(x,y)|x|^{4n-2}\omega(t,x)\omega(t,y)\,dx\,dy.$$

We will study each of these three terms.

First, assume that $(x,y) \in A_1$ and write

$$K(x,y) = \langle y, (x-y)^\perp \rangle \frac{\langle y, 2x - y\rangle}{|x-y|^2|x|^2}.$$

Since $|x-y| \geq |x|/2n$ and $|2x-y| \leq 3|x|$, we have the inequality

$$|K(x,y)| \leq \frac{|y|^2|2x-y|}{|x|^2|x-y|} \leq 6n\,\frac{|y|^2}{|x|^2},$$

and we obtain the bound

$$|\alpha_1(t)| \le \frac{12n^2}{\pi} \iint\limits_{A_1} |x|^{4(n-1)}|y|^2 \omega(t,x)\omega(t,y)\,dx\,dy \le \frac{12n^2}{\pi}\, i_0 m_{n-1}(t).$$

Now, assume that $(x,y) \in A_3$. This implies that $|x-y| \ge |y|/2n$ and $(1-1/2n)|y|/|x| \ge 1$. The kernel $K(x,y)$ may be written as

$$K(x,y) = \frac{\langle x, (x-y)^\perp\rangle}{|x-y|^2} + \frac{\langle x, y^\perp\rangle}{|x|^2},$$

and we deduce that on A_3

$$|K(x,y)| \le \frac{|x|}{|x-y|} + \frac{|y|}{|x|} \le 2n\,\frac{|y|^2}{|x|^2}.$$

It follows that

$$|\alpha_3(t)| \le \frac{4n^2}{\pi}\, i_0 m_{n-1}(t).$$

Finally, we split the integral over A_2 into two terms

$$\alpha_2(t) = I_1(t) + I_2(t)$$

where

$$I_1(t) = -\frac{2n}{\pi} \iint\limits_{A_2} |x|^{4n-2} \frac{\langle x, y^\perp\rangle}{|x-y|^2} \omega(t,x)\omega(t,y)\,dx\,dy,$$

$$I_2(t) = \frac{2n}{\pi} \iint\limits_{A_2} |x|^{4(n-1)} \langle x, y^\perp\rangle \omega(t,x)\omega(t,y)\,dx\,dy.$$

In the region A_2, we have $|x| \le 2|y|$, and we can bound the second contribution, $I_2(t)$, by

$$|I_2(t)| \le \frac{4n}{\pi}\, i_0 m_{n-1}(t).$$

Now, observe that the region A_2 is symmetric with respect to the diagonal and that

$$H(x,y) \equiv \frac{\langle x, y^\perp\rangle}{|x-y|^2} = -H(y,x).$$

The integral $I_1(t)$ can be therefore rewritten as

$$I_1(t) = -\frac{n}{\pi} \iint\limits_{A_2} H(x,y)\left(|x|^{4n-2} - |y|^{4n-2}\right)\omega(t,x)\omega(t,y)\,dx\,dy.$$

To evaluate this integral, we first use the following identity

$$|x|^{4n-2} - |y|^{4n-2} = \langle x-y, x+y \rangle \sum_{j=0}^{2n-2} |x|^{4n-4-2j} |y|^{2j}.$$

Thus, in the region A_2, we find

$$\begin{aligned}\big||x|^{4n-2} - |y|^{4n-2}\big| &\leq 3|y||x-y||x|^{4(n-1)} \sum_{j=0}^{2n-2} \left(1 - \frac{1}{2n}\right)^{-2j} \\ &\leq 6n\,|y||x-y||x|^{4(n-1)}.\end{aligned}$$

On the other hand, we note that

$$|H(x,y)| = \frac{\big|\langle x-y, y^{\perp}\rangle\big|}{|x-y|^2} \leq \frac{|y|}{|x-y|}.$$

Combining the two last estimates yields

$$|I_1(t)| \leq \frac{6n^2}{\pi}\, i_0 m_{n-1}(t).$$

Summing up the bounds for α_1, α_3, I_1, and I_2, and then using Hölder's inequality we get

$$m_n'(t) \leq C_0 i_0 n^2 m_{n-1}(t) \leq C_0 i_0 n^2 m_0^{1/n} m_n(t)^{1-1/n}.$$

It follows that $m_n(t)$ can be estimated as claimed in (3.6). $\square$

3.2 Discussion of other cases

The influence of the boundary on the large time behavior of the vorticity is crucial. The result for the full plane case is clearly false for domains with boundaries. To convince ourselves, it is sufficient to remember that in subsection 2.2.1 it is proved that the center of mass of nonnegative vorticity in the half-plane behaves exactly like $O(t)$, so no confinement is possible. On the other hand, in the latter case not even the diameter can be estimated better than $O(t)$; this is suggested by the discrete example of section 4.2.4. In fact, the compactness of the boundary is extremely important. We discuss next two relevant cases: the exterior domain and the half plane.

3.2.1 The case of the half plane

As noted above, no complete confinement can be true. However, partial confinements can hold. We will discuss this in the following. Let us begin by fixing basic notation. We denote by $\mathbb{H}$ the horizontal half-plane given by $\mathbb{H} = \{x \in \mathbb{R}^2; x_2 > 0\}$. Reflection with respect to $x_2 = 0$ will be denoted by $x = (x_1, x_2) \mapsto \overline{x} = (x_1, -x_2)$. If $z = (z_1, z_2)$ then its perpendicular vector is $z^\perp = (-z_2, z_1)$. We use $L^p_c(\mathbb{H})$ to denote the Lebesgue space of p-th power integrable functions, $p \geq 1$, with compact support in $\mathbb{H}$. The dual of L^p is $L^{p'}$, with the conjugate exponent given by $p' = p/(p-1)$.

Let us fix an initial vorticity ω_0. We will assume in this part that ω_0 is a given non-negative function in $L^p_c(\mathbb{H})$ for some $p > 2$. If $\omega_0 \in L^p_c(\mathbb{H})$, $2 < p < \infty$, then we saw that there exists a weak solution u, ω of (1.1) associated with this initial vorticity (see [29]). Furthermore, $\omega(t,\cdot) \geq 0$, $t \geq 0$, and the L^1 and L^p-norms of $\omega(t,\cdot)$ are bounded by the L^1 and L^p-norms, respectively, of the initial vorticity. Using the method of images we can write the velocity u in terms of vorticity ω as:

$$u(t,x) = \int_{\mathbb{H}} \Big[\frac{(x-y)^\perp}{2\pi|x-y|^2} - \frac{(x-\overline{y})^\perp}{2\pi|x-\overline{y}|^2}\Big]\,\omega(t,y)\,\mathrm{d}y. \tag{3.9}$$

We denote the kernel appearing the integral above by:

$$K = K(x,y) = \frac{(x-y)^\perp}{2\pi|x-y|^2} - \frac{(x-\overline{y})^\perp}{2\pi|x-\overline{y}|^2}, \tag{3.10}$$

whose components are given explicitly by:

$$K_1(x,y) = \frac{y_2[y_2^2 - x_2^2 + (x_1-y_1)^2]}{\pi|x-\overline{y}|^2|x-y|^2} \quad \text{and} \quad K_2(x,y) = \frac{2(x_1-y_1)x_2y_2}{\pi|x-\overline{y}|^2|x-y|^2}. \tag{3.11}$$

It is easy to see that

$$|K(x,y)| \leq \frac{1}{\pi|x-y|}, \tag{3.12}$$

from which we can deduce the fact that, if $p > 2$, then an $L^1 \cap L^p$-vorticity ω gives rise to an L^∞-velocity u with the estimate:

$$\|u\|_{L^\infty(\mathbb{H})} \leq C\|\omega\|_{L^p(\mathbb{H})}^{p'/2}\|\omega\|_{L^1(\mathbb{H})}^{1-p'/2}$$

as can be seen from Lemma 2.2.2.

3.2.1.1 Vertical confinement

We start with a vertical confinement result that was proved in [23].

Theorem 3.2.1 *There exists a constant C such that for $x_2 \leq C(t\log t)^{\frac{1}{3}}$ for all $x \in$* $\operatorname{supp}\omega(t,\cdot)$.

We will show that there exists a constant $C_1 > \sqrt{3}$ such that $|v_2(x)| \leq C_1 x_2^{-2}$ for all x such that $x_2 \geq C_1(t\log t)^{1/3}$ and t sufficiently large. This will imply that no fluid particle can escape the region $x_2 \leq C_1(t\log t)^{1/3}$.

As $|x-\overline{y}|^2 \geq \max(x_2^2, x_2y_2)$ we can estimate by relation (2.3) and by the lemma

$$\begin{aligned}
|v_2(x)| &\leq \frac{2}{\pi}\Big|\int\limits_{y_2<x_2/2} \frac{(x_1-y_1)x_2y_2}{|x-y|^2|x-\overline{y}|^2}\omega(y)\,dy\Big| + \frac{2}{\pi}\Big|\int\limits_{y_2>x_2/2} \frac{(x_1-y_1)x_2y_2}{|x-y|^2|x-\overline{y}|^2}\omega(y)\,dy\Big| \\
&\leq \frac{C}{x_2}\int\limits_{y_2<x_2/2} \frac{y_2}{|x-y|}\omega(y)\,dy + C\int\limits_{y_2>x_2/2} \frac{1}{|x-y|}\omega(y)\,dy \\
&\leq \frac{C}{x_2}\int\limits_{y_2<x_2/2} \frac{y_2}{|x_2-y_2|}\omega(y)\,dy + C\Big(\|\omega\|_{L^\infty}\int\limits_{y_2>x_2/2} \omega(y)\,dy\Big)^{1/2} \\
&\leq \frac{C}{x_2^2}\int\limits_{D} y_2\omega(y)\,dy + C\Big(\|\omega\|_{L^\infty}\int\limits_{y_2>x_2/2} \omega(y)\,dy\Big)^{1/2}.
\end{aligned}$$

The proof of the theorem is completed once the following proposition is proved:

Proposition 3.2.2 *For all $k>0$ there exists a constant C_0 such that*

$$\int\limits_{y_2>x_2/2} \omega(t,y)\,dy \leq \frac{C_0}{x_2^k},$$

for all $x_2 > C_0\big[(1+t)\log(2+t)\big]^{1/3}$.

Proof of the proposition. Let

$$f_r(t) = \int\limits_D \eta\left(\frac{x_2-r}{\lambda r}\right)\omega(t,x)\,dx,$$

where $\lambda = \lambda(r) \in (0,1)$ is to be chosen later and

$$\eta(s) = \frac{e^s}{1+e^s}.$$

We easily see that

$$f_r(t) \geq \eta(0)\int\limits_{x_2>r} \omega(t,x)\,dx.$$

So, to prove the proposition it suffices to estimate

$$f_r(t) \leq \frac{C_0}{r^k},$$

for all $r > C_0[(1+t)\log(2+t)]^{1/3}$. To do that, we will deduce a differential inequality verified by f_r.

The equation for ω as well as the Biot-Savart law (2.3) imply

$$\begin{aligned} f_r'(t) &= \int\limits_D \eta\Big(\frac{x_2-r}{\lambda r}\Big)\partial_t\omega(t,x)\,dx \\ &= -\int\limits_D \eta\Big(\frac{x_2-r}{\lambda r}\Big)v(x)\cdot\nabla\omega(x)\,dx \\ &= \frac{1}{\lambda r}\int\limits_D \eta'\Big(\frac{x_2-r}{\lambda r}\Big)v_2(x)\omega(x)\,dx \\ &= \frac{2}{\pi\lambda r}\iint\limits_{D^2} \eta'\Big(\frac{x_2-r}{\lambda r}\Big)\frac{(x_1-y_1)x_2y_2}{|x-y|^2|x-\overline{y}|^2}\omega(x)\omega(y)\,dx\,dy. \end{aligned}$$

Using the change of variables $(x,y)\longleftrightarrow(y,x)$ we finally get

$$f_r'(t) = \frac{1}{\pi\lambda r}\iint\limits_{D^2}\Big[\eta'\Big(\frac{x_2-r}{\lambda r}\Big)-\eta'\Big(\frac{y_2-r}{\lambda r}\Big)\Big]\frac{(x_1-y_1)x_2y_2}{|x-y|^2|x-\overline{y}|^2}\omega(x)\omega(y)\,dx\,dy. \tag{3.13}$$

The mean value theorem implies

$$\eta'\Big(\frac{x_2-r}{\lambda r}\Big)-\eta'\Big(\frac{y_2-r}{\lambda r}\Big)=\frac{x_2-y_2}{\lambda r}\eta''(\xi),$$

where ξ is situated between $\frac{x_2-r}{\lambda r}$ and $\frac{y_2-r}{\lambda r}$. It is very easy to check that $|\eta''(s)|\le\eta(s)$ for all s and that the function η is nonnegative and increasing. We can therefore conclude that

$$\Big|\eta'\Big(\frac{x_2-r}{\lambda r}\Big)-\eta'\Big(\frac{y_2-r}{\lambda r}\Big)\Big|\le\frac{|x_2-y_2|}{\lambda r}\Big(\eta\Big(\frac{x_2-r}{\lambda r}\Big)+\eta\Big(\frac{y_2-r}{\lambda r}\Big)\Big).$$

Inserting this relation in (3.13) yields

$$\begin{aligned} f_r'(t) &\le \frac{1}{\pi\lambda^2r^2}\iint\limits_{D^2}\Big[\eta\Big(\frac{x_2-r}{\lambda r}\Big)+\eta\Big(\frac{y_2-r}{\lambda r}\Big)\Big]\frac{x_2y_2}{(x_2+y_2)^2}\omega(x)\omega(y)\,dx\,dy \\ &= \frac{2}{\pi\lambda^2r^2}\iint\limits_{D^2}\underbrace{\eta\Big(\frac{x_2-r}{\lambda r}\Big)\frac{x_2y_2}{(x_2+y_2)^2}}_{L(x,y)}\omega(x)\omega(y)\,dx\,dy. \end{aligned}$$

For $x_2<r/2$ we bound $L(x,y)\le e^{-\frac{1}{2\lambda}}$ and for $x_2>r/2$ we estimate $L(x,y)\le\eta\big(\frac{x_2-r}{\lambda r}\big)\frac{y_2}{x_2}\le\frac{2}{r}\eta\big(\frac{x_2-r}{\lambda r}\big)y_2$. Using also the conservation of mass and of $\int\limits_D x_2\omega(x)\,dx$ we obtain the following bound:

$$f_r'(t)\le\frac{C}{\lambda^2r^2}e^{-\frac{1}{2\lambda}}+\frac{C}{\lambda^2r^3}f_r(t).$$

Gronwall's lemma now gives

$$f_r(t)\le f_r(0)e^{\frac{Ct}{\lambda^2r^3}}+re^{\frac{Ct}{\lambda^2r^3}-\frac{1}{2\lambda}}.$$

We obviously have that $f_r(0) \leq Ce^{-\frac{1}{2\lambda}}$ for r large enough. We therefore get

$$f_r(t) \leq Ce^{\frac{Ct}{\lambda^2 r^3} - \frac{1}{2\lambda}}(1+r).$$

If we assume that $t < \frac{\lambda r^3}{4C}$ then

$$f_r(t) \leq Ce^{-\frac{1}{4\lambda}}(1+r).$$

The choice $\lambda = \big[4(k+1)\log r\big]^{-1}$, which leads to $r \geq C_3(t\log t)^{1/3}$, completes the proof of the proposition. $\square$

3.2.1.2 One-sided horizontal confinement

We saw in subsection 2.2.1 that the horizontal component of the center of mass in half-plane flow travels with speed bounded below by a positive constant. This excludes any possible sublinear-in-time horizontal confinement, at least in the direction $x_1 > 0$. On the other hand, half-plane flows with nonnegative vorticity have a tendency to move to the right, resisting left “back flow”. The purpose of this part is to make this statement more precise. The following result was proved in [20]:

Theorem 3.2.3 *Let $\omega_0 \in L^p_c(\mathbb{H})$, $p > 2$, $\omega_0 \geq 0$. Let u and ω be solutions of (1.1) with initial vorticity ω_0. Then there exists a positive constant D depending solely on the initial vorticity such that*

$$\operatorname{supp}\omega(t,\cdot) \subset \{x \in \mathbb{H}\ ;\ x_1 \geq -D(t\log t)^{\frac{1}{2}}\}$$

for all $t > 2$.

