Spectral Theory

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Contents

Chapter 1

Hilbert spaces

1.1 Scalar product

Let E and F be two $\mathbb{C}$-vector spaces. A mapping $f: E \longrightarrow F$ is said to be **antilinear** if, for all $x, y \in E$ and all $\lambda \in \mathbb{C}$ we have $f(x+y) = f(x) + f(y)$ and $f(\lambda x) = \overline{\lambda} f(x)$.

Definition 1.1.1 *Let E be a complex vector space. We call sesqui-linear form on E a mapping $B: E \times E \longrightarrow \mathcal{C}$ such that, for all $y \in E$ the mapping $x \longmapsto B(x, y)$ is linear and the mapping $x \longmapsto B(y, x)$ is anti-linear.*

Proposition 1.1.1 *(polarization identity)*

1. *Let E be a complex vector space and B a sesqui-linear form on E. For all $x, y \in E$ we have*

$$4B(x,y) = B(x+y,x+y)-B(x-y,x-y)+iB(x+iy,x+iy)-iB(x-iy,x-iy).$$

2. *Let E be a real vector space and B a bilinear form on E. For all $x, y \in E$ we have*

$$4B(x,y) = B(x+y,x+y) - B(x-y,x-y).$$

Proof. We have $B(x+y,x+y) - B(x-y,x-y) = 2B(x,y) + 2B(y,x)$. Replacing y by iy, we find $B(x+iy,x+iy) - B(x-iy,x-iy) = 2B(x,iy) + 2B(iy,x) = -2iB(x,y) + 2iB(y,x)$. $\square$

In particular, to determine a symmetric sesqui-linear form B, it suffices to determine $B(x,x)$ for all $x \in E$.

Corollary 1.1.1 *Let E be a complex vector space and B a sesqui-linear form on E. The following are equivalent:*

(i) For all $x, y \in E$ we have $B(y, x) = \overline{B(x, y)}$.

(ii) For all $x \in E$, $B(x, x) \in \mathbb{R}$.

Proof. Set $S(x, y) = B(x, y) - \overline{B(y, x)}$. This define a sesqui-linear form. By the polarization identity, S is zero if and only if, for all $x \in E$, $S(x, x) = 0$. □

Definition 1.1.2 *Let E be a complex vector space. We call* **hermitian form** *on E a sesqui-linear form verifying any of the equivalent conditions of corollary 1.1.1. A hermitian form B on E is said to be positive if, for all $x \in E$, $B(x, x) \geq 0$.*

A symmetric bilinear form B on a real vector space E is said to be positive if, for all $x \in E$, $B(x, x) \geq 0$.
We call **semi-scalar product**, *often denoted by $(x, y) \longmapsto < x, y >$, any symmetric positive form on a real vector space or any positive hermitian form on a complex vector space. It is called* **scalar product** *if, it verify in addition the following property: for all $x \in E$, $< x, x >= 0$ if and only if $x = 0$.*
On appelle **espace prhilbertien** *(rel ou complexe) un espace vectoriel (rel ou complexe) muni d'un produit scalaire.*

Exemples.

1. Let $E = \mathbb{R}^N$. If $a_1, \cdots, a_N$ are positive real numbers, the relation

$$< x, y >:= \sum_{1 \leq i \leq N} a_j x_j y_j$$

   define on E a semi-scalar product, which is a scalar product if and only if all a_j are strictly positive.

2. Let X be metric space locally compact and separable, μ a positive Radon measure on X and $E := \mathcal{D}^0(X, \mathbb{K})$. The relation

$$< f, g >:= \int f(x)\overline{g(x)}\, d\mu(x)$$

   define a semi-scalar product, which is a scalar product if and only if $\mathrm{Supp}\mu = X$.

3. The space $E := C_{2\pi} = \{f : \mathbb{R} \longmapsto \mathbb{K}$ continuous and $2\pi -$ periodical$\}$ with the relation

$$< f, g >:= \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)}\, dx$$

   is a prehilbert space.

4. Let I be a set. Denote, for $p \geq 1$, by $\ell^p(I) \subset \mathbb{K}^I$ the set of sequences $(x_i)_{i\in I}$ such that $|x_i|^p$ is summable. Put on $\ell^p(I)$ the discrete measure m,
$$\int x\,dm = \sum_{i\in I} x_i := \sup_{J\in\mathcal{P}_f(I)} \sum_{i\in J} x_i < \infty,$$
where $\mathcal{P}_f(I)$ is the set of finite parts of I.

   The case $p = 2$ is very particular, $\ell^2(I)$ with the scalar product defined by
$$< x, y >:= \sum_{i\in I} x_i\overline{y_i}$$
is a prehilbert space.

The classical proof is applicable to the prehilbert case for:

Proposition 1.1.2 *(Cauchy-Schwarz inequality) Let E a prehilbert space. For all $x, y \in E$ we have*
$$| < x, y > |^2 \leq < x, x >< y, y > .$$

Corollary 1.1.2 *Let E be a prehilbert space. the mapping $x \longmapsto \sqrt{< x, x >}$ define a semi-norm on E.*

Proof. For all $x, y \in E$, we have $< x + y, x + y >=< x, x > + < y < y > + < x, y > +\overline{< x, y >} \leq< x, x > + < y, y > +2| < x, y > | \leq [\sqrt{< x, x >} + \sqrt{< y, y >}]^2$, by Cauchy-Schwarz inequality. $\square$

Proposition 1.1.3 *Let E be a prehilbert space. For all $x \in E$, the linear form f_x: $y \longmapsto< y, x >$ is continuous. Moreover the mapping $x \longmapsto f_x$ is anti-linear and isometric from E into E^*.*

Proof. Let p be the semi-norm of corollary 1.1.2. For $y \in E$ we have $|f_x(y)| \leq p(x)p(y)$ (Cauchy-Scwarz). So $f_x \in E^*$ and $\|f_x\| \leq p(x)$. Now since $p(x)^2 = f_x(x) \leq \|f_x\|p(x)$, we get that $\|f_x\| = p(x)$. $\square$

In the following we give a case where equality in the Cauchy-Schwarz inequality occur.

Proposition 1.1.4 *Let $x, y \in E$ a prehilbert space. Then*
$| < x, y > | = \|x\| \cdot \|y\|$ if and only if x and y are linearly dependent.

Proof. The condition is clearly sufficient. Assume that $| < x, y > | = \|x\| \, \|y\|$ and let $\varepsilon \in \mathbb{C}$, $|\varepsilon| = 1$ such that $\mathrm{Re}[\varepsilon < x, y >] = | < x, y > |$. Then $\|y\|x\| - \varepsilon\|y\|x\|^2 = 0$. □

A direct consequence of the definition of the norm is

Proposition 1.1.5 (parallelogram identity) *For all $x, y \in E$ we have*

$$\left\| \frac{x+y}{2} \right\|^2 + \left\| \frac{x-y}{2} \right\|^2 = \frac{1}{2} \left(\|x\|^2 + \|y\|^2 \right).$$

Definition 1.1.3 *Let E be a prehilbert space. We say that two elements x and y of E are orthogonal if $< x, y >= 0$. We say that the subsets A and B are orthogonal if every element of A is orthogonal to every element of B. We call orthogonal of a part A of E the set $A^\perp$ of elements of E orthogonal to A.*

It is clear that $A^\perp = \bigcap\limits_{x \in A} \ker f_x$. Hence its is a closed sub-vector space of E.

A direct consequence is

Proposition 1.1.6 (Pythagore's theorem) *If $x, y \in E$ are orthogonal in a prehilbert space, then*

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Definition 1.1.4 *A Hilbert space is a complete prehilbert space for the norm defined by its scalar product.*

Fundamental examples.

1. Every finite dimensional prehilbert space is a Hilbert space.
2. If μ is a measure on a measured space, the space $L^2(\mu)$ define a Hilbert space with the following scalar product:

$$< f, g >:= \int f \overline{g} \, d\mu.$$

1.2 Projection Theorem

One of the fundamental tools of the Hilbert structure is projection theorem. In the following H is a Hilbert space endowed with the scalar product $<,>$ and the associated norm $\|\cdot\|$.

Theorem 1.2.1 *Let C be a nonempty closed and convex set of H. Then for all $x \in H$, there exists a unique $y \in C$ such that*

$$\|x-y\| = d(x,C).$$

This point y, called **projection of** x **on** C *and denoted by $P_C(x)$, is characterized by*

$$y \in C \quad \textit{and for all } z \in C \quad Re < x-y, z-y > \leq 0. \tag{1.1}$$

Proof. Denote by $d := \inf\{\|x-y\|;\, y \in C\}$ the distance to C. Let $y, z \in C$ and set $b := x - \frac{y+z}{2}$ and $c := \frac{y-z}{2}$. Then $d \leq \|b\|$ since $\frac{y+z}{2} \in C$. Since $x - y = b - c$ and $x - z = b + c$, we have

$$\|x-y\|^2 + \|x-z\|^2 = 2\left[\|b\|^2 + \|c\|^2\right] \geq 2d^2 + \frac{\|y-z\|^2}{2}.$$

Thus $\|y-z\|^2 \leq 2[\|x-y\|^2 - d^2] + 2[\|x-z\|^2 - d^2]$.
For $n \in \mathbb{N}$, set $C_n := \{y \in C;\, \|x-y\|^2 \leq d^2 + \frac{1}{n}\}$. C_n is nonempty closed set of H and the diameter of C_n, $\delta(C_n) \leq 2/\sqrt{n}$ hence tends to zero. Since H is complete, the intersection of C_n, that is equal to $\{y \in C;\, \|x-y\| = d\}$, contains a unique point y_0.
Let $y \in C$. For $t \in [0,1]$, we have $y_0 + t(y-y_0) \in C$, hence $\|y_0 + t(y - y_0) - x\| \geq \|y - y_0\|$. Set $f(t) := \|y_0 + t(y-y_0) - x\|^2 = \|y_0 - x\|^2 + 2t\mathrm{Re} < y_0 - x, y - y_0 > + t^2\|y-y_0\|^2$. Since $f(0) \leq f(t)$ for all $t \in [0,1]$, $f'(0) \geq 0$, i.e. $\mathrm{Re} < y_0 - x, y - y_0 > \geq 0$. □

Denote by P_C the projection on C. Condition (1.1) permits to show that P_C is a contraction:

Proposition 1.2.1 *Under the same hypothesis, for all $x, y \in H$, we have*

$$\|P_C(x) - P_C(y)\| \leq \|x-y\|.$$

Proof. Set $u := P_C x$ and $v := P_C y$. We have

$$\begin{aligned}
\mathrm{Re} < x-y, u-v > &= \mathrm{Re} < x-v, u-v > + \mathrm{Re} < v-x, u-v > \\
&= \mathrm{Re} < x-u, u-v > + \|u-v\|^2 + \mathrm{Re} < v-x, u-v > \\
&\geq \|u-v\|^2.
\end{aligned}$$

Hence by Cauchy-Schwarz inequality, $\|u-v\|^2 \leq \|x-y\| \, \|u-v\|$. □

In the case of sub-vector space:

Proposition 1.2.2 *Let E be a closed sub-vector space of H. Then P_E is a linear operator from H to E. If $x \in H$, then $P_E(x)$ is the unique element $y \in H$ such that*

$$y \in E \qquad \text{and } x-y \in E^{\perp}.$$

Proof. Condition (1.1) could be written as

$$y \in E, \text{ and for all } z \in E, \text{ Re} < x-y, z-y > \leq 0.$$

But if $y \in E$ and $\lambda \in \mathbb{C}^*$, the mapping $z' \longmapsto z = y + \bar{\lambda} z'$ is a bijection from E onto itself. Condition (1.1) is then equivalent to

$$y \in E, \text{ and for all } z' \in E, \text{ and all } \lambda \in \mathbb{C}, \text{ Re}[\lambda < x-y, z' >] \leq 0$$

which is equivalent to

$$y \in E \qquad \text{and } x-y \in E^{\perp}.$$

□

Corollary 1.2.1 *For all closed sub-vector space E of H, we have*

$$H = E \oplus E^{\perp}$$

and the projector on E associated to this direct sum is P_E. P_E is called **orthogonal projector** *on E.*

Proof. If $x \in F$, $x = P_E x + (x - P_E x)$ and by proposition 1.2.2, $P_E x \in E$ and $x - P_E x \in E^{\perp}$. From the other hand, if $x \in E \cap E^{\perp}$, then $< x, x >= 0$, so $x = 0$. □

Corollary 1.2.2 *For all sub-vector space E of H, we have*

$$H = \overline{E} \oplus E^{\perp}.$$

In particular, E is dense in H if and only if $E^{\perp} = \{0\}$.

Proof. Remember that $E^{\perp} = \bar{E}^{\perp}$. □

Corollary 1.2.3 *For all sub-vector space E of H, we have*

$$\overline{E} = E^{\perp\perp}.$$

Proof. Clearly $E \subset E^{\perp\perp}$ and hence, since $E^{\perp\perp}$ is closed, $\bar{E} \subset E^{\perp\perp}$. From the other hand we have $H = \bar{E} \oplus E^{\perp}$ and $H = E^{\perp\perp} \oplus E^{\perp}$. □

Proposition 1.2.3 *The anti-linear isometric mapping $x \longmapsto f_x$ of proposition 1.1.3 is a bijection from H onto H^*.*

Proof. Let $x^* \in H^*$ and denote by E its kernel. If $x^* \neq 0$ then $E \neq H$ and $E^{\perp} \neq \{0\}$ (corollary 1.2.2). Let then $x \in E^{\perp}$, $x \neq 0$. So f_x is zero on E. Since $f_x(x) \neq 0$, there exists $\lambda \in \mathbb{K}$ such that $x^*(x) = \lambda f_x(x)$. Since E is a hyperplane and $x \notin E$, we have $H = E \oplus \mathbb{K}x$. Thus x^* and λf_x that coincide on E and on x are equal. Therefore $x^* = f_{\bar{\lambda}x}$. □

Corollary 1.2.4 *Every Hilbert space is reflexive.*

Proof. Let H be a Hilbert space and $\ell \in H^{**}$. The mapping $x \longmapsto \ell(f_x)$ belongs to H^*. By the last proposition, there exists $y \in H$ such that for all $x \in H$ we have $\ell(f_x) = \overline{f_y(x)} =< y, x >= f_x(y)$. Thus for every $x^* \in H^*$ we have $\ell(x^*) = x^*(y)$, i.e ℓ is the image of y by the canonical injection from H to H^{**}. □

1.3 Adjoint of a linear continuous mapping

Recall that $\mathcal{L}(E, F)$ denote the space of linear continuous (operator) from E into F and that $\mathcal{L}(E) = \mathcal{L}(E, E)$. In what follows E and F are Hilbert spaces.

Theorem 1.3.1 *(**Riesz**) The mapping*

$$\begin{cases} E & \longrightarrow & E^* \\ y & \longmapsto & \phi_y : y^* \in E^* \longmapsto \phi_y(y^*) := < y^*, y > \end{cases}$$

is surjective isometry. In other words, for all linear continuous form ϕ on E, there exists a unique $y \in E$ such that $\phi = \phi_y$ and $\|y\| = \|\phi_y\|$.

In the following we study some important applications of Riesz's Theorem.

Proposition 1.3.1 *Let $T \in \mathcal{L}(E, F)$. There exists a unique $T^* \in \mathcal{L}(F, E)$ such that for all $x \in E$ and all $y \in F$ we have*

$$< Tx, y >=< x, T^*y > .$$

T^* is called the **adjoint** *of T.*

Proof. For all $y \in F$, the mapping $x \longmapsto < Tx, y >$ is linear and continuous. There exists then a unique element y^* denoted by $T^*y \in E$ such that for all $x \in E$ we have $< Tx, y >=< x, T^*y >$. Clearly T^* is linear.
Now, for all $x \in E$ and all $y \in F$ we have $| < x, T^*y > | = | < Tx, y > | \leq \|Tx\| \|y\| \leq \|T\| \|x\| \|y\|$. Thus $\|T^*y\| \leq \|T\| \|y\|$. Therefore T^* is continuous and $\|T^*\| \leq \|T\|$. □

Hereafter some properties of adjoint operator:

Proposition 1.3.2 *The mapping $T \longmapsto T^*$ is anti-linear and isometric from $\mathcal{L}(E, F)$ into $\mathcal{L}(F, E)$: for all $T \in \mathcal{L}(E, F)$ we have $T^{**} = T$ and $\|T^* \circ T\| = \|T\|^2$. For all Hilbert space G, all $S \in \mathcal{L}(E, F)$ and all $T \in \mathcal{L}(F, G)$ we have $(T \circ S)^* = S^* \circ T^*$*

Proof. $\|T^*T\| \leq \|T^*\| \|T\| \leq \|T\|^2$. Now, for $x \in E$, with $\|x\| \leq 1$ we have $\|Tx\|^2 =< Tx, Tx >=< x, T^*Tx >\leq \|T^*T\|$ (Cauchy-Schwarz). Hence $\|T\|^2 \leq \|T^*T\|$. □

Proposition 1.3.3 *Let $T \in \mathcal{L}(E, F)$. Then ker $T^* =$ Im $T^\perp$ and $\overline{T^*(F)} =$ (ker $T)^\perp$.*

Proof. Let $y \in F$. $y \in \ker T^*$ if and only if for all $x \in E$, $< Tx, y >=< x, T^*y >= 0$ if and only if $y \in \mathrm{T}^\perp$. From corollary 1.2.3, $\overline{\mathrm{Im} T} = \ker T^{*\perp}$, replace then T by T^*. □

Definition 1.3.1 *An element $U \in \mathcal{L}(E, F)$ is said to be* **unitary** *if $U^* \circ U = Id_E$ and $U \circ U^* = Id_F$. $T \in \mathcal{L}(E)$ is said to be* **normal** *if $T^* \circ T = T \circ T^*$,* **self-adjoint** *if $T = T^*$ and* **positive** *if it is self-adjoint and, for all $x \in E$ we have $< Tx, x >\geq 0$.*

Examples.

1. Let H be a Hilbert space and $P \in \mathcal{L}(H)$ an orthogonal projector and $E := \mathrm{Im}(P)$ its image. For all $x, x' \in E$ and $y, y' \in E^\perp$ we have $< P(x + y), x' + y' >=< x, x' >=< x + y, P(x + y) >$; hence P is self-adjoint. Moreover, $< P(x + y), x + y >=< x, x >\geq 0$ hence P is positive.

2. For all $T \in \mathcal{L}(H)$, TT^* and T^*T are self-adjoint.

3. Consider the Hilbert space $H := L^2(\Omega, \mu)$ where Ω is a measurable space and μ a σ-finite measure (i.e. Ω is countable union of subset of finite measure for μ). Let $K \in L^2(\mu \times \mu)$. For $f \in H$ define

$$T_K(f) := \int K(x,y) f(y)\, d\mu(y)$$

for μ-a.e. x. By Cauchy-Schwarz inequality, $T_K f \in H$ and T_K is a linear continuous operator on H whose norm verify

$$\|T_K\| \leq \|K\|_{L^2(\mu\times\mu)}.$$

By Fubini's theorem, one can verify that

$$< T_K f, g >= f, T_{K^*} g >,$$

where $K^*(x,y) := \overline{K(y,x)}$. Thus $T_K^* = T_{K^*}$. It is easy to verify that T_K is self-adjoint if and only if, $K(x,y) = \overline{K(y,x)}$ for μ-a.e x and y.