Before we give the proof of Theorem 3.2.3 we need a technical lemma, in which we obtain an estimate on the mass of vorticity in the “back flow” region; we see that it is exponentially small.

Lemma 3.2.4 *Given $k \in \mathbb{N}$, there exist positive constants D_1 and D_2, depending only on the initial vorticity and on k, such that*

$$\int_{y_1 < -r} \omega(t,y)\, dy \leq \frac{D_1}{r^k}$$

provided that $r \geq D_2(t\log t)^{\frac{1}{2}}$ and $t \geq 2$.

Proof. Consider the auxiliary function $\eta = \eta(s) = \frac{e^s}{1+e^s}$. It is easy to see that η is nonnegative, increasing and

$$|\eta''(s)| \leq \eta(s). \tag{3.14}$$

Set

$$f_r(t) = \int \eta\big(-\frac{x_1 + r}{\lambda r}\big)\,\omega(t,x)\,\mathrm{d}x,$$

where $\lambda > 0$ will be chosen later. As η is nonnegative and increasing we clearly have:

$$f_r(t) \geq \int_{x_1 \leq -r} \eta\big(-\frac{x_1+r}{\lambda r}\big)\, \omega(t,x)\, \mathrm{d}x \geq \eta(0) \int_{x_1 \leq -r} \omega(t,x)\, \mathrm{d}x, \tag{3.15}$$

where we have used that for $x_1 \leq -r$ we have that $-\frac{x_1+r}{\lambda r} \geq 0$. Therefore it suffices for our purposes to estimate $f_r(t)$.

We will deduce a differential inequality for f_r from which we estimate f_r. To this end we differentiate in time to find:

$$f_r'(t) = -\frac{1}{\lambda r} \int \eta'\big(-\frac{x_1+r}{\lambda r}\big) u_1(t,x)\, \omega(t,x)\, \mathrm{d}x,$$

where we have used the vorticity equation (1.1) and integration by parts to throw derivatives onto η,

$$= -\frac{1}{2\pi\lambda r} \iint \eta'\big(-\frac{x_1+r}{\lambda r}\big) \Big[\frac{x_2+y_2}{|x-\overline{y}|^2} - \frac{x_2-y_2}{|x-y|^2}\Big]\, \omega(t,x)\, \omega(t,y)\, \mathrm{d}x\, \mathrm{d}y,$$

using the Biot-Savart law (3.9),

$$\leq \frac{1}{2\pi\lambda r} \iint \eta'\big(-\frac{x_1+r}{\lambda r}\big) \frac{x_2-y_2}{|x-y|^2}\, \omega(t,x)\, \omega(t,y)\, \mathrm{d}x\, \mathrm{d}y,$$

as η', x_2 and y_2 are positive. Finally, we symmetrize the kernel above by making the change of variables $x \leftrightarrow y$ to obtain:

$$\begin{aligned} f_r'(t) &\leq \frac{1}{4\pi\lambda r} \iint \Big[\eta'\big(-\frac{x_1+r}{\lambda r}\big) - \eta'\big(-\frac{y_1+r}{\lambda r}\big)\Big] \frac{x_2-y_2}{|x-y|^2}\, \omega(t,x)\, \omega(t,y)\, \mathrm{d}x\, \mathrm{d}y \\ &\leq \frac{1}{4\pi\lambda r} \iint \frac{|x_1-y_1|}{\lambda r} |\eta''(\theta_{x,y})| \frac{|x_2-y_2|}{|x-y|^2}\, \omega(t,x)\, \omega(t,y)\, \mathrm{d}x\, \mathrm{d}y, \end{aligned}$$

by the mean value theorem, with $\theta_{x,y}$ some point between $-\frac{x_1+r}{\lambda r}$ and $-\frac{y_1+r}{\lambda r}$.

Next we use (3.14) and the fact that η is nonnegative and increasing to deduce that

$$|\eta''(\theta_{x,y})| \leq |\eta(\theta_{x,y})| \leq \eta\big(-\frac{x_1+r}{\lambda r}\big) + \eta\big(-\frac{y_1+r}{\lambda r}\big).$$

Since $|x_1-y_1|\, |x_2-y_2| \leq |x-y|^2$ we finally obtain the differential inequality:

$$f_r'(t) \leq \frac{1}{4\pi\lambda^2 r^2} \iint \Big[\eta\big(-\frac{x_1+r}{\lambda r}\big) + \eta\big(-\frac{y_1+r}{\lambda r}\big)\Big]\, \omega(t,x)\, \omega(t,y)\, \mathrm{d}x\, \mathrm{d}y = \frac{\|\omega_0\|_{L^1}}{2\pi\lambda^2 r^2} f_r(t),$$

where we have used that the L^1-norm of $\omega(t,\cdot)$ is constant in time. Integration now yields

$$f_r(t) \leq f_r(0) \exp\Big(t \frac{\|\omega_0\|_{L^1}}{2\pi\lambda^2 r^2}\Big).$$

Clearly we may assume, without loss of generality, that $\operatorname{supp}\omega_0 \subset \{x_1 \geq 0\}$. Then

$$f_r(0) = \int \eta\big(-\frac{x_1 + r}{\lambda r}\big)\omega_0(x)\,\mathrm{d}x \leq \eta\big(-\frac{1}{\lambda}\big)\|\omega_0\|_{L^1} \leq \exp\big(-\frac{1}{\lambda}\big)\|\omega_0\|_{L^1}.$$

Hence, we infer that

$$f_r(t) \leq \|\omega_0\|_{L^1} \exp\Big(t\frac{\|\omega_0\|_{L^1}}{2\pi\lambda^2 r^2} - \frac{1}{\lambda}\Big).$$

In view of (3.15), to finish the proof it is now sufficient to choose λ such that

$$\exp\Big(t\frac{\|\omega_0\|_{L^1}}{2\pi\lambda^2 r^2} - \frac{1}{\lambda}\Big) \leq \frac{1}{r^k} = \exp(-k\log r).$$

The choice

$$\lambda = \frac{1}{2k\log r}$$

is convenient provided that the following inequality holds

$$\frac{r^2}{\log r} \geq t\frac{2k\|\omega_0\|_{L^1}}{\pi}. \tag{3.16}$$

Notice that the function $r \mapsto r^2/\log r$ is nondecreasing if $r > e$. Hence, choosing D_2 sufficiently large, it is easy to ensure (3.16) if $r \geq D_2(t\log t)^{\frac{1}{2}}$ and $t \geq 2$. This completes the proof. $\square$

Next we use Lemma 3.2.4 to estimate the horizontal velocity.

Proposition 3.2.5 *Under the hypothesis of Theorem 3.2.3, there exist positive constants D_3 and D_4 such that*

$$|u_1(t,x)| \leq \frac{D_3}{|x_1|} \quad \textit{for all} \quad t \geq 2 \textit{ and } x \in \mathbb{H} \textit{ such that } x_1 \leq -D_4(t\log t)^{\frac{1}{2}}.$$

Proof. We will estimate directly $u_1(t,x)$. From the Biot-Savart law (3.9) and the decay estimate (3.12) it follows that

$$\begin{aligned}
|u_1(t,x)| &\leq \int \frac{1}{\pi|x-y|}\,\omega(t,y)\,\mathrm{d}y \\
&\leq \int_{y_1 < x_1/2} \frac{1}{\pi|x-y|}\,\omega(t,y)\,\mathrm{d}y + \int_{y_1 \geq x_1/2} \frac{1}{\pi|x-y|}\,\omega(t,y)\,\mathrm{d}y \\
&\leq \frac{D_{1,p}}{\pi}\|\omega_0\|_{L^p}^{p'/2}\Big(\int_{y_1 < x_1/2} \omega(t,y)\,\mathrm{d}y\Big)^{1-p'/2} + \frac{2}{\pi|x_1|}\|\omega_0\|_{L_1},
\end{aligned}$$

using Lemma 2.2.2 with $a = 1$. We have also used that both the L^1 and the L^p-norms of $\omega(t,\cdot)$ are bounded by their initial values.

Let

$$k = \left[\frac{2}{2-p'}\right] + 1,$$

where $[a]$ denotes the largest integer smaller than a. Choose D_2 as in Lemma 3.2.4 and let x satisfy $x_1 \leq -D_4(t\log t)^{\frac{1}{2}}$ with $D_4 = 2D_2$. The conclusion then follows from Lemma 3.2.4 with D_3 computed accordingly. □

We finish this section with the proof of the horizontal confinement to the left.

Proof of Theorem 3.2.3. Let D_3 and D_4 be as in Proposition 3.2.5. If need be increase the values of D_3 and D_4 to show that any trajectory which reaches the region $\{x_1 \leq -D_4(t\log t)^{\frac{1}{2}}\}$ does not have enough horizontal velocity to go past the line $x_1 = -2D_4(t\log t)^{\frac{1}{2}}$. This proves that every trajectory lies in the region $\{x_1 \geq -2D_4(t\log t)^{\frac{1}{2}}\}$ (with D_4 depending on the initial position of the trajectory); in particular, the support of the evolved vorticity stays in that region. □

3.2.2 The exterior domain case

Let us now consider the case of an exterior domain. In this setting, the moment of inertia and the center of mass are no longer conserved so a loss of 2 in the final result is to be expected. Indeed, known estimates on the Biot-Savart kernel and a similar proof as in section 3.1 or sub-subsection 3.2.1.1 imply that the propagation of vorticity's support is not faster than $O\big[(t\log t)^{\frac{1}{2}}\big]$, see Marchioro [32]. Furthermore, using the conservation of logarithmic moments of the vorticity, it is possible to improve this estimate up to $O\big[\big(\frac{t}{\log t}\big)^{\frac{1}{2}}\big]$ in some cases.

Depending on the shape of the obstacle, further improvements can be obtained. For instance, if the obstacle is the ball $B(0,1)$, the Biot-Savart kernel can be explicitly expressed as

$$K(x,y) = \frac{1}{2\pi}\nabla_x^{\perp} \log\frac{|x-y|}{|x-y^*||y|}, \qquad y^* = \frac{y}{|y|^2}.$$

As a consequence, it can be checked that the moment of inertia is conserved. However, the center of mass is not conserved, it rather turns around the origin. It can be proved in this situation that the propagation of vorticity's support is not faster than $O\big[(t\log t)^{\frac{1}{3}}\big]$.

3.2.3 Other extensions

We simply note here that other extensions and improvements include unbounded initial vorticity and velocity [28, 19], slightly viscous flows [33] and axisymmetric flows, [4, 30].

Chapter 4

Asymptotics for unsigned vorticity

Incompressible, ideal fluid flow can be described in terms of the behavior of vorticity, the curl of the fluid velocity. This is especially useful in two space dimensions, as in this case (the scalar) vorticity is conserved along particle trajectories. The equations of fluid dynamics can then be recast as the transport of an active scalar with vorticity as the dynamic variable. In this context, the problem of describing the large time behavior of unsigned vorticity is a very natural one, and it is the broad subject we address in this chapter.

Let ω_0 be a compactly supported function in $L^p(\mathbb{R}^2)$, with $p > 2$, and let $\omega = \omega(x,t)$ be the vorticity associated to a weak solution of the incompressible two-dimensional Euler equations in the full plane, with initial vorticity ω_0. Recall that in vorticity form, the Euler equations may be written as an active scalar transport equation:

$$\begin{cases} \omega_t + (K * \omega) \cdot \nabla \omega = 0, \\ \omega(x,0) = \omega_0, \end{cases} \tag{4.1}$$

with K the Biot-Savart vector kernel for the full plane, given by

$$K(x) = K(x_1, x_2) = \frac{1}{2\pi |x|^2}(-x_2, x_1) = \frac{x^\perp}{2\pi |x|^2}. \tag{4.2}$$

We are interested in obtaining information on the behavior of the solution $\omega(\cdot, t)$ as $t \to \infty$, particularly with regards to the spatial distribution of vorticity. We consider a self-similar rescaling of vorticity of the form:

$$\widetilde{\omega}_\alpha(x,t) \equiv t^{2\alpha}\omega(t^\alpha x, t),$$

with $\alpha \in (0,1]$. This scaling preserves the integral of vorticity and its L^1 norm. The large time behavior of $\widetilde{\omega}_\alpha$ carries information on the distribution of vorticity, focusing on a certain asymptotic scale determined by the parameter α. The purpose of this chapter is to prove three results. The first result is that, for any initial data ω_0 and parameter $\alpha > 1/2$, we have $\widetilde{\omega}_\alpha \rightharpoonup m\delta_0$, where $m = \int \omega_0$ and δ_0 is the Dirac measure at the origin. The second result analyzes the behavior in a case when only steady vortex pairs are present, the case of nonnegative vorticity in the half-plane. This result can be formulated as a full plane result stating that, if: (i) the initial vorticity ω_0 is odd with respect to the horizontal axis, (ii) its restriction to the upper half-plane has a distinguished sign and (iii) $\alpha = 1$, then the hypothesis that $|\widetilde{\omega}_1(x,t)| \rightharpoonup \mu$, where μ is a measure (which must be supported in $\{|x_1| \leq M\} \times \{x_2 = 0\}$ for confinement reasons), implies that μ must consist of an at most countable sum of Dirac masses whose supports may only accumulate at the origin. Our last result in this chapter is a generalization of this one to the case of the full plane. We remove conditions (i) and (ii) on ω_0, keeping the same conclusion.