Proposition 1.3.4 *Let T be a self-adjoint operator on H, then*

$$\|T\| = \sup\{< Tx, x > \ \textit{with } x \in E,\ \|x\| = 1\}.$$

Proof. Let $\gamma := \sup\{< Tx, x > \text{ with } x \in E,\ \|x\| = 1\}$. We have $\gamma \leq \|T\|$ and for all $x \in H$, $| < Tx, x > | \leq \gamma \|x\|^2$. Let $y, z \in H$ nad $\lambda \in \mathbb{R}$, then

$$| < T(y \pm \lambda z), y \pm z > | = | < Ty, y > \pm 2\lambda \mathrm{Re} < Ty, z > + \lambda^2 < Tz, z > | \leq \gamma \|y \pm z\|^2.$$

Hence

$$4|\lambda| \mathrm{Re} < Ty, z > \leq \gamma \left[\|y + \lambda z\|^2 + \|y - \lambda z\|^2\right] = 2\gamma \left[\|y\|^2 + \lambda^2 \|z\|^2\right],$$

this is true for all real λ, hence $|\mathrm{Re} < Ty, z > | \leq \gamma \|y\|\,\|z\|$. Choose now $z = Ty$. □

Proposition 1.3.5 *Let $T \in \mathcal{L}(E,F)$. The following conditions are equivalent:*

(i) T *is unitary.*

(ii) T *is surjective and* $T^* \circ T = Id_E$.

(iii) T *is an isometry from* E *to* F.

Proof. (i)$\Rightarrow$(ii): Since $T^*T = Id_F$, T is surjective.
(ii)$\Rightarrow$(iii): If $T^*T = Id_F$, then for all $x \in E$, we have $\|Tx\|^2 = < Tx, Tx > = < x, T^*Tx > = < x, x > = \|x\|^2$.
(iii)$\Rightarrow$(i): Since $(x,y) \longmapsto < x, T^*Ty > = < Tx, Ty >$ is a scalar product on E, by polarization identity we get that, for all $x, y \in E$, $< x, T^*Ty > = < x, y >$. Hence $T^*Ty - y \in E^\perp = \{0\}$. Thus $T^*T = Id_F$ and since T is bijective, $T^* = T^{-1}$. □

Definition 1.3.2 ***(Weak convergence)*** *We say that a sequence $(x_n) \subset E$ converges weakly in E if for all $y \in E$ we have*

$$\lim_{n\to\infty} < x_n, y > = < x, y > .$$

x is called weak limit of the sequence (x_n).

It is clear that a weak limit of a sequence is unique, and by Cauchy-Schwarz inequality, strong convergence implies weak convergence.

As a direct application of Riesz's Theorem one can deduce the following version of Banach-Alaoglu's Theorem in Hilbert space.

Theorem 1.3.2 *From every bounded sequence of E one can extract a weakly convergent subsequence.*

The existence of the adjoint of an arbitrary linear continuous operator gives the following property.

Proposition 1.3.6 *Let (x_n) be a sequence of E that converges weakly to $x \in E$. Then for all $T \in \mathcal{L}(E)$, the sequence Tx_n converges weakly to Tx.*

Proof. For all $y \in E$ we have

$$\lim_n < Tx_n, y > = < x_n, T^*y > = < x, T^*y > = < Tx, y > .$$

□

1.4 Hilbert basis

In this section E will denote a prehilbert space. A system $(x_i)_{i\in I}$ of E is said to be **orthogonal system** if for all $i \neq j$, $x_i \perp x_j$. Recall that, by Pythagore's theorem, we have, for all finite subset J of I

$$\left\| \sum_{i\in J} x_i \right\|^2 = \sum_{i\in J} \|x_i\|^2.$$

We get then directly the following proposition.

Proposition 1.4.1 *An orthogonal system in which all elements are non zero is a free system.*

Proof. Let $J \subset I$ a finite part and $(\lambda_j)_{j\in J} \subset \mathbb{K}$ such that $\sum_{j\in J} \lambda_j x_j = 0$. Then

$$0 = \left\| \sum_{j\in J} \lambda_j x_j \right\|^2 = \sum_{j\in J} |\lambda_j|^2 \|x_j\|^2,$$

and so $\lambda_j = 0$ for all $j \in J$. □

Definition 1.4.1 *An orthogonal system whose elements are of norm 1 is called* **orthonormal basis** *(or orthonormed). A total orthonormal basis of E is called* **Hilbert basis** *of E.*

Examples.

1. Let $T > 0$ and C_T the space of T-periodic continuous functions from $\mathbb{R}$ into $\mathbb{K}$ which is a prehilbert space. For $n \in \mathbb{Z}$ set

   $$e_n(x) := e^{\frac{2i\pi nx}{T}}.$$

   It is easy to see that the class $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal system of C_T. Moreover this system is total in C_T endowed with supremum norm. Since the norm associated to the scalar product is less than or equal to the supremum norm, this system is a Hilbert basis.

2. Consider the space $E = \ell^2(I)$. Define for $j \in I$, the element $e_j \in E$ by $e_j(j) = 1$ et $e_j(i) = 0$ for $i \neq j$. The system $(e_j)_{j\in I}$ is orthonormal (evident). Let's show that it is total. For this, let $x \in E$ and $\varepsilon > 0$. By definition, and since $\sum_{i\in I} |x_i|^2 < \infty$, there exists a finite part $J \subset I$ such that

   $$\sum_{i\in I, i\neq J} |x_i|^2 = \sum_{i\in I} |x_i|^2 - \sum_{i\in J} |x_i|^2 \leq \varepsilon^2.$$

   This implies that

   $$\left\| x - \sum_{i\in J} x_i e_i \right\|^2 \leq \varepsilon^2.$$

   Thus $\ell^2(I)$ is a Hilbert space and $(e_i)_{i\in I}$ is a Hilbert basis of $\ell^2(I)$.

Proposition 1.4.2 *Let $(e_i)_{i\in I}$ a finite orthonormal system of E and let F be the vector space generated by this system. For all $x \in E$, the orthogonal projection $P_F(x)$ is given by*

$$P_F(x) = \sum_{i\in I} < x, e_i > e_i.$$

Consequently,

$$\|x\|^2 = \left\| x - \sum_{i\in I} < x, e_i > e_i \right\|^2 + \sum_{i\in I} | < x, e_i > |^2.$$

Proof. For the first point, it suffices to show that $y := \sum_{j\in J} < x, e_j > e_j$ verify the properties of proposition 1.2.2. It is clear that $y \in F$ and for all $j \in J$, $< x - y, e_j >= 0$, so $x - y \in F^{\perp}$. For the rest apply Pythagore's theorem. □

A first consequence:

Proposition 1.4.3 Bessel's inequality *Let $(e_i)_{i\in I}$ be an orthonormal system of E. Then for all $x \in E$ we have*

$$\sum_{i\in I} | < x, e_i > |^2 \leq \|x\|^2.$$

In particular, $(< x, e_i >)_{i\in I}$ is an element of $\ell^2(I)$.

The equality in the previous inequality is characterized by

Theorem 1.4.1 Bessel-Parseval *Let $(e_i)_{i\in I}$ an orthonormal system of E. The following properties are equivalent:*

(i) The system $(e_i)_{i\in I}$ is a Hilbert Basis.

*(ii) For all $x \in E$, $\|x\|^2 = \sum_{i\in I} | < x, e_i > |^2$ (**Bessel's equality**).*

(iii) For all $x, y \in E$, $< x, y >= \sum_{i\in I} < x, e_i >< e_i, y >$.

Thus, if $(e_i)_{i\in I}$ is a Hilbert basis of E, the mapping from E into $\ell^2(I)$ defined by $x \longmapsto (< x, e_i >)_{i\in I}$ is a linear isometry. This isometry is surjective if and only if E is Hilbert space.

Proof. (i)⇒(ii): Let $x \in E$. For all $\varepsilon > 0$, there exists a finite subset $J \subset I$ s.t. the distance between x and $\mathrm{span}(e_j,\, j \in J)$ is less than ε or equal. By proposition 1.4.2,

$$\|x\|^2 - \varepsilon^2 \leq \sum_{j \in J} | < x, e_j > |^2 \leq \sum_{j \in i} | < x, e_j > |^2.$$

Making $\varepsilon \to 0$ and using Bessel's inequality we get the result.
(ii)⇒(i): Conversely, for all $x \in E$, and all $\varepsilon > 0$, there exists a finite subset $J \subset I$ such that $\|x\|^2 - \varepsilon^2 \leq \sum_{j \in J} | < x, e_j > |^2$ and then by proposition 1.4.2

$$\left\| x - \sum_{j \in J} < x, e_j > e_j \right\| \leq \varepsilon.$$

Thus (e_i) is total.
The equivalence between (ii) and (iii) is direct from the definition of the scalar product in terms of the norm:

$$< x, y > := \quad \tfrac{1}{2} \left[\|x + y\|^2 - \|x\|^2 - \|y\|^2 \right] \quad + \frac{i}{2} \left[\|x + iy\|^2 - \|x\|^2 - \|y\|^2 \right].$$

If the isometry is surjective then E is isometric to $\ell^2(I)$ and hence complete. Now assume that E is a Hilbert space and let $(x_i)_{i \in I} \in \ell^(I)$. Set $a := \sum |x_i|^2$. There exists then an increasing sequence (J_n) of finite subsets of I such that for all $n \in \mathbb{N}$, $\sum_{J_n} |x_i|^2 \geq a - 2^{-n}$. Set $u_n := \sum_{J_n} x_i e_i$. Then, if $p < n$,

$$\|u_p - u_n\|^2 = \sum_{j \in J_p, j \notin J_n} |x_i|^2 \leq 2^{-n}.$$

Thus (u_n) converges to some $x \in E$. Since $a = \sum_{i \in \cap_n J_n} |x_i|^2$, for all $i \notin \cap J_n$, $x_i = 0$ and $< x, e_i >= \lim_{n \to \infty} < u_n, e_i >= 0$. If $i \in \cap J_n$, then $< x, e_i >= \lim_{n \to \infty} < u_n, e_i >= x_i$. Thus $< x, e_i >= x_i$ for all i, which proves the surjectivity. □

As a consequence we get

Theorem 1.4.2 *Let $(e_i)_{i \in I}$ a Hilbert system of E. Then for all all $x \in E$ we have*

$$x = \sum_{i \in I} < x, e_i > e_i.$$

Proof. By proposition 1.4.2 we know that for every finite subset $J \subset I$ we have

$$\left\| x - \sum_{j \in J} < x, e_j > e_j \right\|^2 = \|x\|^2 - \sum_{j \in J} | < x, e_j > |^2.$$

It suffices then to apply the definitions and the second property of the last theorem. □

Proposition 1.4.4 Schmidt orthonormalization procedure *Let* $N \in \{1, 2, \cdots\} \cup \{+\infty\}$ *and* $(f_n)_{0 \le n \le N}$ *a free system of* E. *There exists an orthonormal system* $(f_n)_{0 \le n \le N}$ *of* E, *such that, for all* $p < N$, *the systems* $(f_n)_{0 \le n \le p}$ *and* $(e_n)_{0 \le n \le p}$ *generate the same sub-vector spaces of* E.

Proof. Left as an exercise to the reader.

Using this procedure, one can directly show the following

Corollary 1.4.1 *A prehilbert space is separable if and only if it admits a countable Hilbert basis.*

Two prehilbert spaces are said to be isometric if there exists a surjective isometry from one of them to the other. Another consequence of theorem 1.4.1:

Corollary 1.4.2 *An infinite dimensional Hilbert space is separable if and only if it is iometric to the Hilbert space* ℓ^2.

Chapter 2

Spectrum of a bounded operator

In this chapter we give elementary definitions and properties concerning the spectrum of a linear operator on a Banach or Hilbert space.

2.1 Spectrum

If E is a Banach space on $\mathbb{K} = \mathbb{C}$, denote by $\mathcal{L}(E)$ the Banach algebra (non commutative) of linear continuous mappings from E into itself. The product of two elements T, S is the composition: $TS := T \circ S$. An element $T \in \mathcal{L}(E)$ is said to be invertible, if it admits an inverse in $\mathcal{L}(E)$. In other terms, if T is invertible, it is bijective and its inverse in $\mathcal{L}(E)$ is unique and equal to T^{-1}. Indeed, a direct application of the open mapping theorem is that the inverse of a linear bijective continuous operator is always continuous.

We start by a simple but useful lemma.

Lemma 2.1.1 *Let $T \in \mathcal{L}(E)$ with $\|T\| < 1$. Then $Id + T$ is invertible. The series of general term $(-T)^n$ converges and its sum is $(Id + T)^{-1}$.*

Proof. Set $S_n := \sum_{0 \le k \le n} (-T)^k$. Since $\|T^n\| \le \|T\|^n$, we have, for $p \le q$,

$$\|S_q - S_p\| = \left\| \sum_{p+1 \le k \le q} (-T)^k) \right\| \le \frac{\|T\|^p}{(1 - \|T\|)},$$

thus S_n is a Cauchy sequence in the complete space $\mathcal{L}(E)$. Let S be its limit. For all n, $S_{n+1} = Id - TS_n = Id - S_nT$. Making $n \to \infty$ we get the equality: $S = I - ST = Id - TS$, thus $Id + T$ is invertible and $(Id + T)^{-1} = S$. $\square$

Proposition 2.1.1 *Let E, F be two Banach spaces. The set $U \subset \mathcal{L}(E,F)$ of linear continuous and invertible mapping is an open of $\mathcal{L}(E,F)$. The mapping $\Phi: T \longmapsto T^{-1}$ is continuous is differentiable from U into $\mathcal{L}(F,E)$ and its differential is $(d\Phi)_T: S \longmapsto -T^{-1}ST^{-1}$.*

Proof. If $F = E$ and by the last lemma, V the set of linear continuous invertible mapping in $\mathcal{L}(E)$, is a neighborhood of Id and the mapping $\psi T \longmapsto T^{-1}$ is continuous and differentiable and $d\psi_{Id}h = -h$.
In general. Let $T \in U$. Observe that S is invertible if and only if $T^{-1}S$ is invertible. In this case, $S^{-1} = (T^{-1}S)^{-1}T^{-1}$. In other terms, denoting by $f: \mathcal{L}(E,F) \to \mathcal{L}(E)$, $f(S) := T^{-1}S$ and $g: \mathcal{L}(E) \to \mathcal{L}(E,F)$, $g(S) := ST^{-1}$, we have $U = f^{-1}(V)$ and for all $S \in U$, $\phi(S) = g(\psi(f(S)))$. Therefore, U is a neighborhood of T and since f and g are linear and continuous, ϕ is differentiable at T and $d\phi_T = g \circ d\psi_{Id} \circ f$, i.e. $d\phi_T(h) = -T^{-1}hT^{-1}$. □

Definition 2.1.1 *Let $T \in \mathcal{L}(E)$. We call* **resolvant of** *T, denoted by $\rho(T)$ the set of $\lambda \in \mathbb{C}$ such that $\lambda Id - T$ is invertible. We call* **spectrum of** *T, denoted by $\sigma(T)$, the complementary of the resolvant: $\sigma(T) := C \backslash \rho(T)$. Finally, we call* **resolvent of** *T the mapping that to $\lambda \in \rho(T)$ associate $(\lambda Id - T)^{-1}$, denoted by $R(\lambda)$ or $R(\lambda, T)$.*

Proposition 2.1.2 (Resolvent equation)
Let $T \in \mathcal{L}(E)$. Then, for all $\lambda, \mu \in \rho(T)$, we have

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) = (\mu - \lambda)R(\mu)R(\lambda).$$

Proof. Direct calculation. □

Theorem 2.1.1 *Let E be a non trivial Banach space and $T \in \mathcal{L}(E)$. The spectrum of T is a nonempty compact of $\mathbb{C}$, the resolvent is analytic from $\rho(T)$ into $\mathcal{L}(E)$ and for all $\lambda \in \rho(T)$, we have $R'(\lambda) = R(\lambda)^2$.*

Proof. Let $U \subset \mathcal{L}(E)$ be the set of invertible operators. The mapping $f_\lambda: \lambda \longmapsto T - \lambda$ is continuous, hence the inverse image of U is open. Thus $\sigma(T)$ is closed. Let $\phi: U \to \mathcal{L}(E)$ defined by $g(S) := S^{-1}$. We have $R(\lambda) = \phi \circ f_\lambda$ hence by proposition 2.1.1, R_λ is continuous and differentiable and $R'(\lambda) = d\phi_{f_\lambda(\lambda)} f'_\lambda(\lambda)$, so since $f'_\lambda(\lambda) = -Id$ then $R'(\lambda) = -R(\lambda)(-Id)R(\lambda) = R(\lambda)^2$.
Now let $|\lambda| > \|T\|$. By lemma 2.1.1, $Id - \lambda^{-1}T$ is invertible, hence $\lambda - T$ is invertible and $\lambda R(\lambda) = -R(\lambda^{-1})$. Therefore $\sigma(T)$ is bounded and so a compact of $\mathbb{C}$. Moreover $\lim_{\lambda \to \infty} \lambda R(\lambda) = -Id$. It is clear that $\lambda \longmapsto R(\lambda)$ is analytic on $\rho(T)$. If $\sigma(T)$ is empty then R would be entire, and since

$\lim_{\lambda\to\infty} R(\lambda) = 0$, by Liouville's theorem, $R \equiv 0$, so $\|Id\| \leq \|\lambda - T\|\|(\lambda - T)^{-1}\| = 0$, which is impossible unless E is trivial. □

Remark. If $dim\, E < \infty$, the spectrum of T could be empty in the case where $\mathbb{K} = \mathbb{R}$. For this, it suffices that the characteristic polynomial does not admit real solutions, but this is false in the case where $\mathbb{K} = \mathbb{C}$.

Example. Let $E := C([0,1])$ and T the operator defined for all $f \in E$ by

$$Tf(x) := \int_0^x f(t)\, dt.$$

It is easy to see that $\ker T = \{0\}$ and $\mathrm{Im} T = \{g \in C^1([0,1]);\ g(0) = 0\}$. T is injective but not surjective, in other terms $0 \in \sigma(T)$ and $0 \notin \sigma_p(T)$. Let's show that 0 is the unique spectral value of T: For this take $\lambda \neq 0$ and $g \in E$. If f verify the equation

$$\lambda f - Tf = g, \tag{2.1}$$

then the function $h := Tf \in C^1([0,1])$ and verify

$$h(0) = 0 \qquad \text{and } \lambda h' - h = g. \tag{2.2}$$

Conversely, if $h \in C^1([0,1])$ verify (2.2), then the function $f := h'$ is solution of (2.1). One can see directly that the unique solution of the differential equation (2.2) is given by

$$h(x) = \frac{e^{x/\lambda}}{\lambda} \int_0^x g(t) e^{-t/\lambda}\, dt.$$

Therefore,

$$\lambda f - Tf = g \iff f(x) = \frac{1}{\lambda}\left[g(x) + \frac{e^{x/\lambda}}{\lambda} \int_0^x g(t) e^{-t/\lambda} \right],$$

hence $\lambda \in \rho(T)$ and

$$(\lambda - T)^{-1} g\,(x) = \frac{1}{\lambda}\left[g(x) + \frac{e^{x/\lambda}}{\lambda} \int_0^x g(t) e^{-t/\lambda} \right].$$

Proposition 2.1.3 *Let $T \in \mathcal{L}(E)$. The limit $\lim_{n\to\infty} \|T^n\|^{1/n}$ exists and*

$$\lim_{n\to\infty} \|T^n\|^{1/n} = \inf_{n\in\mathbb{N}^*} \|T^n\|^{1/n}.$$

This value will be denoted by $r(T)$ and called spectral radius of T. Moreover, for all $\lambda \in \sigma(T)$, we have $|\lambda| \leq r(T)$ and

$$r(T) = \max\{|\lambda|; \qquad \lambda \in \sigma(T)\}.$$

In particular $r(T) \leq \|T\|$ and for all $\lambda \in \sigma(T)$, $|\lambda| \leq \|T\|$.