The confinement results from the previous chapter basically control the rate at which vorticity is spreading. The present chapter is an attempt to go beyond controlling this rate, actually describing the way in which vorticity is spreading.

If the initial vorticity does not have a distinguished sign, the best confinement one may expect in general is at the rate $a = 1$, as we saw in subsections 2.2.1 and 2.2.2. This means that the self-similar scale of interest is $\alpha = 1$, and the time asymptotic behavior of $|\widetilde{\omega}_1|$ is what would give a reasonably complete description of vorticity scattering in this case.

The remainder of this chapter is divided into four sections. In the next section we discuss the result on the asymptotic behavior of $\widetilde{\omega}_\alpha$. The following section contains the result for nonnegative vorticity in the half-plane. The third section deals with the theorem on $|\widetilde{\omega}_1|$. We end the chapter with some final conclusions.

All the results from this chapter can be found in [20, 21].

4.1 Confinement of the net vorticity

Let $\omega_0 \in L^p_c(\mathbb{R}^2)$, for some $p > 2$ and consider $\omega = \omega(x,t)$ a solution of (4.1) with initial data ω_0. Our basic problem is to describe the spatial distribution of the vorticity $\omega(\cdot,t)$ for large t. Clearly, if ω_0 is single-signed, the known results on confinement tell us that, for any $\alpha > 1/4$, the support of $\widetilde{\omega}_\alpha$ is contained in a disk centered at the origin whose radius vanishes as $t \to \infty$. What happens when the vorticity is allowed to change sign?

Let $\widetilde{u}_\alpha \equiv K * \widetilde{\omega}_\alpha$, with K given by (4.2). It is a straightforward calculation to verify that $\widetilde{\omega}_\alpha$ and $\widetilde{u}_\alpha$ satisfy the equation

$$\frac{\partial \widetilde{\omega}_\alpha}{\partial t} - \frac{\alpha}{t} \operatorname{div}\,(x\widetilde{\omega}_\alpha) + \frac{1}{t^{2\alpha}} \operatorname{div}\,(\widetilde{u}_\alpha \widetilde{\omega}_\alpha) = 0. \tag{4.3}$$

We are now ready to state and prove our first result.

Theorem 4.1.1 *Let $\alpha > 1/2$ and set $m = \int \omega_0(x)\,dx$. Then $\widetilde{\omega}_\alpha(\cdot,t) \rightharpoonup m\delta_0$ weak-$*$ in $\mathcal{BM}(\mathbb{R}^2)$ as $t \to \infty$.*

Proof. We will begin by considering the linear part of the evolution equation (4.3) with initial condition at $t = 1$:

$$\begin{cases} \dfrac{\partial f}{\partial t} - \dfrac{\alpha}{t} \operatorname{div}\,(xf) = 0 \\ \\ f(x,1) = g(x). \end{cases}$$

The solution f is given by the (multiplicative) semigroup $f(x,t) = S_t[g](x) \equiv t^{2\alpha} g(t^\alpha x)$, interpreted in the sense of distributions. We then write (4.3) as an inhomogeneous version of this linear equation, with source term given by

$$h(x,t) \equiv -\frac{1}{t^{2\alpha}} \operatorname{div}\,(\widetilde{u}_\alpha \widetilde{\omega}_\alpha).$$

With this we can write the solution $\widetilde{\omega}_\alpha$ of (4.3), with initial data $\widetilde{\omega}_\alpha(x,1) = \omega(x,1) \equiv g(x)$, using Duhamel's formula:

$$\widetilde{\omega}_\alpha(x,t) = S_t[g](x) + \int_1^t S_{t/s}[h](x,s)\,\mathrm{d}s. \tag{4.4}$$

(In the integral above the semigroup is acting in the spatial variable only.) Of course (4.4) must be interpreted in the sense of distributions. We now turn to the analysis of each term in (4.4). Let $\varphi \in C_c^\infty(\mathbb{R}^2)$. We then have:

$$\int_{\mathbb{R}^2} \varphi(x)\widetilde{\omega}_\alpha(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^2} \varphi\left(\frac{y}{t^\alpha}\right) g(y)\,\mathrm{d}y + \int_1^t \int_{\mathbb{R}^2} \varphi\left(\frac{s^\alpha y}{t^\alpha}\right) h(y,s)\,\mathrm{d}y\,\mathrm{d}s \equiv I_1 + I_2.$$

First note that, as $t \to \infty$,

$$I_1 \to \left(\int_{\mathbb{R}^2} g(y)\,\mathrm{d}y\right)\varphi(0),$$

by the Lebesgue Dominated Convergence Theorem. Next, recall that the total integral of vorticity is conserved and hence the proof will be concluded once we establish that $I_2 \to 0$. We compute directly, integrating by parts and using the relation between $\widetilde{u}_\alpha$ and $\widetilde{\omega}_\alpha$:

$$\begin{aligned} I_2 &= -\int_1^t \int_{\mathbb{R}^2} \varphi\left(\frac{s^\alpha y}{t^\alpha}\right) \frac{1}{s^{2\alpha}} \operatorname{div}\,(\widetilde{u}_\alpha \widetilde{\omega}_\alpha)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_1^t \int_{\mathbb{R}^2} \frac{1}{s^\alpha t^\alpha} \nabla\varphi\left(\frac{s^\alpha y}{t^\alpha}\right) \cdot (\widetilde{u}_\alpha \widetilde{\omega}_\alpha)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_1^t \frac{1}{s^\alpha t^\alpha} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla\varphi\left(\frac{s^\alpha y}{t^\alpha}\right) \cdot K(y-z)\widetilde{\omega}_\alpha(z,s)\widetilde{\omega}_\alpha(y,s)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

We now use the antisymmetry of the Biot-Savart kernel K to obtain:

$$I_2 = \frac{1}{2}\int_1^t \frac{1}{s^\alpha t^\alpha} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H_\varphi(s,t,z,y)\widetilde{\omega}_\alpha(z,s)\widetilde{\omega}_\alpha(y,s)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s,$$

where

$$H_\varphi(s,t,z,y) \equiv \left(\nabla\varphi\left(\frac{s^\alpha y}{t^\alpha}\right) - \nabla\varphi\left(\frac{s^\alpha z}{t^\alpha}\right)\right) \cdot K(y-z).$$

Let us observe that

$$|H_\varphi| \le \frac{s^\alpha}{t^\alpha}\|D^2\varphi\|_{L^\infty}|y-z||K(y-z)| \le C(\varphi)\frac{s^\alpha}{t^\alpha}.$$

Hence we arrive finally at

$$|I_2| \leq C(\varphi) \left(\int_{\mathbb{R}^2} |\omega_0| \right)^2 \frac{t-1}{t^{2\alpha}},$$

which clearly converges to 0 as $t \to \infty$ as long as $2\alpha > 1$. This concludes the proof. $\square$

Remark 4.1.2 *Surprisingly, this result seems to be optimal in the sense that it is false for* $\alpha = \frac{1}{2}$*. This is suggested by the discrete example from subsection 2.1.5 where the vortices have non-vanishing total mass and stay in a region exactly like* $O(t^{\frac{1}{2}})$*.*

Remark 4.1.3 *The particular way in which we use the antisymmetry of the Biot-Savart kernel together with the bilinearization of the nonlinearity of the Euler equations is due to J.-M. Delort, who used it in his proof of existence of weak solutions for 2D Euler with vortex sheet initial data, see [12].*

Remark 4.1.4 *This result does not say anything new if the initial vorticity has a distinguished sign. As we mentioned in the introduction, if the vorticity has a distinguished sign, the support of vorticity is contained in a ball whose radius grows like* $\mathcal{O}(t^{\alpha})$*, with* $1/4 < \alpha$*. From that, Theorem 4.1.1 follows immediately.*

Remark 4.1.5 *What new information is contained in the conclusion of Theorem 4.1.1? Imagine that we are given initial vorticity* $\omega_0 = \omega_0^+ - \omega_0^-$*, which are the positive and negative parts of the initial vorticity. Let* $\omega = \omega^+ - \omega^-$ *be the solution of 2D Euler with initial vorticity* ω_0*. Due to the nature of vortex dynamics, both* ω^+ *and* ω^- *are time-dependent rearrangements of* ω_0^+ *and* ω_0^- *respectively, and hence their integrals, which we may call* m^+ *and* m^-*, are constant in time. Additionally, a consequence of Theorem 4.1.1 is that the integral of vorticity in a ball of radius* t^{α} *converges to* $m^+ - m^-$*, for any* $\alpha > 1/2$*. This is weak confinement of the imbalance between the positive and negative parts of vorticity in a ball of sublinear radius. This weak confinement is consistent with the conjectural picture that the only way for the support of vorticity to grow fast is through the shedding of vortex pairs.*

4.2 Asymptotic behavior of nonnegative vorticity in the half-plane

Let $\omega = \omega(t,x)$ be the vorticity associated to a solution of the incompressible two-dimensional Euler equations on the upper half-plane with an initial vorticity ω_0 which is bounded, compactly supported and nonnegative. We consider a rescaling $\widetilde{\omega} = \widetilde{\omega}(t,x) = t^2\omega(t,tx)$, whose time-asymptotic behavior encodes information on the scattering of ω into traveling wave solutions of the 2D Euler system on the half-plane. This choice of rescaling was also made in view of the fact that the horizontal velocity of the center of vorticity is bounded away from zero from below (see Theorem 2.2.1). The rescaling $\widetilde{\omega}$ is weakly compact as a time-dependent family of measures. The main purpose of this section is to present a structure theorem, stating that if the rescaling $\widetilde{\omega}$ is actually weakly convergent to a measure then this measure must be of the form $\sum m_i \delta(x_1 - \alpha_i) \otimes \delta(x_2)$, with $m_i > 0$, α_i a discrete set of points on an interval of the form $[0, M]$ whose only possible accumulation point is $x_1 = 0$, and where δ denotes the one-dimensional Dirac measure centered at 0.

Results on confinement of vorticity are rigorous actualizations of the rough idea that single signed 2D vorticity tends to rotate around, but not to spread out. This is false if the vorticity is not single signed, which can be seen by considering the behavior of *vortex pairs*, vorticity configurations that tend to translate to infinity with constant speed due to their self-induced velocity, see subsection 2.2.1. Due to the traveling wave behavior of vortex pairs, vorticity scattering in two dimensions may become complicated, and interesting, when vorticity is allowed to change sign. In the previous section we have proved confinement, in a weak sense, of the net vorticity in a region with roughly square-root in time growth in its diameter. From the point of view of scattering, this result accounts for the behavior of the net vorticity, but says very little about the behavior of vortex pairs, because these tend to be weakly self-canceling when looked at from a large spatial scale. It one wants to study vortex scattering, the relevant information is the large-time behavior of $|\widetilde{\omega}_a(t, \cdot)|$, mainly in the case $a = 1$. The present section is directed precisely at this problem, with the simplifying assumption that the vorticity be odd with respect to a straight line, single-signed on each side of the symmetry line. Another way of expressing this is to say that in this section will study the scattering of co-axial, unidirectional vortex pairs.

Let $\omega = \omega(t, x)$ be the solution of the half-plane problem defined for all time, associated to initial data ω_0, which we assume, for simplicity, to be smooth, compactly supported and nonnegative. The confinement results proved in subsection 3.2.1 implies that the support of $\omega(t, \cdot)$ is contained in a rectangle of the form $(a_1 - b_1 t^\alpha, ct) \times (0, a_2 + b_2 t^\beta)$, with a_i real constants, $b_i, c > 0$ and $0 \le \alpha, \beta < 1$. We wish to examine the asymptotic behavior of the vorticity on the linearly growing horizontal scale that is naturally associated with the motion of vortex pairs. The approach we use is inspired on work on the asymptotic behavior of solutions of systems of conservation laws due to G. Q. Chen and H. Frid, see [10]. Let $\widetilde{\omega}(t, x) \equiv t^2 \omega(t, tx)$. The function $\widetilde{\omega}$ has bounded L^1 norm and will be shown to have support in a rectangle of the form $(-b_1 t^{\alpha-1}, c) \times (0, b_2 t^{\beta-1})$. Hence the family of measures $\{\widetilde{\omega}(t, \cdot)\}_{t>0}$ is weak-$*$ precompact and any weak limit of subsequences of this family is of the form $\mu \otimes \delta_0$, with μ a nonnegative measure supported on the interval $[0, c]$. We will refer to such a measure μ as an *asymptotic velocity density*. Our main result may be stated in the following way.

Theorem 4.2.1 *Suppose that the initial data ω_0 for the half-plane problem is such that there exists a* **unique** *asymptotic velocity density μ, i.e., $\widetilde{\omega}(t, \cdot) \rightharpoonup \mu \otimes \delta_0$ when $t \to \infty$. Then μ is the sum of an at most countable set of Diracs whose supports may only accumulate at zero.*

The proof involves writing the PDE for the evolution of $\widetilde{\omega}$ and using the *a priori* estimates available and the structure of the nonlinearity in a way that is characteristic of weak convergence methods, see [17]. We will briefly discuss the physical meaning of both the hypothesis that $\widetilde{\omega}(t,\cdot)$ converges weakly and the conclusion regarding the structure of μ.

The study of the wavelike behavior of vortex pairs goes back to Pocklington in [43], with more recent interest going back to work of Norbury, Deem and Zabusky and Pierrehumbert, see [11, 39, 42]. The existence (and abundance) of steady vortex pairs, which are traveling wave solutions of the 2D incompressible Euler equations, i.e. vorticity shapes which propagate with constant speed without deforming, has been established in the literature in several ways, see [6, 24, 39]. Steady vortex pairs have been object of an extensive literature, from asymptotic studies, see [50] and numerical studies, see [44] and even experimental work, see [16]. Although some analytical results (see [38]) and numerical evidence, [40], point to the orbital stability of steady vortex pairs under appropriate conditions, this stability is an interesting, largely open problem, see [45].

Compactly supported vortex pairs interact in a way such that the intensity of the interaction decays with the inverse of the square of the distance between them. Hence, vortex pairs moving with different speeds tend to behave like individual particles, decoupling after a large time. This is what makes the study of vortex scattering interesting in this context. Let us illustrate the point of view we want to pursue with the example of the Korteweg-deVries equation. Nonlinear scattering for the KdV is well-understood, as solutions of KdV with smooth, compactly supported initial data are expected to resolve into a scattering state composed of an N-soliton plus a slowly decaying dispersive tail. This fact was first formulated as a conjecture by P. Lax in [26] and broadly explored through the method of inverse scattering since then. The conclusion of Theorem 4.2.1 may be regarded as a weak, or averaged form of Lax's conjecture for vortex pair dynamics. Note that steady vortex pairs correspond to classical solitons in this analogy, but no existence for the multi-bump solutions that would be associated to the classical N-solitons has been rigorously established.

Let us call shape space the space of smooth compactly supported vorticity configurations, identifying configurations which are related through horizontal translations. Steady vortex pairs correspond to stationary shapes with respect to Euler dynamics. There are solutions of the two-dimensional incompressible Euler equations that describe periodic loops in shape space. Two examples of this behavior are: 1) a pair of like-signed point vortices on a half plane, which orbit one another periodically as they translate horizontally, called leapfrogging pairs, and 2) Deem and Zabusky's translational V-states, which are vortex patches with discrete symmetry, see [11]. From the point of view of scattering such solutions represent another kind of asymptotic state or, in other words, another kind of particle. Furthermore, one may well imagine solutions with quasiperiodic or chaotic behavior in shape space. Although there is no example of either case in the literature, the passive tracer dynamics of the leapfrogging pair is known to be chaotic, see [41]. Possible chaotic shapes represent an interesting illustration of Theorem 4.2.1, as both the hypothesis of weak convergence and the conclusion are clearly related to the ergodicity of shape dynamics and the self-averaging of the velocity of the center of vorticity of such generalized vortex pairs. Finally, we must mention the work of Overman and Zabusky [40], where they do numerical experiments on the short term scattering of pairs of translational V-states, the first (and only) study to date on the interaction of coaxial vortex pairs, which is the main point of the present work.

We now turn to our main concern in this section, the rigorous study of the asymptotic behavior of flows with nonnegative vorticity in the half-plane. We divide this section in several subsections. In the first one we introduce the self-similar rescaling of the flow which encodes the scattering information we wish to study, we write an evolution equation for the rescaled vorticity and we interpret the vortex confinement information obtained in the previous section in terms of the new scaling. The second subsection is the technical heart of this section, where we study the behavior of the nonlinearity in the equations with respect to the self-similar scaling. In the third subsection we use the information obtained to prove our main result. We then end this half-plane paranthesis with a discrete example and some comments and conclusions.

4.2.1 Rescaled vorticity and asymptotic densities

One key feature of vortex dynamics in a half-plane is nonlinear wave propagation. In order to examine wave propagation it is natural to focus on a self-similar rescaling of physical space, as has been performed by Chen and Frid in the context of systems of conservation laws, see [10]. Let us fix, throughout this section, a nonnegative function $\omega_0 \in L^p_c(\mathbb{H})$, $p > 2$, and $\omega = \omega(t,\cdot)$, $u = u(t,\cdot)$, solutions with initial vorticity ω_0. Set

$$\widetilde{\omega}(t,y) = t^2\omega(t,ty) \quad \text{and} \quad \widetilde{u}(t,y) = tu(t,ty), \tag{4.5}$$

the rescaled vorticity and velocity, respectively. The scaling above respects the elliptic system relating velocity and vorticity so that we still have

$$\begin{cases} \operatorname{div} \widetilde{u} = 0 \\ \operatorname{curl} \widetilde{u} = \widetilde{\omega}. \end{cases}$$

It is immediate that $\widetilde{u}_2(t,x_1,0) = 0$ and therefore we can recover $\widetilde{u}$ from $\widetilde{\omega}$ by means of the Biot-Savart law for the half-plane:

$$\widetilde{u}(t,x) = \int_{\mathbb{H}} K(x,y)\,\widetilde{\omega}(t,y)\,\mathrm{d}y, \tag{4.6}$$

with K defined in (3.10).

Let $M = \|u\|_{L^\infty(\mathbb{R}_+\times\mathbb{H})}$. Then the confinement estimates for vorticity in the half-plane, in particular Theorems 3.2.3 and 3.2.1 and the fact that the vorticity ω is transported by the velocity u, imply that there exists a constant $C > 0$ such that:

$$\operatorname{supp}\omega(t,\cdot) \subset \left[-C(t\log t)^{\frac{1}{2}}, C_0 + Mt\right] \times \left[0, C(t\log t)^{\frac{1}{3}}\right] \quad \text{for all } t \geq 2,$$

where $C_0 = \sup\{x_1 \ ; \ x \in \operatorname{supp}\omega_0\}$. This in turn implies an asymptotic localization of $\operatorname{supp}\widetilde{\omega}(t,\cdot) = \frac{1}{t}\operatorname{supp}\omega(t,\cdot)$, namely:

$$\operatorname{supp} \widetilde{\omega}(t,\cdot) \subset \Big[-C\big(\frac{\log t}{t}\big)^{\frac{1}{2}}, \frac{C_0}{t} + M\Big] \times \Big[0, C\big(\frac{\log t}{t^2}\big)^{\frac{1}{3}}\Big]. \tag{4.7}$$

Next, from the vorticity equation one may derive a transport equation for the evolution of $\widetilde{\omega}(t,y)$, which takes the form:

$$\partial_t \widetilde{\omega}(t,y) - \frac{1}{t}\operatorname{div}\big[y\,\widetilde{\omega}(t,y)\big] + \frac{1}{t^2}\operatorname{div}\big[\widetilde{u}(t,y)\,\widetilde{\omega}(t,y)\big] = 0. \tag{4.8}$$

Using the scaling (4.5) we find

$$\|\,\widetilde{\omega}(t,\cdot)\|_{L^q} = t^{2(1-\frac{1}{q})}\|\omega(t,\cdot)\|_{L^q} \le t^{2(1-\frac{1}{q})}\|\omega_0\|_{L^q} \quad \forall q \in [1,p]. \tag{4.9}$$

Furthermore, the L^1-norm of $\widetilde{\omega}$ is conserved in time. We wish to treat $\widetilde{\omega}$ as a bounded L^1-valued function of time, possessing nonnegative measures as weak-$*$ limits for large time. The confinement estimate (4.7) implies that any weak-$*$ limit of $\widetilde{\omega}$ must have the structure $\mu \otimes \delta_0(x_2)$, with the support of μ contained in the interval $[0,M]$.

It is in the nature of the self-similar rescaling (4.5) that much of the scattering behavior of the flow is encoded in the measure μ. This measure is the main subject of the remainder of this section, and, as such, deserves an appropriate name.

Definition 4.2.2 *Let $\mu \in BM([0,M])$ be a nonnegative measure such that there exists a sequence of times $t_k \to \infty$ for which*

$$\widetilde{\omega}(t_k,\cdot) \rightharpoonup \mu \otimes \delta_0 \text{ in the weak-$*$ topology of bounded measures, as } t_k \to \infty.$$

Then we call μ an **asymptotic velocity density** *associated to ω_0.*

It can be readily checked that, if $\omega(t,x) = \omega_0(x_1 - \sigma t, x_2)$, then there exists a unique asymptotic velocity density μ, which is a Dirac delta at position $(\sigma, 0)$ with mass given by the integral of ω_0. For a general flow an asymptotic velocity density encodes information on typical velocities with which different portions of vorticity are traveling.

4.2.2 The key estimate

Our purpose in this part is to understand the structure of the asymptotic velocity densities. To do so we make use of the evolution equation (4.8) for $\widetilde{\omega}$ and we examine the behavior for large time of each of its terms. The main difficulty in doing so is understanding the behavior of the nonlinear term $\operatorname{div}(\widetilde{u}\,\widetilde{\omega})$, which is our goal in this subsection.

We begin with two general measure-theoretical lemmas which will be needed in what follows. These are standard exercises in real analysis and we include the proofs only for the sake of completeness. Recall that a measure is called *continuous* if it attaches zero mass to points.

Lemma 4.2.3 *Let μ be a finite and compactly supported nonnegative measure on $\mathbb{R}$. Then μ is the sum of a nonnegative continuous measure ν and a countable sum of positive Dirac measures (the discrete part of μ). Moreover, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if I is an interval of length less than δ, then $\nu(I) \leq \varepsilon$.*

Proof. Let $A = \{x\ ;\ \mu(\{x\}) \neq 0\}$. Then A is countable; indeed, $A = \bigcup_n A_n$ and each $A_n = \{x\ ;\ \mu(\{x\}) \geq 1/n\}$ must be finite because μ is finite. Hence we may write $A = \{x_1, x_2, \dots\}$ and $m_j = \mu(\{x_j\})$. Of course, $\nu = \mu - \sum_j m_j \delta_{x_j}$ is a continuous, nonnegative measure.

Let J be a compact interval containing the support of μ. For each $x \in J$, it follows that, since $\nu(\{x\}) = 0$, there exists $\delta_x > 0$ such that $\nu([x - \delta_x, x + \delta_x]) \leq \varepsilon/2$. Let $J_x = [x - \delta_x, x + \delta_x]$. Then $J \subset \bigcup_{x\in J} J_x$ so that, using the fact that J is compact, we can extract a finite subcover, $J \subset J_{x_1} \cup \dots \cup J_{x_n}$. The interval J is now divided in a finite number of disjoint intervals (not necessarily $J_{x_1}, J_{x_2}, \dots, J_{x_n}$), each having ν–measure less than $\varepsilon/2$. It is then sufficient to choose δ equal to one half of the minimum length of these intervals. For that choice of δ, it is clear that an interval of length δ cannot intersect more than two of the disjoint intervals constructed above so that its ν–measure will be less than ε. ☐

Lemma 4.2.4 *Let γ_n be a sequence of nonnegative Radon measures on $\overline{\mathbb{H}}$, converging weakly to some measure γ, and having the supports uniformly bounded in the vertical direction. Then, for every compact interval $[a, b]$ one has that*

$$\limsup_{n\to\infty} \gamma_n([a, b] \times \mathbb{R}_+) \leq \gamma([a, b] \times \mathbb{R}_+).$$

Proof. Fix $\varepsilon > 0$. Since

$$\gamma([a, b] \times \mathbb{R}_+) = \lim_{\delta\to 0} \gamma([a - \delta, b + \delta] \times \mathbb{R}_+),$$

there exists $\delta > 0$ such that

$$\gamma([a - \delta, b + \delta] \times \mathbb{R}_+) < \gamma([a, b] \times \mathbb{R}_+) + \varepsilon.$$

Let φ be a continuous function supported in $(a - \delta, b + \delta)$ and such that $0 \leq \varphi \leq 1$ and $\varphi\big|_{[a,b]} = 1$. According to the hypothesis, we have that $\big\langle \gamma_n(y), \varphi(y_1)\big\rangle \to \big\langle \gamma(y), \varphi(y_1)\big\rangle$, so there exists N such that

$$\big\langle \gamma_n(y), \varphi(y_1)\big\rangle \leq \big\langle \gamma(y), \varphi(y_1)\big\rangle + \varepsilon \quad \forall n \geq N.$$

From the hypothesis on the test function φ it follows that, for all $n \geq N$,

$$\begin{aligned}\gamma_n([a, b] \times \mathbb{R}_+) &\leq \big\langle \gamma_n(y), \varphi(y_1)\big\rangle \leq \big\langle \gamma(y), \varphi(y_1)\big\rangle + \varepsilon \\ &\leq \gamma([a - \delta, b + \delta] \times \mathbb{R}_+) + \varepsilon \leq \gamma([a, b] \times \mathbb{R}_+) + 2\varepsilon.\end{aligned}$$

We deduce that

$$\limsup_{n\to\infty} \gamma_n([a,b]\times\mathbb{R}_+) \le \gamma([a,b]\times\mathbb{R}_+) + 2\varepsilon.$$

The desired conclusion follows by letting $\varepsilon \to 0$. $\square$

Let us now return to the study of the asymptotic behavior of vorticity. Let $\omega_0 \ge 0$ be a fixed function in $L^p_c(\mathbb{H})$, for some $p > 2$, and let u, ω be solutions of the half-plane problem, with $\widetilde{u}$, $\widetilde{\omega}$ defined in (4.5). Let μ be an asymptotic velocity density associated to ω_0. Then μ is a nonnegative measure in $BM([0,M])$, with $M = \|u\|_{L^\infty([0,\infty)\times\mathbb{H})}$, and by Lemma 4.2.3, μ can be written as

$$\mu = \nu + \sum_{i=1}^{\infty} m_i \delta_{\alpha_i}, \tag{4.10}$$

where ν is the continuous part of μ and $\alpha_i \in [0,M]$. As $\omega_0 \ge 0$ it follows that $m_i \ge 0$ and, as μ is a bounded measure, $\sum_{i=1}^{\infty} m_i < \infty$. Furthermore we can assume without loss of generality that $\alpha_i \ne \alpha_j$ in the decomposition (4.10).

Let $\{t_k\}$ be a sequence of times approaching infinity such that

$$\widetilde{\omega}(t_k,\cdot) \rightharpoonup \mu \otimes \delta_0(x_2),$$

as $k \to \infty$, weak-$*$ in $BM(\overline{\mathbb{H}})$. The following proposition is what we refer to as the key estimate in the title of this subsection.

Proposition 4.2.5 *Let $\psi \in C^0(\mathbb{R})$. Then there exists a constant $D > 0$, depending only on p, such that the following estimate holds:*

$$\limsup_{k\to\infty} \left| \int_{\mathbb{H}} \psi(y_1) \frac{\widetilde{u}_1(t_k,y)}{t_k} \widetilde{\omega}(t_k,y)dy \right| \le D\|\omega_0\|_{L^p}^{\frac{p'}{2}} \sum_{i=1}^{\infty} m_i^{2-\frac{p'}{2}} |\psi(\alpha_i)|. \tag{4.11}$$

Before giving the proof of Proposition 4.2.5, let us motivate the statement with the following example. Consider a steady vortex pair with vorticity given by $\omega(t,x) = \omega_0(x_1 - \sigma t, x_2)$ and velocity $u(t,x) = u_0(x_1 - \sigma t, x_2)$. Then it is easy to see that the rescaled nonlinear term $\frac{\widetilde{u}_1}{t}\,\widetilde{\omega}$ converges to $\sigma m \delta_\sigma \otimes \delta_0$ where $m = \int \omega_0 \, dx$. Based on this example, one would expect the right-hand side of (4.11) to be $\sum_i \alpha_i m_i |\psi(\alpha_i)|$ instead. On the other hand, for the steady vortex pair, it can be easily checked that

$$\sigma = \frac{1}{\int \omega_0 \, dx} \int (u_0)_1 \omega_0 \, dx \le \|u_0\|_{L^\infty}.$$

Using Lemma 2.2.2 we infer that

$$|\sigma| \le D\|\omega_0\|_{L^p}^{p'/2} m^{1-p'/2}.$$

which then implies that, as measures, the weak limit of $\frac{\widetilde{u}_1}{t}\,\widetilde{\omega}$ is less than $D\|\omega_0\|_{L^p}^{p'/2} m^{2-p'/2} \delta_\sigma \otimes \delta_0$. Hence, in light of this example we see that estimate (4.11) is weaker than what might be expected, but nevertheless it is consistent with the behavior of steady vortex pairs.