Proof. Set $a := \inf_{n\in\mathbb{N}^*} \|T^n\|^{1/n}$. We have

$$a \leq \liminf_{n\to\infty} \|T^n\|^{1/n}.$$

Let $\varepsilon > 0$ and $n_0 > 0$ such that $\|T^{n_0}\|^{n_0} \leq a+\varepsilon$. Let $n > 0$ and p, q integers with $0 \leq q \leq n_0$ and $n = n_0p + q$. Hence

$$\|T^n\| \leq \|T^{n_0}\|^p \|T\|^q.$$

Since $\lim_{n\to\infty} \frac{q}{n} = 0$ and $\lim \frac{p}{n} = \frac{1}{n_0}$, we deduce that

$$\limsup_{n\to\infty} \|T^n\|^{1/n} \leq \|T^{n_0}\|^{1/n_0} \leq a + \varepsilon.$$

Since this is valid for all $\varepsilon > 0$, we get $\lim_{n\to\infty} \|T^n\|^{1/n} = a$.
Now let λ with $|\lambda| > r(T)$ and $r \in]r(T), |\lambda|[$. Since $r > r(T)$, there is $n_0 > 0$ such that for all $n \geq n_0$, $\|T^n\| \leq r^n$. The series $\sum_{n\geq 0} \lambda^{-n-1}T^n$ is then normally convergent in $\mathcal{L}(E)$ and it is easy to see that

$$(\lambda - T)\left(\sum_{n\geq 0} \lambda^{-n-1}T^n\right) = \left(\sum_{n\geq 0} \lambda^{-n-1}T^n\right)(\lambda - T) = Id$$

hence $\lambda \in \rho(T)$.
Let $\rho := \max\{|\lambda|;\ \lambda \in \rho(T)\}$. We know that $\rho \leq r(T)$. Set for $n > 0$ and $t > \rho$

$$J_n(t) := \frac{1}{2\pi}\int_0^{2\pi} (t\exp(i\theta))^{n+1} R(te^{i\theta})\, d\theta.$$

Since

$$\frac{\partial}{\partial\theta}\left[(t\exp(i\theta))^{n+1}R(te^{i\theta})\right] = it\frac{\partial}{\partial\theta}\left[(t\exp(i\theta))^{n+1}R(te^{i\theta})\right],$$

we see that

$$\frac{dJ_n}{dt} = \frac{1}{2it\pi}\int_0^{2\pi} \frac{\partial}{\partial\theta}\left[(t\exp(i\theta))^{n+1}R(te^{i\theta})\right] d\theta = 0$$

on $]\rho, \infty[$. Hence (expanding $R(\lambda) = \sum \lambda^{-n-1}T^n$) for all $t > \rho$, $J_n(t) = T^n$, thus $\|T^n\| = \|J_n(t)\| \leq t^{n+1}M_t$, where M_t is the maximum of $\|R(te^{i\theta})\|$, for $\theta \in [0, 2\pi]$. Therefore, for all $t > \rho$, $r(T) \leq t$, since $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$, and so $r(T) \leq \rho(T)$. □

We will often use the following (simple) proposition

Proposition 2.1.4 *Let E, F two Banach spaces and $T \in \mathcal{L}(E, F)$. The following are equivalent:*

(i) T is injective and its image is closed.

(ii) There exists $K > 0$ such that for all $x \in X$ we have $\|Tx\| \geq K\|x\|$.

(iii) There is no sequence $(x_n) \subset E$ such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|Tx_n\| = 0$.

Proposition 2.1.5 *Let E, F be two Banach spaces and $T \in \mathcal{L}(E, F)$. Then ${}^tT \in \mathcal{L}(F^*, E^*)$ is invertible if and only if T is invertible.*

Proof. If T is invertible then $T^{-1}T = Id_E$ and $TT^{-1} = Id_F$. This gives that ${}^tT{}^t(T^{-1}) = Id_{E^*}$ and ${}^t(T^{-1}){}^tT = Id_{F^*}$. Hence ${}^tT$ is invertible and $({}^tT)^{-1} = {}^t(T^{-1})$.
Conversely, if ${}^tT$ is invertible. Let $x \in E$ and $x^* \in E^*$ (by Hahn-Banach) with $\|x^*\| \leq 1$ and $x^*(x) = \|x\|$. Set $y^* := ({}^tT)^{-1}x^*$. Then $x^* = {}^tTy^* = y^* \circ T$ and $\|y^*\| \leq K\|x^*\| \leq K$, where $K := \|({}^tT)^{-1}\|$. Thus $\|x\| = x^*(x) = y^*(Tx) \leq K\|Tx\|$. Hence T is injective and its image is closed in F.
By Hahn-Banach theorem there exists $A \subset F^*$ such that $\mathrm{Im}T = \cap_{y^*\in A}\ker y^*$. So, for all $y^* \in A$, y^* is zero on $\mathrm{Im}T$. Hence ${}^tTy^*$ $(= y^* \circ T)$ is zero, and since ${}^tT$ is bijective, $y^* = 0$. Therefore $A \subset \{0\}$, i.e. $\mathrm{Im}T = F$. □

We get directly:

Corollary 2.1.1 *$\sigma(T) = \sigma({}^tT)$.*

Definition 2.1.2 *Let $T \in \mathcal{L}(E)$ and $\lambda \in \sigma(T)$. We distinguish three possibilities:*

1. *λ is an eigenvalue, i.e. $\lambda - T$ is not injective. We say that λ is in the* **point spectrum** *$\sigma_p(T)$ of T.*

2. *$\lambda - T$ is injective but $Im(\lambda - T)$ is not dense in E. We say that λ is in the* **residue spectrum** *$\sigma_r(T)$ of T.*

3. *$\lambda - T$ is injective but its image is not closed. We say that λ is in the* **continuous spectrum** *$\sigma_c(T)$ of T.*

Remarks

1. $\lambda \in \sigma_r(T)$ means that λ is an eigenvalue of ${}^tT$, but not of T, i.e $\lambda - T$ is injective but $\lambda - {}^tT$ is not: there exists then $x^* \in E^*$ such that $(\lambda - {}^tT)x^* = 0$ hence $x^* \circ (\lambda - T) = 0$ which implies that $\mathrm{Im}(\lambda - T) \subset \ker x^*$. Then $\mathrm{Im}(\lambda - T)$ is not dense in E.

2. $\lambda \in \sigma_c(T)$ means that $\lambda \in \sigma(T)$ but λ is not eigenvalue of T or of ${}^tT$.

3. We have $\sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \sigma_c(T)$.

2.2 Hilbert case

In this section we consider the particular case where H is a non trivial Hilbert space. Some properties of bounded self-adjoint operators are given.

From proposition 1.3.3 we deduce directly:

Corollary 2.2.1 *Let $T \in \mathcal{L}(H)$, then*

$$\sigma(T^*) = \overline{\sigma(T)} = \{\overline{\lambda},\ \lambda \in \sigma(T)\}.$$

If $\lambda \in \rho(T)$, then $\overline{\lambda} \in \rho(T^*)$ and

$$R(\overline{\lambda}, T) = [R(\lambda, T)]^*.$$

Moreover

$$\sigma_r(T) = \{\lambda \in \mathbb{C} \backslash \sigma_p(T);\ \bar{\lambda} \in \sigma_p(T^*)\}$$

Proposition 2.2.1 *The residue spectrum of a normal operator is empty.*

Proof. Let $T \in \mathcal{L}(H)$ a normal operator. For all $x \in H$, we have $\|T^*x\|^2 =< T^*x, T^*x >=< x, TT^*x >=< x, T^*Tx >=< Tx, Tx >= \|Tx\|^2$. So $\ker T^* = \ker T$. Since for all λ, $\lambda - T$ is normal, we have $\ker(\bar{\lambda} - T^*) = \ker(\lambda - T)$. Thus $\bar{\lambda}\sigma_p(T^*)$ if and only if $\lambda \in \sigma_p(T)$. We get the result applying the last corollary. $\square$

There is no relation between eigenvalues of T and those of T^*:

Example. Let $E = \ell^2(\mathbb{N})$ and T the operator right shift, defined by $(Tu) = v$ where v is the sequence defined by $v_0 = 0$ and for all $i \geq 1$, $v_i = u_{i-1}$. T does not admit eigenvalues: $\sigma_p(T) = \emptyset$. It is easy to verify that the adjoint of T is the conjugate of the operator left shift and that $\sigma_p(T^*) = D(0, 1)$ the open unit disc.

Proposition 2.2.2 *For all $T \in \mathcal{L}(H)$ we have $\|T^*T\| = \|TT^*\| = \|T\|^2$.*

Proof. Since $\|T^*\| = \|T\|$ we have $\|T^*T\| \leq \|T\|^2$. From the other hand, $\|Tx\|^2 =< Tx, Tx >=< x, T^*T, x >\leq \|x\|^2\|T^*T\|$. Hence $\|T\|^2 \leq \|T^*T\|$. Thus $\|T\|^2 = \|T^*T\|$. $\square$

Proposition 2.2.3 *The spectral radius of a normal operator is equal to its norm.*

Proof. If T is self-adjoint, then $< Tx, Tx >=< x, T^2x >$, thus $\|T^2\| = \|T\|^2$ and $\|T^{2^n}\| = \|T\|^{2^n}$. Hence $r(T) = \lim_{n\to\infty} \|T^{2^n}\|^{2^{-n}} = \|T\|$.
Now if T is normal then we have $\|Tx\|^2 =< Tx, Tx >=< x, T^*Tx >$, hence $\|T^*T\| = \|T\|^2$. By induction, $\|(T^*T)^n\| = \|T^n\|^2$ and then $\|T^*T\| = \rho(T^*T) = \rho(T)^2 = \|T\|^2$. $\square$

This gives directly

Corollary 2.2.2 *Let $T \in \mathcal{L}(H)$, then*

$$\|T\| = \sqrt{r(TT^*)} = \sqrt{r(T^*T)}.$$

Proposition 2.2.4 *Let T be a self-adjoint operator on H. Then*

1. *$\sigma_p(T) \subset \mathbb{R}$.*
2. *For all $\lambda \in \mathbb{C}$, $\overline{Im(\lambda - T)} = [ker(\overline{\lambda} - T)]^\perp$.*
3. *Eigen-spaces associated to distinct eigenvalues are orthogonal.*

Proof. 1. Let $\lambda \in \sigma_p(T)$ and $x \in H_\lambda$, i.e. $x \neq 0$, $Tx = \lambda x$. Then $\lambda\|x\|^2 =< Tx, x >\in \mathbb{R}$ since T is self-adjoint, hence $\lambda \in \mathbb{R}$.
2. direct from proposition 1.3.3.
3. If $\lambda \neq \mu$ are two eigenvalues of T and $x \in H_\lambda$ and $y \in H_\mu$, then $\lambda < x, y >=< Tx, y >=< x, Ty >= \mu < x, y >$. Thus $< x, y >= 0$. $\square$

The following theorem states that, in fact, the whole spectrum is real.

Theorem 2.2.1 *Let T be a bounded self-adjoint operator on H. Then*

$$\sigma(T) \subset [m, M],$$

$m \in \sigma(T)$ and $M \in \sigma(T)$, where

$$m = \inf\{< Tx, x >, \text{ with } x \in E,\ \|x\| = 1\}$$

and

$$M = \sup\{< Tx, x >, \text{ with } x \in E,\ \|x\| = 1\}$$

Proof. Set, for $\lambda \in \mathbb{C}$, $d(\lambda)$ the distance from λ to the interval $[m, M]$. For all $x \in H$, $x \neq 0$, we have

$$< \lambda x - Tx, x >= \|x\|^2 [\lambda- < Ty, y >],$$

where $y := x/\|x\|$. Then by Cauchy-Schwarz inequality we have

$$d(\lambda)\|x\|^2 \leq | < \lambda x - Tx, x > | \leq \|x\| \, \|\lambda x - Tx\|. \tag{2.3}$$

Now if $\lambda \notin [m, M]$, then $d(\lambda) > 0$ and then $\lambda - T$ is injective. Let's show that ImT is closed. If $(\lambda x_n - T x_n)$ is a sequence that converges to $y \in H$, then by equation (2.3), (x_n) is a Cauchy sequence hence convergent to some $x \in H$. Clearly $\lambda x - Tx = y$ hence $y \in \mathrm{Im}\ (\lambda - T)$. By proposition 2.2.4, we have $\overline{\mathrm{Im}(\lambda - T)} = \ker(\bar{\lambda} - T)^{\perp}$. Since $\bar{\lambda} \notin [m, M]$, $\overline{\mathrm{Im}(\lambda - T)} = H$ and hence $\lambda - T$ is bijective. Therefore $\lambda \in \rho(T)$.
Remainder to show that $m, M \in \sigma(T)$. Let's show, for example, that $m \in \sigma(T)$ (for M consider $-T$). Set $S := T - m$, then S is positive. The mapping $(x, y) \longmapsto < Sx, y >$ is a scalar product on H. Cauchy-Schwarz inequality for this scalar product gives, for all $x, y \in H$

$$| < Sx, y > |^2 \leq | < Sx, x > | \, | < Sy, y > |. \tag{2.4}$$

Now by definition of m, there is a sequence (x_n), $\|x_n\| = 1$, with $\lim_{n\to\infty} | < Sx_n, x_n > | = 0$. Hence by (2.4)

$$\|Sx_n\|^2 \leq | < Sx_n, x_n > |^{\frac{1}{2}} | < Sx_n, Sx_n > |^{\frac{1}{2}} \leq | < Sx_n, x_n > |^{\frac{1}{2}} \|S\|^{\frac{1}{2}} \|Sx_n\|.$$

Therefore $\|Sx_n\| \leq | < Sx_n, x_n > |^{\frac{1}{2}} \|S\|^{\frac{1}{2}}$ hence tends to zero. If $m \notin \sigma(T)$, S is invertible and hence $x_n \to 0$ which is impossible. □

Using proposition 1.3.4 we get

Corollary 2.2.3 *Let T be a self-adjoint operator on H. Then T is positive if and only if $\sigma(T) \subset \mathbb{R}_+$. In this case $\|T\| \in \sigma(T)$.*

Proof. Since T is self-adjoint, $\|T\| = \sup_{\|x\|=1} | < Tx, x > |$ hence $\|T\| = \max\{|m|, |M|\}$. T positive implies that $0 \leq m \leq M$ and so $\|T\| = M \in \sigma(T)$. □

Chapter 3

Symbolic Calculus

One of the most important aims of spectral theory is the symbolic calculus: Given a linear operator A, find the functional space $\mathcal{A}$ (the best possible) on which one can define $f(A)$, $f \in \mathcal{A}$. A good functional space is for example $\mathcal{H}(O)$ the space of analytic functions on the open O of the complex plane that contains the spectrum of A. But also, in the case where the spectrum is real, the space $\mathcal{C}(\mathbb{R})$. With a functional space we can "translate properties of functions to the operators".

In this chapter we will define such symbolic calculus in the case where A is bounded, and then in the Hilbert case where A is self-adjoint. Later we will deal with the case of unbounded self-adjoint operator...

3.1 Case of bounded operator

In all this section X is a Banach space and A a bounded operator, $A \in \mathcal{L}(X)$. Denote by $R_A(X)$ the set of rational fractions without poles in $\sigma(A)$, i.e. the set of fractions $\frac{p}{q}$, where $p, q \in \mathbb{C}(X)$ with $\text{Zero}(q) \cap \sigma(A) = \emptyset$. This space will play an important role, since we can define $p(A)$ in a naturel way and hence $\frac{p}{q}(A)$. Note that R_A is a ring with identity (1) and for all $p, q \in R_A$, $p(A)q(A) = q(A)p(A)$.

Proposition 3.1.1 *There exists a unique linear mapping* $\Phi\colon R_A \to \mathcal{L}(X)$ *homomorphism of rings verifying* $\Phi(1) = Id$ *and* $\Phi(X) = A$.

Proof. *existence.* For all polynomial $p(x) = \sum a_k X^k \in \mathbb{C}[X]$, set $\Phi(p) := \sum a_k A^k \in \mathcal{L}(X)$. It is obvious that Φ is linear and $\Phi(pq) = \Phi(p)\Phi(q)$.
Now if p is a polynomial ($\neq 0$) with $\text{Zero}(p) \cap \sigma(A) = \emptyset$, then $\Phi(A)$ is invertible: Indeed, it suffices to write $p(X) = a\Pi(X - r_k)$, where the r_k's are the roots of p (counted with their multiplicity). Since the r_k's are not in $\sigma(A)$,

each $A - r_k$ is invertible and so $\Phi(A)$.
Now if $f = \frac{p}{q} \in R_A$ then $\Phi(f) = p(A)[q(A)]^{-1}$. Of course $\Phi(f)$ is independent of the choice of p and q.
The linearity of Φ as well as the homomorphism is direct.
uniqueness. If $\Psi\colon R_A \to \mathcal{L}(X)$ is another mapping verifying the same properties, we can show by induction that $\Psi(x^n) = A^n$, so by linearity Ψ and Φ coincide on $\mathbb{C}[X]$. $\square$

The uniqueness of the mapping Φ justify the following notation:

Notation. The operator $\Phi(f)$ will be denoted: $f(A)$.

Remark. This justify the appellation symbolic calculus. In fact, if for all nonnegative integer n, x^n is the function $x \longmapsto x^n$, then $x^n(A) = A^n$. From this we get that for all polynomial $p(x) = \sum a_i x^i$, $p(A)$ defined by use of Φ is the same as the "classical" $p(A)$.

Theorem 3.1.1 Spectral mapping theorem
For all $f \in R_A$ we have

$$\sigma(f(A)) = f(\sigma(A)),$$

and for all $g \in R_{f(A))}$, we have $g(f(A)) = [g \circ f](A)$.