Proof of Proposition 4.2.5. Let us denote the integral we wish to estimate by B_k, so that

$$B_k \equiv \int_{\mathbb{H}} \psi(y_1)\frac{\widetilde{u}_1(t_k,y)}{t_k}\widetilde{\omega}(t_k,y)dy. \tag{4.12}$$

Fix $\varepsilon > 0$ throughout. Since $\sum_{i=1}^{\infty} m_i < \infty$ there exists $N = N(\varepsilon)$ such that

$$\sum_{i>N} m_i < \frac{\varepsilon}{4}.$$

Additionally, it is easy to find $\delta = \delta(\varepsilon) > 0$ such that, if I is an interval, $|I| \leq \delta$, then

$$\nu(I) < \frac{\varepsilon}{4}, \tag{4.13}$$

by using Lemma 4.2.3, and also

$$\mu([\alpha_i - 2\delta, \alpha_i + 2\delta]) < m_i(1+\varepsilon), \quad i = 1, \dots, N, \tag{4.14}$$

$$[\alpha_i - \delta, \alpha_i + \delta] \cap [\alpha_j - \delta, \alpha_j + \delta] = \emptyset, \quad i \neq j \in \{1, \dots, N\}, \tag{4.15}$$

$$|\psi(y_1) - \psi(\alpha_i)| < \varepsilon \quad \forall\, y_1 \in [\alpha_i - \delta, \alpha_i + \delta],\ i = 1, \dots, N. \tag{4.16}$$

In view of Lemma 4.2.4 and relation (4.14), there exists K_0 such that, if $k > K_0$ then

$$\int\limits_{[\alpha_i - 2\delta, \alpha_i + 2\delta] \times \mathbb{R}_+} \widetilde{\omega}(t_k, y)\, \mathrm{d}y < m_i(1+\varepsilon) \qquad \forall i = 1, \dots, N. \tag{4.17}$$

Consider now an interval $I \subset \mathbb{R} \setminus \bigcup\limits_{i=1}^{N} (\alpha_i - \frac{\delta}{2}, \alpha_i + \frac{\delta}{2})$ of length at most δ. According to relation (4.13)

$$\nu(I) < \frac{\varepsilon}{4}.$$

On the other hand $\mu - \nu$, the discrete part of μ, restricted to I avoids the Diracs at $\alpha_1, \dots, \alpha_N$ so that

$$(\mu - \nu)(I) \leq \sum_{i>N} m_i < \frac{\varepsilon}{4}.$$

Therefore

$$\mu(I) < \frac{\varepsilon}{2}. \tag{4.18}$$

Given a compact interval $\mathcal{J} \subset \mathbb{R} \setminus \bigcup\limits_{i=1}^{N} (\alpha_i - \frac{\delta}{2}, \alpha_i + \frac{\delta}{2})$ of length at most δ we can use (4.18) and Lemma 4.2.4 together with the fact that $\widetilde{\omega}(t_k, \cdot) \rightharpoonup \mu \otimes \delta$ to find K_0 large enough so that, in addition to (4.17), we have

$$\int_{\mathcal{J} \times \mathbb{R}_+} \widetilde{\omega}(t_k, y)\, \mathrm{d}y < \frac{\varepsilon}{2},$$

for any $k > K_0$. We wish to show that this K_0 can be chosen *independently* of $\mathcal{J}$, but we shall have to pay a price, namely the estimate above will hold with ε on the right-hand-side, instead of $\varepsilon/2$.

Let J be a compact interval such that $J \times \mathbb{R}_+$ contains the support of $\widetilde{\omega}(t,\cdot)$ for all t. We write the set $J \setminus \bigcup_{i=1}^{N}(\alpha_i - \frac{\delta}{2}, \alpha_i + \frac{\delta}{2})$ as a finite disjoint union of intervals I_j, each of which we subdivide into intervals of length exactly δ, together with an interval of size at most δ, this being the right-most subinterval of I_j. This way the set $J \setminus \bigcup_{i=1}^{N}(\alpha_i - \frac{\delta}{2}, \alpha_i + \frac{\delta}{2})$ can be written as the union of intervals $J_1, \ldots, J_l$ of length precisely δ plus some remaining intervals $J_{l+1}, \ldots, J_L$ of length strictly less than δ. According to (4.18), we have that

$$\mu(J_i) < \frac{\varepsilon}{2} \qquad \forall i = 1, \ldots, L.$$

Next we apply Lemma 4.2.4 and use the fact that $\widetilde{\omega}(t,\cdot) \rightharpoonup \mu \otimes \delta$, to obtain K_0 such that (4.17) is satisfied together with:

$$\int\limits_{J_i \times \mathbb{R}_+} \widetilde{\omega}(t_k, y)\,\mathrm{d}y < \frac{\varepsilon}{2} \qquad \forall i = 1, \ldots, L, \quad k > K_0. \tag{4.19}$$

Let I be a subinterval of $\mathbb{R} \setminus \bigcup_{i=1}^{N}(\alpha_i - \frac{\delta}{2}, \alpha_i + \frac{\delta}{2})$ of length less than δ. It is easy to see that I can intersect at most two of the intervals J_i as otherwise, by construction, this would imply it had to contain an interval of length precisely δ. According to (4.19) we deduce that $\int_{I \times \mathbb{R}_+} \widetilde{\omega}(t_k, y)\,\mathrm{d}y < \varepsilon$ for all $k > K_0$. We have just shown that, if I is an interval of length at most δ, $I \subset \mathbb{R} \setminus \bigcup_{i=1}^{N}(\alpha_i - \frac{\delta}{2}, \alpha_i + \frac{\delta}{2})$ then

$$\int\limits_{I \times \mathbb{R}_+} \widetilde{\omega}(t_k, y)\,\mathrm{d}y < \varepsilon, \quad \forall k > K_0. \tag{4.20}$$

Let $k > K_0$ and set

$$E_i = [\alpha_i - \delta, \alpha_i + \delta] \times \mathbb{R}_+, \quad F_i = [\alpha_i - 2\delta, \alpha_i + 2\delta] \times \mathbb{R}_+, \quad E = E_1 \cup \cdots \cup E_N.$$

According to (4.15), the sets $E_1, \ldots, E_N$ are disjoint, so we can write B_k, defined in (4.12), as:

$$B_k = \underbrace{\sum_{i=1}^{N} \int_{E_i} \psi(y_1) \frac{\widetilde{u}_1(t_k, y)}{t_k} \widetilde{\omega}(t_k, y)\,\mathrm{d}y}_{B_{k1}} + \underbrace{\int_{E^c} \psi(y_1) \frac{\widetilde{u}_1(t_k, y)}{t_k} \widetilde{\omega}(t_k, y)\,\mathrm{d}y}_{B_{k2}}.$$

We will estimate separately B_{k1} and B_{k2}. Note that both estimates rely in an essential way on the Biot-Savart law and the fact that the kernel can be estimated by $|x-y|^{-1}$ (see (3.12)). In the remainder of this proof we will denote by C a constant which is independent of ε and t.

Estimate of B_{k1}. Using the Biot-Savart law (4.6) and relation (3.12), one can bound B_{k1} as follows:

$$
\begin{aligned}
|B_{k1}| &\leq \sum_{i=1}^{N} \iint\limits_{\substack{x\in\mathbb{H}\\ y\in E_i}} \frac{|\psi(y_1)|}{\pi|x-y|}\frac{\widetilde{\omega}(t_k,x)}{t_k}\widetilde{\omega}(t_k,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{t_k}\sum_{i=1}^{N} \iint\limits_{\substack{|x-y|\geq\delta\\ x\in\mathbb{H},y\in E_i}} \frac{|\psi(y_1)|}{\pi|x-y|}\widetilde{\omega}(t_k,x)\widetilde{\omega}(t_k,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad\qquad + \frac{1}{t_k}\sum_{i=1}^{N} \iint\limits_{\substack{|x-y|<\delta\\ x\in\mathbb{H},y\in E_i}} \frac{|\psi(y_1)|}{\pi|x-y|}\widetilde{\omega}(t_k,x)\widetilde{\omega}(t_k,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \frac{\sup|\psi|}{\pi t_k\delta}\|\,\widetilde{\omega}\|_{L^1}\sum_{i=1}^{N}\int_{E_i}\widetilde{\omega}(t_k,y)\,\mathrm{d}y + \frac{1}{t_k}\sum_{i=1}^{N} \iint\limits_{\substack{|x-y|<\delta\\ x\in\mathbb{H},y\in E_i}} \frac{|\psi(y_1)|}{\pi|x-y|}\widetilde{\omega}(t_k,x)\widetilde{\omega}(t_k,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \frac{C}{\delta t_k} + \sum_{i=1}^{N} \iint\limits_{\substack{|x-y|<\delta\\ x\in\mathbb{H},y\in E_i}} \frac{|\psi(y_1)|}{\pi t_k|x-y|}\widetilde{\omega}(t_k,x)\widetilde{\omega}(t_k,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

According to (4.16), for $y\in E_i$ we have that $|\psi(y_1)-\psi(\alpha_i)|<\varepsilon$. We therefore deduce that

$$
|B_{k1}| \leq \frac{C}{\delta t_k} + \sum_{i=1}^{N}\frac{|\psi(\alpha_i)|+\varepsilon}{\pi t_k} \iint\limits_{\substack{|x-y|<\delta\\ x\in\mathbb{H},y\in E_i}} \frac{1}{|x-y|}\widetilde{\omega}(t_k,x)\widetilde{\omega}(t_k,y)\,\mathrm{d}x\,\mathrm{d}y.
$$

Applying Lemma 2.2.2 yields

$$
\begin{aligned}
\iint\limits_{\substack{|x-y|<\delta\\ x\in\mathbb{H},y\in E_i}} \frac{1}{|x-y|}\widetilde{\omega}(t_k,x)\widetilde{\omega}(t_k,y)\,\mathrm{d}x\,\mathrm{d}y &\leq \int_{E_i}\Big(\int\limits_{[y_1-\delta,y_1+\delta]\times\mathbb{R}_+}\frac{\widetilde{\omega}(t_k,x)}{|x-y|}\,\mathrm{d}x\Big)\widetilde{\omega}(t_k,y)\,\mathrm{d}y \\
&\leq D_{1,p}\int_{E_i}\Big(\int\limits_{[y_1-\delta,y_1+\delta]\times\mathbb{R}_+}\widetilde{\omega}(t_k,x)\,\mathrm{d}x\Big)^{1-\frac{p'}{2}}\|\,\widetilde{\omega}\|_{L^p}^{\frac{p'}{2}}\,\widetilde{\omega}(t_k,y)\,\mathrm{d}y.
\end{aligned}
$$

Now, if $y \in E_i$ then $[y_1-\delta, y_1+\delta] \subset [\alpha_i - 2\delta, \alpha_i + 2\delta]$, so that $[y_1-\delta, y_1+\delta] \times \mathbb{R}_+ \subset F_i$. Hence

$$\iint\limits_{\substack{|x-y|<\delta \\ x\in\mathbb{H}, y\in E_i}} \frac{1}{|x-y|}\widetilde{\omega}(t_k,x)\widetilde{\omega}(t_k,y)\,\mathrm{d}x\,\mathrm{d}y \le D_{1,p}\Big(\int_{F_i}\widetilde{\omega}(t_k,x)\,\mathrm{d}x\Big)^{1-\frac{p'}{2}} t_k\|\omega_0\|_{L^p}^{\frac{p'}{2}}\left(\int_{E_i}\widetilde{\omega}(t_k,y)\,\mathrm{d}y\right)$$

$$\le t_k D_{1,p}\big[m_i(1+\varepsilon)\big]^{2-\frac{p'}{2}}\|\omega_0\|_{L^p}^{\frac{p'}{2}},$$

where we have used (4.9) and (4.17). We conclude that

$$|B_{k1}| \le \frac{C}{\delta t_k} + C_1\sum_{i=1}^{N}(|\psi(\alpha_i)|+\varepsilon)\big[m_i(1+\varepsilon)\big]^{2-\frac{p'}{2}}, \tag{4.21}$$

with $C_1 = D_{1,p}\|\omega_0\|_{L^p}^{\frac{p'}{2}}\pi^{-1}$.

Estimate of B_{k2}. We estimate directly, similarly to what was done with B_{k1}:

$$\begin{aligned}|B_{k2}| &\le \frac{C}{\delta t_k} + \iint\limits_{\substack{|x-y|<\delta/3 \\ x\in\mathbb{H}, y\in E^c}} \frac{|\psi(y_1)|}{\pi t_k|x-y|}\widetilde{\omega}(t_k,x)\widetilde{\omega}(t_k,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\le \frac{C}{\delta t_k} + \frac{\|\psi\|_{L^\infty}}{\pi t_k}\iint\limits_{\substack{|x-y|<\delta/3 \\ x\in\mathbb{H}, y\in E^c}} \frac{1}{|x-y|}\widetilde{\omega}(t_k,x)\widetilde{\omega}(t_k,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Lemma 2.2.2 implies in the same way that

$$\begin{aligned}|B_{k2}| &\le \frac{C}{\delta t_k} + \frac{D_{1,p}}{\pi t_k}\|\psi\|_{L^\infty}\int_{E^c}\Big(\int\limits_{[y_1-\frac{\delta}{3},y_1+\frac{\delta}{3}]\times\mathbb{R}_+}\widetilde{\omega}(t_k,x)\,\mathrm{d}x\Big)^{1-\frac{p'}{2}}\|\,\widetilde{\omega}\|_{L^p}^{\frac{p'}{2}}\,\widetilde{\omega}(t_k,y)\,\mathrm{d}y \\ &\le \frac{C}{\delta t_k} + \frac{D_{1,p}}{\pi}\|\psi\|_{L^\infty}\|\omega_0\|_{L^p}^{\frac{p'}{2}}\|\omega_0\|_{L^1}\sup_{y\in E^c}\Big(\int\limits_{[y_1-\frac{\delta}{3},y_1+\frac{\delta}{3}]\times\mathbb{R}_+}\widetilde{\omega}(t_k,x)\,\mathrm{d}x\Big)^{1-\frac{p'}{2}}.\end{aligned}$$

For $y \in E^c$, the interval $[y_1-\frac{\delta}{3}, y_1+\frac{\delta}{3}]$ is of length less than δ and included in $\mathbb{R}\setminus \bigcup_{i=1}^{N}(\alpha_i-\frac{\delta}{2},\alpha_i+\frac{\delta}{2})$. We deduce from (4.20) that

$$\int\limits_{[y_1-\frac{\delta}{3},y_1+\frac{\delta}{3}]\times\mathbb{R}_+}\widetilde{\omega}(t_k,x)\,\mathrm{d}x < \varepsilon.$$

which implies that

$$|B_{k2}| \le \frac{C}{\delta t_k} + C_2\varepsilon^{1-\frac{p'}{2}}, \tag{4.22}$$

with $C_2 = C_1 \|\psi\|_{L^\infty} \|\omega_0\|_{L^1}$.