Proof. Let $\lambda \in \mathbb{C}$. If $f - \lambda$ does not vanish on the spectrum of A then $h : (f-\lambda)^{-1} \in R_A$ and since $(f-\lambda)h = 1$ then $(f-\lambda)(A)h(A) = Id$. Thus $(f-\lambda)(A) = f(A) - \lambda Id$ is invertible and so $\sigma(f(A)) \subset f(\sigma(A))$.
Now let $\lambda \in \mathbb{C}$, that is not a pole of f, there exists then $h \in R_A$ such that $f - f(\lambda) = (x-\lambda)h$. Then $f(A) - f(\lambda) = (A-\lambda)h(A) = h(A)(A-\lambda)$. If $f(\lambda) - f(A)$ is invertible of inverse R then $(T-\lambda)h(A)S = Id = Sh(A)(A-\lambda)$ and so $A - \lambda$ is invertible, i.e. $\lambda \notin \sigma(A)$. Thus $f(\sigma(A)) \subset \sigma(f(A))$.
To terminate, notice that the two mappings $R_{f(A)} \to \mathcal{L}(X)$ defined by $g \longmapsto g(f(A))$ and $g \longmapsto [g \circ f](A)$ verify the conditions of proposition 3.1.1, thus they coincide. $\square$

Other type of symbolic calculus could be defined in this framework: Since the spectrum of A is compact hence bounded, let γ be an arbitrary path, that is bounded and turns around $\sigma(T)$. γ oriented positively. Briefly, note that the theory of integrals on paths could be generalized for analytic functions defined on a neighborhood O of Imγ into $\mathcal{L}(X)$. Notice also that the residue formula (Cauchy) still valid. Therefore, if $f \in H(O)$, an analytic function on O valued in $\mathcal{L}(X)$ then the formula

$$\int_\gamma f(z)\, dz,$$

define a bounded operator in $\mathcal{L}(X)$. In this framework we can show that the Dunford integral:

$$\Phi(f) := \frac{1}{2i\pi}\int_\gamma f(z)R(z,A)\,dz$$

define symbolic calculus on $H(O)$ (that extend the one defined above).

3.2 Case of a bounded self-adjoint operator

In this section, H will be a Hilbert space.

Proposition 3.2.1 *Let $A \in \mathcal{L}(X)$ and $f \in R_A$. Then $f(A)^* = \tilde{f}(A^*)$, where $\tilde{f}$ is defined by $\tilde{f}(\overline{\lambda}) = \overline{f(\lambda)}$ for all $\lambda \in \mathbb{C}$ that is not a pole of f.*

Proof. The mapping $R_A \to \mathcal{L}(H)$ defined by $f \longmapsto \tilde{f}(A^*)^*$ verify the conditions of proposition 3.1.1. □

Proposition 3.2.2 *If $A \in \mathcal{L}(H)$ is normal then for all $f \in R_A$, $f(A)$ is normal.*

Proof. For all $Y \subset \mathcal{L}(H)$ denote by $Y' := \{S \in \mathcal{L}(H);\ ST = TS\ \forall T \in Y\}$. Y' is a closed subspace and a sub-ring of $\mathcal{L}(H)$. Moreover, if $S \in Y'$ and S invertible then $S^{-1}T = S^{-1}TSS^{-1} = S^{-1}STS^{-1} = TS^{-1}$, for all $T \in Y$. In other words, $S^{-1} \in Y'$. Therefore, if $S \in Y'$ and $f \in R_S$, then $f(S) \in Y'$.
Let $Y = \{A, A^*\}$ and $Z = Y'$. Since all elements of Y commutes with all elements of Z, we see that $Y \subset Z'$. So $f(A), \tilde{f}(A^*) \in Z$. Since A is normal, $Y \subset Z$, then $Z' \subset Y' = Z$. Thus $f(A) \in Z'$ and $\tilde{f}(A^*) = f(A)^* \in Z' \subset Z$ so they commute, i.e. $f(A)$ is normal. □

Proposition 3.2.3

1. *The spectrum of any unitary operator of $\mathcal{L}(H)$ is included into the unit circle $C(0,1)$ of the complex plane.*
2. *The spectrum of any self-adjoint operator of $\mathcal{L}(H)$ is included into the real line $\mathbb{R}$.*

Proof. 1. Let $U \in \mathcal{L}(H)$ and $\lambda \in \sigma(U)$. Since $\|U\| \leq 1$ the spectral radius of U is less than 1 or equal, hence $|\lambda| \leq 1$, and since U is bijective $\lambda \neq 0$ and by theorem 3.1.1, $\lambda^{-1} \in \sigma(U^{-1})$. But $U^{-1} = U^*$ so $\|U^{-1}\| \leq 1$ thus $|\lambda^{-1}| \leq 1$, i.e. $|\lambda| = 1$.
2. Let $A \in \mathcal{L}(H)$ a self-adjoint operator. For all real t, with $t > \|A\|$, $A \pm tId$ are invertible. Denote by f the mapping $X \longmapsto (X + ti)/(X - ti)$. Since $\tilde{f} = f^{-1}$, by proposition 3.2.1, $f(A)^* = \tilde{f}(A) = f(A)^{-1}$ thus $f(A)$ is

unitary and by the first point $\sigma(f(A)) \subset C(0,1) = f(\mathbb{R})$. Using theorem 3.1.1 we get the result. $\square$

Notations. If K is a compact space, denote by $C(K)$ the Banach space of continuous functions from K into $\mathbb{C}$ with the supremum norm:
$\|f\|_\infty := \sup\{|f(x)|;\, x \in K\}$.
If K is a compact of $\mathbb{C}$, denote by z^n, for all integer n, the mapping $\lambda \longmapsto \lambda^n$.

Theorem 3.2.1 *Let H be a Hilbert space and $A \in \mathcal{L}(H)$ a self-adjoint or unitary operator. There exists a unique linear continuous mapping*
$\Phi: C(\sigma(A)) \to \mathcal{L}(H)$ verifying $\Phi(1) = Id$, $\Phi(z) = A$ and for all $f, g \in C(\sigma(A))$, we have $\Phi(fg) = \Phi(f)\Phi(g)$. For all $f \in R_A$, we have $\Phi(f) = f(A)$. Moreover, Φ is an isometry, and for all $f \in C(\sigma(A))$ we have $\Phi(f)^* = \Phi(\overline{f})$.

Proof. Let $\phi: R_A \to C(\sigma(A))$ defined by $\phi(f) = f_{|_{\sigma(A)}}$ and $\Psi: R_A \to \mathcal{L}(H)$ defined by $\Psi(f) = f(A)$. For all $f \in R_A$, $f(A)$ is normal by proposition 3.2.2 hence $\|f(A)\|$ is equal to its spectral radius. Thus by theorem 3.1.1, $\|f(A)\| = \sup\{|\lambda|,\, \lambda \in \sigma(f(A))\} = \sup\{|f(\lambda)|,\, \lambda \in \sigma(A)\}$. Therefore $\|\Psi(f)\| = \|\phi(f)\|_\infty$.
Now if A is unitary then $\bar{z} = z^{-1} = \phi(X^{-1}) \in \phi(R_A)$, and if A is self-adjoint then $\bar{z} = z = \phi(X) \in \phi(R_A)$. In both cases $\bar{z} \in \phi(R_A)$. Now for all $f \in R_A$, $\overline{f(z)} = \tilde{f}(\bar{z}) \in \phi(R_A)$. Therefore $\phi(R_A)$ is sub-vector space and sub-ring of $C(\sigma(A))$ that contains constants (since $1 = \phi(1)$) stable under conjugate and separate points of $\sigma(A)$ (since $z \in \phi(R_A)$), so by Stone-Weierstrass theorem, $\phi(R_A)$ is dense in $C(\sigma(A))$. Therefore there exists a unique linear continuous mapping $\Phi: C(\sigma(A)) \to \mathcal{L}(H)$ such that $\Psi = \Phi \circ \phi$.
We have $\Phi(1) = \Phi(\phi(1)) = \Psi(1) = Id$. $\Phi(z) = \Phi(\phi(X)) = \Psi(X) = A$. The mappings that to $(f,g) \in C(\sigma(A)) \times C(\sigma(A))$ associates respectively $\Phi(fg)$ and $\Phi(f)\Phi(g)$ coincide on $\Phi(R_A) \times \Phi(R_A)$ so they are equals. Moreover the set of functions $f \in C(\sigma(A))$ with $\|f(A)\| = \|f\|_\infty$ is closed and contains $\Phi(A)$. Hence Φ is an isometry. Finally, for $f \in R_A$, we have $f(A)^* = \tilde{f}(A^*)\Phi(\tilde{f}(\bar{z})) = \Phi(\overline{f(z)})$. The set of functions $f \in C(\sigma(A))$ such that $\Phi(f)^* = \Phi(\bar{f})$ is closed and contains $\phi(A)$, hence this true for all $f \in C(\sigma(A))$.
Remains to show uniqueness. If Φ_1 is another one, then $\Phi \circ \phi$ and $\Phi_1 \circ \phi$ coincide on $\phi(R_A)$. By density we get $\Phi = \Phi_1$. $\square$

Notation. For all $f \in C(\sigma(A))$, denote by $\Phi(f) = f(A)$.