Collecting the estimates for B_{k1} and B_{k2} (relations (4.21) and (4.22)) yields the following bound for B_k:

$$|B_k| \leq \frac{C}{\delta t_k} + C_1 \Big\{ \|\psi\|_{L^\infty} \|\omega_0\|_{L^1} \varepsilon^{1-\frac{p'}{2}} + \sum_{i=1}^{N} (|\psi(\alpha_i)| + \varepsilon) \big[m_i (1+\varepsilon) \big]^{2-\frac{p'}{2}} \Big\}. \tag{4.23}$$

Take the lim sup as $k \to \infty$ above to obtain:

$$\limsup_{k\to\infty} |B_k| \leq C_1 \Big\{ \|\psi\|_{L^\infty} \|\omega_0\|_{L^1} \varepsilon^{1-\frac{p'}{2}} + \sum_{i=1}^{\infty} (|\psi(\alpha_i)| + \varepsilon) \big[m_i (1+\varepsilon) \big]^{2-\frac{p'}{2}} \Big\}.$$

Next, send $\varepsilon \to 0$ in order to reach the desired conclusion. □

4.2.3 Large time asymptotics

We will now make use of the equation for $\widetilde{\omega}$ given in relation (4.8) together with Proposition 4.2.5 to deduce an inequality for the limit measure μ, given by (4.27). Surprisingly, this estimate alone will be sufficient to deduce the main result of this part, Theorem 4.2.8. Let us begin with an outline of the proof of (4.27). One begins with the equation for the evolution for $\widetilde{\omega}$ (4.8), taking the product with a fixed test function and integrating in space. The resulting equation has three terms. The first one, when integrated from 0 to t, is uniformly bounded in t. Now, if $\operatorname{div}\big[y\,\widetilde{\omega}(t,y)\big]$ is weakly convergent as $t \to \infty$, then the integral in time of the second term will, in principle, diverge like $\log t$ as $t \to \infty$. As for the third term, it is not difficult to see that it is $\mathcal{O}(1/t)$. The dominant part of the third term must balance the logarithmic blow-up in time of the second term. The aim of Proposition 4.2.5 is precisely to estimate this dominant part of the third term.

We will begin with a lemma, relating asymptotics on the linear part of the evolution equation for $\widetilde{\omega}$ (4.8) to the nonlinear part. To this end fix $\psi \in C^0(\mathbb{R})$ and define the quantities

$$A[t;\psi] \equiv \int_{\mathbb{H}} \psi(y_1) y_1 \widetilde{\omega}(t,y)\,\mathrm{d}y, \text{ and} \tag{4.24}$$

$$B[t;\psi] \equiv \int_{\mathbb{H}} \psi(y_1) \frac{\widetilde{u}_1(t,y)}{t} \widetilde{\omega}(t,y)\,\mathrm{d}y. \tag{4.25}$$

Note that, as the support of $\widetilde{\omega}$ is contained in a compact set independent of t, it will not matter whether the support of ψ is compact.

Lemma 4.2.6 *The following estimate holds:*

$$\limsup_{t\to\infty} (B[t;\psi] - A[t;\psi]) \geq 0. \tag{4.26}$$

Proof. Let $\varphi \in C^1(\mathbb{R})$ be a primitive of ψ so that $\varphi' = \psi$. Define

$$f(t) \equiv \int_{\mathbb{H}} \varphi(y_1)\, \widetilde{\omega}(t,y)\, \mathrm{d}y,$$

a bounded function, since $\widetilde{\omega}(t,\cdot)$ is bounded in L^1. Differentiating f with respect to t and using the equation (4.8) for $\widetilde{\omega}$ we get, after integration by parts,

$$\begin{aligned} f'(t) &= \int \varphi(y_1)\partial_t \widetilde{\omega}(t,y)\, \mathrm{d}y = \frac{1}{t}\int \varphi(y_1)\, \mathrm{div}(y\,\widetilde{\omega})\, \mathrm{d}y - \frac{1}{t^2}\int \varphi(y_1)\, \mathrm{div}(\widetilde{u}\,\widetilde{\omega})\, \mathrm{d}y \\ &= \frac{1}{t}\int \psi(y_1)\frac{\widetilde{u}_1}{t}\,\widetilde{\omega}\, \mathrm{d}y - \frac{1}{t}\int \psi(y_1) y_1\, \widetilde{\omega}\, \mathrm{d}y \equiv \frac{1}{t}B[t;\psi] - \frac{1}{t}A[t;\psi]. \end{aligned}$$

Integrating from t to t^2 we obtain:

$$f(t^2) - f(t) = \int_t^{t^2} \frac{B[s;\psi] - A[s;\psi]}{s}\, \mathrm{d}s.$$

Let $L = \limsup_{s\to\infty}(B[s;\psi] - A[s;\psi])$. Recall that $\|\widetilde{u}(t,\cdot)/t\|_{L^\infty}$ and $\|\,\widetilde{\omega}(t,\cdot)\|_{L^1}$ are bounded independently of t, and $\widetilde{\omega}(t,\cdot)$ has compact support uniformly in t, so that $L < \infty$. Then, for any $\varepsilon > 0$, there exists $M > 0$ such that, if $s > M$ then $B[s;\psi] - A[s;\psi] < L + \varepsilon$. In particular, if $t > M$ above then

$$f(t^2) - f(t) < (L+\varepsilon)\log t,$$

so that

$$0 = \lim_{t\to\infty} \frac{f(t^2) - f(t)}{\log t} \le L + \varepsilon.$$

The result follows by taking $\varepsilon \to 0$. □

Remark 4.2.7 *Note that, exchanging ψ by $-\psi$ above gives the estimate:*

$$\liminf_{t\to\infty}(B[t;\psi] - A[t;\psi]) \le 0.$$

We will not use this inequality in what follows.

Let us now impose a major hypothesis on the flow, namely that there exists a **unique** asymptotic velocity density, so that

$$\widetilde{\omega}(t,\cdot) \rightharpoonup \mu \otimes \delta(x_2),$$

as $t \to \infty$. We use Lemma 4.2.3 to write

$$\mu = \nu + \sum_{i=1}^{\infty} m_i \delta_{\alpha_i}.$$

Then, for any $\psi \in C^0(\mathbb{R})$, it follows that

$$A[t;\psi] \to \big\langle y_1\mu, \psi(y_1)\big\rangle,$$

as $t \to \infty$. Next use Proposition 4.2.5 to deduce that

$$\limsup_{t\to\infty} |B(t;\psi]| \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} \sum_{i=1}^{\infty} m_i^{2-\frac{p'}{2}} |\psi(\alpha_i)|.$$

We therefore deduce from Lemma 4.2.6 that:

$$\big\langle y_1\mu, \psi(y_1)\big\rangle \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} \sum_{i=1}^{\infty} m_i^{2-\frac{p'}{2}} |\psi(\alpha_i)|.$$

Exchanging ψ for $-\psi$ yields:

$$\big|\big\langle y_1\mu, \psi(y_1)\big\rangle\big| \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} \sum_{i=1}^{\infty} m_i^{2-\frac{p'}{2}} |\psi(\alpha_i)|. \tag{4.27}$$

The relevant fact is that the exponent $2 - \frac{p'}{2} > 1$.

Let δ_P denote the Dirac delta measure at position P. We are now ready to re-state our main result, giving a more precise formulation of Theorem 4.2.1.

Theorem 4.2.8 *Suppose that the nonnegative initial vorticity $\omega_0 \in L^p_c(\mathbb{H})$ is such that there exists a* **unique** *asymptotic velocity density μ associated to ω_0. Then μ must be of the form:*

$$\mu = \sum_{i=1}^{\infty} m_i\ \delta_{\alpha_i}$$

where:

(a) $\alpha_i \neq \alpha_j$ if $i \neq j$ and $\alpha_i \to 0$ as $i \to \infty$;

(b) the masses m_i are nonnegative and verify $\sum_{i=1}^{\infty} m_i = \|\omega_0\|_{L^1}$;

(c) for all i, $\alpha_i \in [0, M]$, where $M = \|u\|_{L^\infty([0,\infty)\times\mathbb{H})}$;

(d) there exists a constant $D > 0$, depending solely on p, such that, for all i with $m_i \neq 0$ we have

$$\alpha_i \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}}\ m_i^{1-\frac{p'}{2}}.$$

Furthermore, there exists i_0 such that $\alpha_{i_0} \neq 0$ and $m_{i_0} \neq 0$.

We will need two lemmas before we give the proof of Theorem 4.2.8.

Lemma 4.2.9 *Let $m_i\delta_{\alpha_i}$ be a Dirac from the discrete part of μ. The following inequality holds true:*

$$\alpha_i \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} m_i^{1-\frac{p'}{2}}.$$

Proof. Eventually changing the order in the summation of the Diracs, we can assume that $i = 1$. Furthermore, the conclusion is trivial if $\alpha_1 = 0$, so we can assume that $\alpha_1 > 0$ as well. Let $\varepsilon > 0$ be fixed. We follow the same construction as in the beginning of the proof of Proposition 4.2.5, using Lemma 4.2.3, to conclude that there exists $\delta \in (0, \alpha_1)$ such that the following inequality holds:

$$\mu([\alpha_1 - \delta, \alpha_1 + \delta]) \leq m_1 + \varepsilon.$$

If $\alpha_i \in [\alpha_1 - \delta, \alpha_1 + \delta]$, $i \geq 2$, then $m_1\delta_{\alpha_1} + m_i\delta_{\alpha_i} \leq \mu$ on $[\alpha_1 - \delta, \alpha_1 + \delta]$, so we must have that $m_i \leq \varepsilon$.

Let $\psi \in C^0(\mathbb{R})$ be a nonnegative function supported in $(\alpha_1 - \delta, \alpha_1 + \delta) \subset \mathbb{R}_+$ which attains its maximum at α_1. By (4.27) and using the nonnegativity of μ and $y_1\psi(y_1)$ we find

$$m_1\alpha_1\psi(\alpha_1) \leq \big\langle \mu, y_1\psi(y_1)\big\rangle \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} \big[\psi(\alpha_1) m_1^{2-\frac{p'}{2}} + \sum_{i=2}^{\infty} \psi(\alpha_i) m_i^{2-\frac{p'}{2}}\big].$$

We observed that if $\alpha_i \in (\alpha_1 - \delta, \alpha_1 + \delta)$, $i \geq 2$, then $m_i \leq \varepsilon$. If $\alpha_i \notin (\alpha_1 - \delta, \alpha_1 + \delta)$ then $\psi(\alpha_i) = 0$. In both cases

$$\psi(\alpha_i) m_i^{2-\frac{p'}{2}} \leq \psi(\alpha_i)\varepsilon^{1-\frac{p'}{2}} m_i \leq \psi(\alpha_1)\varepsilon^{1-\frac{p'}{2}} m_i.$$

We infer that

$$m_1\alpha_1\psi(\alpha_1) \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} \psi(\alpha_1)\big[m_1^{2-\frac{p'}{2}} + \varepsilon^{1-\frac{p'}{2}} \sum_{i=2}^{\infty} m_i\big],$$

that is

$$m_1\alpha_1 \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} \big[m_1^{2-\frac{p'}{2}} + \varepsilon^{1-\frac{p'}{2}} \sum_{i=2}^{\infty} m_i\big].$$

Letting $\varepsilon \to 0$ we get that

$$m_1\alpha_1 \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} m_1^{2-\frac{p'}{2}}$$

which implies the desired result. □

Lemma 4.2.10 *Suppose that μ has no discrete part in some interval $(a, b) \subset \mathbb{R} \setminus \{0\}$. Then $\mu\big|_{(a,b)} = 0$.*

Proof. Let $\psi \in C^0(\mathbb{R})$ with support in (a, b). According to the hypothesis,

$$\operatorname{supp} \psi \cap \{\alpha_1, \alpha_2, \dots\} = \emptyset$$

so that, for this choice of ψ, the right-hand side of (4.27) vanishes. Therefore (4.27) implies

$$\big\langle \mu(y_1), y_1\psi(y_1)\big\rangle = 0$$

that is

$$y_1\mu\big|_{(a,b)} = 0$$

which implies the desired conclusion by recalling that $0 \notin (a,b)$. □

Proof of Theorem 4.2.8.. We begin by noting that Lemma 4.2.9 implies that $\alpha_i \xrightarrow{i\to\infty} 0$. Indeed, $\sum_{i=1}^{\infty} m_i < \infty$ implies that $m_i \xrightarrow{i\to\infty} 0$. According to the conclusion of Lemma 4.2.9 this immediately implies that $\alpha_i \xrightarrow{i\to\infty} 0$.

Next, observe that Lemma 4.2.10 implies that the continuous part ν vanishes. Indeed, $\operatorname{supp}\nu \subset [0,\infty)$ since $\operatorname{supp}\mu \subset [0,M]$. If $\alpha > 0$, as α is not an accumulation point of the set $\{\alpha_1, \alpha_2, \dots\}$, there exists $\delta \in (0,\alpha)$ such that $\{\alpha_1, \alpha_2, \dots\} \cap \big[(\alpha-\delta, \alpha+\delta) \setminus \{\alpha\}\big] = \emptyset$. According to Lemma 4.2.10, the measure μ vanishes in $(\alpha-\delta,\alpha)$ and $(\alpha,\alpha+\delta)$, so the same is true for ν. Since ν is continuous we deduce that ν must vanish in $(\alpha-\delta,\alpha+\delta)$. We proved that ν vanishes in the neighborhood of each point of $(0,\infty)$. This implies that ν vanishes on $(0,\infty)$. Therefore, ν vanishes on $\mathbb{R}\setminus\{0\}$ and is continuous. We conclude that $\nu = 0$.

We have just proved that

$$\mu = \sum_{i=1}^{\infty} m_i\ \delta_{\alpha_i} \otimes \delta_0$$

and also assertion (a) of Theorem 4.2.8. Assertion (b) follows from the positivity of μ (as limit of positive measures) and from the fact that the total mass of μ is $\|\omega_0\|_{L^1}$. Assertion (c) is a consequence of the support of μ being included in $[0,M]$ and (d) is proved in Lemma 4.2.9. Finally, as previously noted, it was shown in [23] that $\int x_1\omega(t,x)\,\mathrm{d}x \geq Ct$ for some positive constant C. This implies that $\int x_1\widetilde{\omega}(t,x)\,\mathrm{d}x \geq C$, which in turn yields $\sum_i m_i\alpha_i = \big\langle \mu, x_1\big\rangle \geq C$. This completes the proof of Theorem 4.2.8. □

4.2.4 Another discrete example: separation of two vortices above a flat wall

Steady vortex pairs provide smooth examples of vorticities for which the corresponding asymptotic velocity densities consist of a single Dirac mass. We would like to give such an example with at least two different Dirac masses in the asymptotic velocity density. As we already pointed out, the existence of multibump solutions in this situation is an interesting open problem, but we can offer a discrete example in order to illustrate this issue. In this section we will give a sufficient condition for linear separation of two vortices above a flat wall which will in turn give us an example of unique asymptotic velocity density concentrating at two distinct Dirac masses.

Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be two vortices above the wall $\{y = 0\}$ of positive masses m_1, resp. m_2. For notational convenience we will assume that we start at time $t = 1$ instead of $t = 0$. Let L be defined by

$$L = m_1 y_1 + m_2 y_2, \tag{4.28}$$

a quantity which is conserved by the motion of the vortices. We will prove the following proposition.

Proposition 4.2.11 *Suppose there exists a positive constant M such that the following relations hold true:*

$$x_2(1) - x_1(1) > M, \tag{4.29}$$

$$L > m_2 y_2(1) + \frac{L^2}{\pi M^3} \tag{4.30}$$

and

$$\frac{m_2}{2\big(y_2(1) + \frac{L^2}{\pi m_2 M^3}\big)} - \frac{m_1^2}{2\big(L - m_2 y_2(1) - \frac{L^2}{\pi M^3}\big)} - \frac{2\max(m_1, m_2)}{M} > 2\pi M. \tag{4.31}$$

Then, the two vortices z_1 and z_2 linearly separate. More precisely,

$$x_2(t) - x_1(t) > Mt \tag{4.32}$$

for all times $t \geq 1$.

Remark 4.2.12 *Let m_1, m_2 and L be some fixed arbitrary positive constants. Then we can always find $x_1(1)$, $y_1(1)$, $x_2(1)$, $y_2(1)$ and M such that relations* (4.28), (4.29), (4.30) *and* (4.31) *are satisfied. Indeed, we first choose $x_1(1)$ and $x_2(1)$ such that* (4.29) *holds. We next note that* (4.30) *and* (4.31) *are satisfied for large enough M and small enough $y_2(1)$. For example, if $y_2(1) = 0$, then* (4.31) *has a left-hand side of order M^3 so it is verified for M large enough; and since, for that choice of M, it is satisfied for $y_2(1) = 0$, it will be satisfied for small enough $y_2(1)$, too. Once $y_2(1)$ and M are chosen, it remains to choose $y_1(1)$ such that* (4.28) *is satisfied for $t = 1$.*

Proof of Proposition 4.2.11. It is sufficient to prove that, as long as (4.32) holds, then

$$(x_2 - x_1)'(t) \geq M. \tag{4.33}$$

Indeed, the result then follows by a contradiction argument: if T is the first time when $x_2(T) - x_1(T) = MT$, then necessarily $T > 1$ and

$$MT = (x_2 - x_1)(T) = x_2(1) - x_1(1) + \int_1^T (x_2 - x_1)' > M + M(T - 1) = MT$$

which is a contradiction.

We will therefore assume in the following that (4.32) holds and try to prove (4.33).

It follows from the method of images that the motion of these vortices can be computed from the full plane flow due to these two vortices together with their images:

$$z_3 = \overline{z}_1 = (x_1, -y_1) \qquad \text{and} \qquad z_4 = \overline{z}_2 = (x_2, -y_2)$$

with masses $m_3 = -m_1$, resp. $m_4 = -m_2$. Therefore, the equations of motion are given by:

$$2\pi z_1' = \frac{(z_1 - z_2)^\perp}{|z_1 - z_2|^2} m_2 + \frac{(z_1 - z_3)^\perp}{|z_1 - z_3|^2} m_3 + \frac{(z_1 - z_4)^\perp}{|z_1 - z_4|^2} m_4,$$

i.e.,

$$\begin{aligned} 2\pi z_1' &= 2\pi(x_1', y_1') \\ &= \Big(\frac{m_1}{2y_1}, 0\Big) + \frac{m_2}{|z_1 - z_2|^2}(y_2 - y_1, x_1 - x_2) + \frac{m_2}{|z_1 - \overline{z}_2|^2}(y_1 + y_2, x_2 - x_1). \end{aligned} \tag{4.34}$$

Interchanging the indexes 1 and 2 we also get

$$\begin{aligned} 2\pi z_2' &= 2\pi(x_2', y_2') \\ &= \Big(\frac{m_2}{2y_2}, 0\Big) + \frac{m_1}{|z_1 - z_2|^2}(y_1 - y_2, x_2 - x_1) + \frac{m_1}{|z_1 - \overline{z}_2|^2}(y_1 + y_2, x_1 - x_2). \end{aligned} \tag{4.35}$$

Let us now estimate y_2. From relation (4.35) it follows that

$$2\pi y_2' = m_1(x_2 - x_1)\Big(\frac{1}{|z_1 - z_2|^2} - \frac{1}{|z_1 - \overline{z}_2|^2}\Big) = \frac{m_1(x_2 - x_1)4y_1y_2}{|z_1 - z_2|^2|z_1 - \overline{z}_2|^2}.$$

In view of (4.28), we can bound $m_1y_1 \le L$ and $y_2 \le L/m_2$ so that, using also relation (4.32),

$$|y_2'| \le \frac{2L^2}{\pi m_2|x_1 - x_2|^3} \le \frac{2L^2}{\pi m_2 M^3 t^3}. \tag{4.36}$$

We deduce that

$$|y_2(t) - y_2(1)| = \Big|\int_1^t y_2'\Big| \le \frac{L^2}{\pi m_2 M^3}\int_1^t \frac{2}{s^3}\,\mathrm{d}s = \frac{L^2}{\pi m_2 M^3}\Big(1 - \frac{1}{t^2}\Big) \le \frac{L^2}{\pi m_2 M^3},$$

which implies that

$$y_2(t) \le y_2(1) + \frac{L^2}{\pi m_2 M^3}. \tag{4.37}$$

Next, from (4.34), (4.35) and (4.28) we have that

$$\begin{aligned} (x_2 - x_1)' &= \frac{1}{2\pi}\Big[\frac{m_2}{2y_2} - \frac{m_1}{2y_1} + \frac{(m_1 + m_2)(y_1 - y_2)}{|z_1 - z_2|^2} + \frac{(m_1 - m_2)(y_1 + y_2)}{|z_1 - \overline{z}_2|^2}\Big] \\ &\ge \frac{1}{2\pi}\Big[\frac{m_2}{2y_2} - \frac{m_1^2}{2(L - m_2y_2)} - \frac{(m_1 + m_2)}{|z_1 - z_2|} - \frac{|m_1 - m_2|}{|z_1 - \overline{z}_2|}\Big]. \end{aligned}$$

Both $|z_1 - z_2|$ and $|z_1 - \overline{z}_2|$ are bounded from below by $|x_1 - x_2| > Mt \geq M$. Furthermore, the first two terms of the right-hand side of the last relation are decreasing with respect to y_2. We therefore deduce from (4.37) that

$$\begin{aligned}(x_2 - x_1)' &\geq \frac{1}{2\pi}\Big[\frac{m_2}{2\big(y_2(1) + \frac{L^2}{\pi m_2 M^3}\big)} - \frac{m_1^2}{2\big(L - m_2 y_2(1) - \frac{L^2}{\pi M^3}\big)} - \frac{(m_1 + m_2)}{M} - \frac{|m_1 - m_2|}{M}\Big] \\ &= \frac{1}{2\pi}\Big[\frac{m_2}{2\big(y_2(1) + \frac{L^2}{\pi m_2 M^3}\big)} - \frac{m_1^2}{2\big(L - m_2 y_2(1) - \frac{L^2}{\pi M^3}\big)} - \frac{2\max(m_1, m_2)}{M}\Big] \\ &\geq M,\end{aligned}$$

where we have used (4.31). This completes the proof. □

Remark 4.2.13 *The conclusion that $x_2(t) - x_1(t) \geq Mt$, for some $M > 0$, always implies the existence of a unique asymptotic velocity density which concentrates on a pair of Dirac masses. In order to see this, first note that, from* (4.36)*, we have that $|y_2'| = \mathcal{O}(1/t^3)$, which implies that $y_2(t)$ converges as $t \to \infty$ and similarly for y_1. From the conservation of energy we have that*

$$2m_1 m_2 \log \frac{|z_1 - z_2|}{|z_1 - \overline{z}_2|} - m_1^2 \log(2y_1) - m_2^2 \log(2y_2)$$

is constant in time. Since $x_2(t) - x_1(t) \geq Mt$ we also know that $\frac{|z_1 - z_2|}{|z_1 - \overline{z}_2|} \to 1$ as $t \to \infty$. We deduce that $\lim_{t\to\infty} y_2(t) \neq 0$ and $\lim_{t\to\infty} y_1(t) \neq 0$. Now, from relations (4.34) *and* (4.35) *we immediately obtain that both x_1' and x_2' converge to a finite limit given by*

$$\alpha_1 \equiv \lim_{t\to\infty} x_1'(t) = \frac{m_1}{4\pi \lim_{t\to\infty} y_1(t)} \quad \text{and} \quad \alpha_2 \equiv \lim_{t\to\infty} x_2'(t) = \frac{m_2}{4\pi \lim_{t\to\infty} y_2(t)}.$$

Observe next that $\lim_{t\to\infty} \frac{x_1(t)}{t} = \lim_{t\to\infty} x_1'(t) = \alpha_1$ and similarly for $\frac{x_2(t)}{t}$. Finally, let us remark that the rescaled vorticity is given in this case by $m_1\delta_{z_1/t} + m_2\delta_{z_2/t}$ so that it clearly converges weakly to $\big(m_1\delta_{\alpha_1} + m_2\delta_{\alpha_2}\big) \otimes \delta_0$. Moreover, $x_2(t) - x_1(t) \geq Mt$ implies that $\alpha_2 - \alpha_1 \geq M > 0$.

4.2.5 Extensions and Conclusions

We end this section with some comments regarding the results obtained here.

(a) The only instance of use of the energy estimate in this work is the observation that, for any asymptotic velocity density μ we have $\big\langle \mu, x_1 \big\rangle > C > 0$, which appears when proving the last part of Theorem 4.2.8. The constant C depends on the kinetic energy of the initial data, as was derived in [23]. It would be interesting to know whether kinetic energy partitions itself in a way that is consistent with the partitioning of vorticity, but we were not able to prove that, at least using only the hypothesis of uniqueness of the asymptotic velocity density.

(b) We only used the hypothesis of uniqueness of the asymptotic velocity density when we derived (4.27). The estimate on the behavior of the nonlinear term given in Proposition 4.2.5 always holds, which raises the possibility of it being exploited further.

(c) The hypothesis that the initial vorticity be p-integrable, with $p > 2$ is used to ensure that the velocity is globally bounded. In principle, with vorticity in L^p, $p \leq 2$, we loose control over the loss of vorticity to infinity, and Lemma 2.2.2 is no longer true. In fact, we do not even know the correct scaling to analyze in this case.

We would like to add a remark on the choice of the scaling $x = ty$. If the scaling $x_1 \equiv ty_1$ in the horizontal direction is motivated by the fact that the first component of the center of vorticity behaves exactly like $O(t)$, the scaling $x_2 \equiv ty_2$ is not justified because the second component of the center of vorticity is constant. Ideally we should not make any rescaling in the vertical direction but then we would have to assume that $t\omega(tx_1, x_2)$ converges weakly, which we found excessive because of the oscillations that may appear in the vertical direction. We could also consider an intermediate scaling of the form $x_2 \equiv f(t)y_2$ where $f(t) \to \infty$ as $t \to \infty$. This last problem is in fact equivalent to the one we consider in this section. If f is such a function, then the weak limits of $\widetilde{\omega}_f(t, y) = tf(t)\omega\big(t, ty_1, f(t)y_2\big)$ are independent of f. Indeed, let ν_f be the weak limit of $\widetilde{\omega}_f(t, y)$ as $t \to \infty$ and choose a test function $h \in C_0^\infty(\overline{\mathbb{H}})$. Then

$$\begin{aligned}
\big\langle \nu_f, h \big\rangle &= \lim_{t\to\infty} \int_{\mathbb{H}} \widetilde{\omega}_f(t, y) h(y)\,\mathrm{d}y \\
&= \lim_{t\to\infty} \int_{\mathbb{H}} \omega(t, x) h\Big(\frac{x_1}{t}, \frac{x_2}{f(t)}\Big)\,\mathrm{d}x \\
&= \lim_{t\to\infty} \Big(\int_{\mathbb{H}} \omega(t, x) h\Big(\frac{x_1}{t}, 0\Big)\,\mathrm{d}x + O\Big(\frac{\|\partial_2 h\|_{L^\infty}}{f(t)}\Big) \int_{\mathbb{H}} x_2 \omega(t, x)\,\mathrm{d}x \Big) \\
&= \lim_{t\to\infty} \int_{\mathbb{H}} \omega(t, x) h\Big(\frac{x_1}{t}, 0\Big)\,\mathrm{d}x
\end{aligned}$$

since we know that $\int_{\mathbb{H}} x_2\omega(t, x)\,\mathrm{d}x = cst.$ and $f(t) \to \infty$ as $t \to \infty$. The last term does not depend on f anymore. Here, we have made the choice $f(t) = t$ only for the sake of simplicity. This means that we study the asymptotic behavior of solutions in horizontal direction but not in the vertical one.

We would like to comment on a few problems that arise naturally from the work presented here. The first is to remove the hypothesis of uniqueness of the asymptotic velocity profile, perhaps with weaker conclusions. Also, we can try to extend this line of reasoning to other fluid dynamical situations with similar geometry, such as flow on an infinite flat channel, axisymmetric flow (smoke ring dynamics), and water wave problems. We may also ask the same questions with respect to full two-dimensional scattering, allowing for vortex pairs moving off to infinity in different directions. Finally, one might try to examine the issue of actually proving the uniqueness of asymptotic velocity densities in special cases, for example, for point vortex dynamics. The case of three point vortices on the half-plane is still open.

4.3 Vortex scattering

Let us now return to the case of an unsigned vorticity in the full plane. Let $\omega = \omega(x,t)$ be a solution of the incompressible 2D Euler equations (4.1) with initial vorticity $\omega_0 \in L^p_c(\mathbb{R}^2)$, for some $p > 2$. The simplest picture consistent with what is known regarding large-time vortex dynamics would have ω_0 scattering into a confined part, which would remain near the center of motion for all time, plus a number of soliton-like vortex pairs, traveling with roughly constant speed. Denoting

$$\widetilde{\omega} = \widetilde{\omega}(x,t) \equiv \widetilde{\omega}_1(x,t) = t^2\omega(tx,t),$$

the result in Section 4.1 implies that $\widetilde{\omega} \rightharpoonup m\delta_0$ when $t \to \infty$, but the weak convergence completely ignores the scattering of vortex pairs, due to their linear-scale self-cancellation. The large-time behavior of $|\widetilde{\omega}|$ provides a useful rough picture of vortex scattering.

First note that $|\widetilde{\omega}(\cdot,t)|$ is a bounded one-parameter family in $L^1(\mathbb{R}^2)$. Since the velocity $K * \omega$ is *a priori* globally bounded, the family $|\widetilde{\omega}(\cdot,t)|$ has its support contained in a single disk D. One can therefore extract a sequence of times $t_k \to \infty$ such that $|\widetilde{\omega}(\cdot,t_k)| \rightharpoonup \mu$, for some measure $\mu \in \mathcal{B}M_+(D)$. It follows from Theorem 4.1.1 that $\mu \geq |m|\delta_0$, where $m = \int \omega_0$. Indeed, if φ is a nonnegative test function,

$$\langle |m|\delta_0, \varphi\rangle = \lim_{k\to\infty} \left| \int \varphi(x)\widetilde{\omega}(x,t_k)dx \right| \leq \lim_{k\to\infty} \int \varphi(x)|\widetilde{\omega}(x,t_k)|dx = \langle \mu, \varphi\rangle.$$

Our purpose is to obtain more information about the measure μ. The result we will present is a generalization of the previous result in the half-plane, which described the structure of the measure μ in the situation of half-plane vortex scattering, and under an important restriction, which we will have to impose in the present context as well. We introduced above the terminology *asymptotic velocity density* for any measure μ which is a limit of $|\widetilde{\omega}(\cdot,t_k)|$ for some sequence $t_k \to \infty$. In fact, due to a sign restriction, the explicit use of the absolute value in the definition of asymptotic velocity densities was not needed in the previous section, and because scattering in the half-plane is a one-dimensional affair, the density μ in the previous section was a measure on the real line, describing the asymptotic density only of the relevant component of velocity. Our result about the structure of μ proved above and the result we will present here only applies to initial vorticities which have a *unique* asymptotic velocity density, i.e. those initial vorticities for which $|\widetilde{\omega}|(\cdot,t)$ converges weakly to a measure μ, rather than being merely weakly compact.

Theorem 4.3.1 *Suppose that the initial vorticity $\omega_0 \in L^p_c(\mathbb{R}^2)$, $p > 2$ has a unique asymptotic velocity density μ. Then μ must be of the form:*

$$\mu = \sum_{i=1}^{\infty} m_i \; \delta_{\alpha_i}$$

where:

(a) $\alpha_i \neq \alpha_j$ *if* $i \neq j$ *and* $\alpha_i \to 0$ *as* $i \to \infty$*;*

(b) the masses m_i *are nonnegative and verify* $\sum_{i=1}^{\infty} m_i = \|\omega_0\|_{L^1}$*;*

(c) for all i*,* $|\alpha_i| \in [0, M]$*, where* $M = \|u\|_{L^\infty([0,\infty)\times\mathbb{R}^2)}$*;*

(d) there exists a constant $D > 0$*, depending solely on* p*, such that, for all* i *with* $m_i \neq 0$ *we have*

$$|\alpha_i| \leq D \|\omega_0\|_{L^p}^{\frac{p'}{2}} \, m_i^{1-\frac{p'}{2}}.$$

Remark 4.3.2 *In the statement above, the masses* m_i *are allowed to vanish only to include the case when the limit measure contains a finite number of Diracs. For notational convenience, in the case when there are only a finite number of Dirac masses, we artificially added a countable number of Dirac masses with zero masses and positions converging to 0.*

Proof. The proof we will present here has much in common with the special case done in Section 4.2, so that we will concentrate on the aspects of the proof which differ from the original case, briefly outlining the remainder.

We first note that, since ω is transported by the velocity u, the same holds for $|\omega|$. This means that $|\omega|$ satisfies, in the weak sense, the equation

$$\partial_t |\omega| + \operatorname{div}(u|\omega|) = 0.$$

The equation for the absolute value of the rescaled vorticity is then given by

$$\partial_t |\widetilde{\omega}(y,t)| - \frac{1}{t} \operatorname{div}\big[y |\widetilde{\omega}(t,y)|\big] + \frac{1}{t^2} \operatorname{div}\big[\widetilde{u}(y,t) |\widetilde{\omega}(t,y)|\big] = 0,$$

where $\widetilde{u}(y,t)$ denotes the rescaled velocity $\widetilde{u}(y,t) = tu(ty,t)$.

Let us take the product with a test function $\varphi \in C^1(\mathbb{R}^2)$ and integrate in space:

$$\partial_t \int |\widetilde{\omega}(y,t)| \, \varphi(y) \, \mathrm{d}y = -\frac{1}{t} \int |\widetilde{\omega}(y,t)| \, y \cdot \nabla\varphi(y) \, \mathrm{d}y + \frac{1}{t^2} \int |\widetilde{\omega}(y,t)| \, \widetilde{u}(y,t) \cdot \nabla\varphi(y) \, \mathrm{d}y. \quad (4.38)$$

We now recall the following argument that was used above. The left-hand side of (4.38), when integrated from 1 to t, is uniformly bounded in t. By hypothesis, we know that

$$\lim_{t\to\infty} \int |\widetilde{\omega}(y,t)| \, y \cdot \nabla\varphi(y) \, \mathrm{d}y = \langle y\mu, \nabla\varphi \rangle,$$

so that the integral from 1 to t of the first term on the right-hand side of (4.38) behaves like $\langle y\mu, \nabla\varphi \rangle \log t$. As for the third term, it is not difficult to see that it is $\mathcal{O}(1/t)$. The dominant part of the third term must balance the logarithmic blow-up in time of the second term. This argument implies, adapting Lemma 4.2.6 to the present situation, that the following inequality must hold:

$$\limsup_{t\to\infty}\Big(\frac{1}{t}\int |\,\widetilde{\omega}(y,t)|\,\widetilde{u}(y,t)\cdot\nabla\varphi(y)\,\mathrm{d}y\Big) \geq \langle y\mu, \nabla\varphi\rangle. \tag{4.39}$$

A straightforward adaptation of Lemma 4.2.3 to compactly supported nonnegative finite measures on the plane yields a decomposition of μ into the sum of a discrete part plus a continuous part, i.e.:

$$\mu = \nu + \sum_{i=1}^{\infty} m_i \delta_{\alpha_i}. \tag{4.40}$$

On the other hand, it was also proved in Proposition 4.2.5 a key estimate that in the present case reads

$$\limsup_{t\to\infty}\Big|\frac{1}{t}\int |\,\widetilde{\omega}(y,t)|\,\widetilde{u}(y,t)\cdot\nabla\varphi(y)\,\mathrm{d}y\Big| \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} \sum_{i=1}^{\infty} m_i^{2-\frac{p'}{2}}|\nabla\varphi(\alpha_i)| \tag{4.41}$$

where $\sum_{i=1}^{\infty} m_i\delta_{\alpha_i}$ is the discrete part in the decomposition (4.40). The proof of Proposition 4.2.5 valid in the case of the half-plane can be adapted in a straightforward manner to the full plane context due to the fact that the key estimate in the original proof is the inequality below, which relates the rescaled velocity to the rescaled vorticity:

$$|\widetilde{u}(x,t)| \leq \int \frac{C}{|x-y|}\,|\,\widetilde{\omega}(y,t)|\,\mathrm{d}y,$$

and this inequality holds in the case of the full space as well.

It follows from (4.39) and (4.41) that

$$\langle y\mu, \nabla\varphi\rangle \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} \sum_{i=1}^{\infty} m_i^{2-\frac{p'}{2}}|\nabla\varphi(\alpha_i)|.$$

Substituting φ by $-\varphi$ we obtain

$$|\langle y\mu, \nabla\varphi\rangle| \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} \sum_{i=1}^{\infty} m_i^{2-\frac{p'}{2}}|\nabla\varphi(\alpha_i)|. \tag{4.42}$$

Next we will use (4.42) to deduce that

$$|\alpha_i| \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} m_i^{1-\frac{p'}{2}}. \tag{4.43}$$

To this end, let us fix $i_0 \in \mathbb{N}$ and choose $\varphi \in C_c^\infty(\mathbb{R}^2)$ such that $\nabla\varphi(0) = \alpha_{i_0}$. Define $\varphi_\varepsilon(x) = \varepsilon\varphi\big(\frac{x-\alpha_{i_0}}{\varepsilon}\big)$ and use it as test function in (4.42) to obtain

$$\Big|\big\langle y\mu, \nabla\varphi\big(\frac{y-\alpha_{i_0}}{\varepsilon}\big)\big\rangle\Big| \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} \sum_{i=1}^{\infty} m_i^{2-\frac{p'}{2}}\Big|\nabla\varphi\big(\frac{\alpha_i-\alpha_{i_0}}{\varepsilon}\big)\Big|. \tag{4.44}$$

The series on the right-hand side converges uniformly for $\varepsilon > 0$ and hence, when $\varepsilon \to 0$, it converges to

$$D\|\omega_0\|_{L^p}^{\frac{p'}{2}} m_{i_0}^{2-\frac{p'}{2}} |\alpha_{i_0}|.$$

As for the left-hand side, first we note that the functions $\nabla\varphi\big(\frac{y-\alpha_{i_0}}{\varepsilon}\big)$ converge pointwise to $\alpha_{i_0}\chi_{\{\alpha_{i_0}\}}$ (which does not vanish μ-almost everywhere, since μ attaches positive mass to α_{i_0}). Also, these functions are bounded uniformly with respect to ε and have supports contained in a single disk. The Lebesgue Dominated Convergence Theorem therefore implies that

$$\Big\langle y\mu, \nabla\varphi\Big(\frac{y-\alpha_{i_0}}{\varepsilon}\Big)\Big\rangle \to \big\langle y\mu, \alpha_{i_0}\chi_{\{\alpha_{i_0}\}}\big\rangle = |\alpha_{i_0}|^2 m_{i_0}$$

as $\varepsilon \to 0$. Putting these arguments together yields (4.43) in the limit, as $\varepsilon \to 0$.

We just proved part (d) of Theorem 4.3.1. Part (a) also follows at once by remarking that we have $m_i \to 0$ so, by (4.43), $\alpha_i \to 0$ as $i \to \infty$ too. Part (c) is a trivial consequence of the fact that the support of the vorticity is transported by the flow of u. Finally, part (b) is a direct consequence of the nonnegativity of the measure μ and also from the conservation of the L^1 norm of $|\widetilde{\omega}|$, once we established that the continuous part of μ vanishes.

We now go to the last part of the argument, i.e. the proof that the continuous part of the measure μ vanishes. Here is where the present proof requires a more substantial modification of the original one.

Let D be a strip of the form $D = \{c \le ay_1 + by_2 \le d\}$ disjoint with the set $A \equiv \{0\}\bigcup_{i\ge1}\{\alpha_i\}$. We prove that the measure μ must necessarily vanish in the interior of such a strip. First, since $0 \notin D$ we must have that $cd > 0$. We assume without loss of generality that $c, d > 0$. Let $[c', d']$ a subinterval of (c, d) and choose a smooth function $h \in C^\infty(\mathbb{R})$ such that $h' \in C_c^\infty(c, d)$, $h' \ge 0$ and $h'(s) = 1/s$ for all $s \in [c', d']$. Choose now $\varphi(y_1, y_2) = h(ay_1 + by_2)$ as test function in (4.42). Since $\operatorname{supp}\nabla\varphi \subset D$ we have that $\operatorname{supp}\nabla\varphi \cap A = \emptyset$, which implies in turn that the right-hand side of (4.42) vanishes for this choice of test function. Therefore the left-hand side must vanish too:

$$0 = \big\langle y\mu, \nabla\big(h(ay_1 + by_2)\big)\big\rangle = \big\langle \mu, (ay_1 + by_2)h'(ay_1 + by_2)\big\rangle. \tag{4.45}$$

The function $y \mapsto (ay_1 + by_2)h'(ay_1 + by_2)$ is nonnegative and it is equal to 1 on the strip $\{c' \le ay_1 + by_2 \le d'\}$. Since the measure μ is nonnegative too, we deduce from (4.45) that μ vanishes on the strip $\{c' \le ay_1 + by_2 \le d'\}$. Also, since $[c', d']$ was an arbitrary subinterval of (c, d), we finally deduce that μ vanishes in the interior of the strip D.

In order to conclude the proof of Theorem 4.3.1, we only need to show that the measure μ vanishes in the neighborhood of each point of A^c. Let $y_0 \in A^c$. Since the only possible accumulation point of the set A is 0, there exists a line $\{ay_1 + by_2 = c\}$ passing through y_0 and which does not cross A. A continuity argument using again that the points α_i can only accumulate at $\{0\}$ shows that there exists a strip $\{c - \varepsilon \le ay_1 + by_2 \le c + \varepsilon\}$ disjoint of A. But we proved that the measure μ must vanish on such a strip. This implies that μ vanishes in the neighborhood of y_0 and this completes the proof of Theorem 4.3.1.

□

4.4 Conclusions

First we observe that Theorem 4.1.1 does not require that the initial vorticity ω_0 belong to L^p. The argument works just as well if the initial vorticity is a bounded (signed) Radon measure, as long as the existence of a (global in time) weak solution is provided. The estimate itself only depends on the total mass of the initial vorticity.

We note also that Theorem 4.3.1 draws a much stronger conclusion than Theorem 4.1.1, but it relies on the hypothesis that the initial vorticity $\omega_0 \in L^p_c$, with $p > 2$ have a unique asymptotic velocity density. This hypothesis clearly deserves further scrutiny.

One natural question arising from this work is the role of the critical exponent $\alpha = 1/2$ in Theorem 4.1.1. This exponent is far from the known critical exponent $\alpha = 1/4$ for the vorticity confinement in the distinguished sign case. In the vortex confinement literature, the critical exponent $\alpha = 1/2$ appears naturally when one does not have *a priori* control over moments of vorticity, see [32], whereas the sharper estimates are obtained when using the conserved moments of vorticity. Using just the moment of inertia one obtains critical exponent $\alpha = 1/3$, in the case of the full plane, see [31], and in the case of the exterior of a disk, see [32]. Using both the moment of inertia and the center of vorticity, we obtain, in the case of the full plane, the critical exponent $\alpha = 1/4$, see Section 3.1. It is therefore reasonable to expect that we might improve the condition on α in Theorem 4.1.1 by using the conserved moments of vorticity, but this would require a new approach.

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