Theorem 3.2.2 Spectral mapping theorem
Let A be a self-adjoint, or unitary operator and $f \in C(\sigma(A)$. Then

1. *$f(A)$ is normal and $\sigma(f(A)) = f(\sigma(A))$.*

2. *If $f(\sigma(A)) \subset \mathbb{R}$ then $f(A)$ is self-adjoint. If $f(\sigma(A)) \subset C(0,1)$ then $f(A)$ is unitary. Moreover, in these cases, for all $g \in C(\sigma(f(A)))$ we have $g \circ f(A) = g(f(A))$.*

Proof. 1. We have $f(A)^* = \bar{f}(A)$, so $f(A)f(A)^* = [f\bar{f}](A) = f(A)^*f(A)$, so $f(A)$ is normal.
Now, if $\lambda \notin f(\sigma(A))$, let $h \in C(\sigma(A))$ the function $s \longmapsto 1/[f(s) - \lambda]$. Since $h(f - \lambda) = (f - \lambda)h = 1$, then $h(A)[f(A) - \lambda] = [f(A) - \lambda]h(A) = Id$ and so $\lambda \notin \sigma(f(A))$.
Conversely, if $\lambda \in f(\sigma(A))$, for $\varepsilon > 0$, set $f_1 := f - \varepsilon$ and $g := \varepsilon/(|f_1| + \varepsilon)$. Notice that $\|g\|_\infty = 1$, and since $|f_1 g|(t) = \varepsilon|f_1(t)|/[|f_1(t)| + \varepsilon]$ so $\|f_1 g\| < \varepsilon$. Since Φ is isometry we have $\|g(A)\| = 1$ and $\|f_1(A)g(A)\| < \varepsilon$. Since $\|g(A)\| = 1 > \|f_1 g\|_\infty/\varepsilon$, there exists $x \in H$ such that $\|g(A)x\| > \|f_1 g\|_\infty \|x\|/\varepsilon$ and so $\|f_1(A)g(A)x\| \leq \|f_1(A)g(A)\|\|x\| < \varepsilon\|g(A)x\|$. Thus there is $y = g(A)x$ such that $\|f_1(A)y\| < \varepsilon\|y\|$. Thus $f_1(A) = f(A) - \lambda$ is not injective hence $\lambda \in \sigma(f(A))$.
2. If $f = \bar{f}$ then $f(A) = \bar{f}(A) = f(A)^*$. If $f(\sigma(A)) \subset C(0,1)$ then $f\bar{f} = 1$, hence $f(A)f(A)^* = f(A)^*f(A) = [f\bar{f}](A) = Id$; $f(A)$ is unitary. Finally the mapping $g \longmapsto [g \circ f](A)$ verifies the conditions of the last theorem, hence coincides with $g \longmapsto g(f(A))$. $\square$

Theorem 3.2.3 *Let $A \in \mathcal{L}(H)$. The following conditions are equivalent:*

(i) For all $x \in H$, $< Ax, x > \in \mathbb{R}_+$.

(ii) There exists $S \in \mathcal{L}(H)$, $A = S^*S$.

(iii) There exists $S \in \mathcal{L}(H)$ self-adjoint, $A = S^2$.

(iv) A is self-adjoint and $\sigma(A) \subset \mathbb{R}_+$.

Proof. (ii)$\Rightarrow$(i): $< Tx, x > = < S^*Sx, x > = < Sx, Sx > \geq 0$.
(iii)$\Rightarrow$(ii) is direct.
(iv)$\Rightarrow$(iii): Assume that $A^* = A$ and $\sigma(T) \subset \mathbb{R}^+$. Denote by $f\colon t \longmapsto \sqrt{t}$. Then by the last theorem, we have $f(A) = f(A)^*$, moreover $f(A)^2 = A$.
(i)$\Rightarrow$(iv): The mapping $(x, y) \longmapsto < Ax, y >$ is sesqui-linear and $< Ay, x > = \overline{< Ax, y >}$ hence A is self-adjoint. Thus $\sigma(A) \subset \mathbb{R}$. Let $t < 0$ and let's show that $A - t$ is bijective. For all $x \in H$, we have

$$-t\|x\|^2 \leq -t\|x\|^2 + < Ax, x > = < (A - t)x, x > \leq \|(A - t)x\|\|x\|$$

so $-t\|x\| \leq \|(A - t)x\|$ and so $A - t$ is injective with closed graph. Now since the residual spectrum of every normal operator is empty, we get the result. $\square$

Definition 3.2.1 (Fractional powers) *If $A \in \mathcal{L}(H)$ is self-adjoint and positive and $\alpha \in]0, +\infty[$, set $A^\alpha = f_\alpha(A)$ where f_α is the mapping $t \longmapsto t^\alpha$.*

Remark. By theorem 3.2.2, we have, for all $\alpha, \beta > 0$

- $A^1 = A$,
- $(A^\alpha)^\beta = A^{\alpha\beta}$,
- $A^\alpha A^\beta = A^{\alpha+\beta}$.

Corollary 3.2.1 (Square root) *For all positive self-adjoint operator $A \in \mathcal{L}(H)$, square root of A, $A^{\frac{1}{2}}$ is a positive self-adjoint operator.*

Chapter 4

Compact operators

In this chapter we will study spectral properties of some particular type of operators: compact operators and Hilbert-Schmidt operators. We will see also Fredholm alternative.

4.1 General properties

In all this section, E and F are two Banach spaces.

Definition 4.1.1 *$A \in \mathcal{L}(E,F)$ is called compact if the image of the closed unit ball of E, $A(\overline{B_E(0,1)})$ is relatively compact in F. Denote by $\mathcal{K}(E,F)$ the set of compact operators from E into F and $\mathcal{K}(E) = \mathcal{K}(E,E)$.*

Remarks.

1. $A \in \mathcal{K}(E,F)$ if and only if the image by A of any bounded subset of E is relatively compact in F.

2. $A \in \mathcal{K}(E,F)$ if and only if the image by A of any bounded sequence of E is a sequence of F with convergent subsequences.

3. Riesz theorem becomes: $Id \in \mathcal{K}(E,E)$ if and only if the dimension of E is finite.

Examples.

1. Every operator T of **finite rank**, i.e. $\dim \mathrm{Im}T < \infty$ is compact. In fact, the image $T(\bar{B})$ is bounded in a finite dimensional space hence relatively compact in $\mathrm{Im}T$ hence relatively compact in F.

2. Let X, Y be two compact metric spaces, $K \in C(X \times Y)$ and μ any Radon measure on Y. Define the **kernel operator** T_K, for all $f \in C(Y)$ by
$$(T_K f)(x) := \int K(x,y) f(y)\, d\mu(y).$$
T_K is compact operator.

3. Let $a < b$, $K \in C([a,b]^2)$ and α, β two continuous functions from $[a,b]$ into itself. For $f \in C([a,b])$ and $x \in [a,b]$ set
$$Tf(x) := \int_{\alpha(x)}^{\beta(x)} K(x,y) f(y)\, dy.$$
The operator T is compact: $T \in \mathcal{K}(C([a,b])$.
In fact, for all $f \in E$ where $E := C([a,b])$, we have
$$\|Tf\| \leq M \|K\| \ \|f\|,$$
where $M := \sup_{x\in[a,b]} |\beta(x) - \alpha(x)|$. Hence $T(\bar{B})$ is a bounded in E. From the other hand, for all $x, y \in [a,b]$ and all $f \in E$ we have
$$|Tf(x) - Tf(y)| \leq M_{x,y} \|f\|,$$
where
$$\begin{aligned} M_{x,y} := \quad & \|K\| \left(|\beta(x) - \beta(y)| + |\alpha(x) - \alpha(y)| \right) \\ & + (\|\alpha\|_\infty + \|\beta\|_\infty) \sup_{z\in[a,b]} |K(x,z) - K(y,z)|. \end{aligned}$$
Uniform continuity of K on $[a,b]^2$ implies that $T(\bar{B})$ is equicontinuous in E. We conclude using Ascoli's theorem.

4. Integration operator
$$Tf(x) := \int_a^x f(t)\, dt$$
is a compact operator on $C([a,b])$.

Proposition 4.1.1 *Let $R \in \mathcal{K}(E,F)$, $T \in \mathcal{L}(E_1, E)$, $S \in \mathcal{L}(F, F_1)$ where E_1 and F_1 are normed spaces. Then SRT is a compact operator.*

Proof. Indeed,
$$SRT(\overline{B_E}) \subset \|T\| S \left(\overline{R(\overline{B_E})} \right).$$
Continuous image of a compact being a compact, we get the result. □

Proposition 4.1.2 *$\mathcal{K}(E,F)$ is a closed sub-vector space of $\mathcal{L}(E,F)$.*

Proof. Let T, S be two compact operators from E to F and $\lambda, \mu \in \mathbb{K}$. Then

$$(\lambda T + \mu S)(\overline{B_E}) \subset \lambda\overline{T(\overline{B_E})} + \mu\overline{S(\overline{B_E})},$$

and this last set is compact since if K and H are two compacts then $\lambda K + \mu H$ is compact as continuous image of $K \times H$. To show that $\mathcal{K}(E,F)$ is closed, let (T_n) be a sequence of compact operators that converges to T (in $\mathcal{L}(E,F)$). It suffices to show that $T\bar{B}_E$. Let $\varepsilon > 0$, and $n \in \mathbb{N}$ such that $\|T - T_n\| \leq \varepsilon/3$. Let $f_1, \cdots, f_k \in \bar{B}_E$ such that the balls $B(T_n f_i, \varepsilon/3)$ is a cover of $T_n\bar{B}_E$. Let then $f \in \bar{B}_E$ and let $j \leq k$ such that $\|T_n f - T_n f_j\| \leq \varepsilon/3$. By triangle inequality we get $\|Tf - Tf_j\| < \varepsilon$. Hence

$$T\bar{B}_E \subset \bigcup_{1 \leq j \leq k} B(Tf_j, \varepsilon),$$

thus $T\bar{B}_E$ is precompact. □

Since every finite rank operator is compact, we get

Corollary 4.1.1 *Every limit of operators of finite rank is a compact operator.*

We terminate this section by the Schauder Theorem:

Theorem 4.1.1 *Let $T \in \mathcal{L}(E,F)$. T is compact if and only if ${}^tT$ is compact.*

Proof.

4.2 Spectral properties of compact operators

In all this section E is a Banach space and T a compact operator.

Lemma 4.2.1 *Let F be a closed sub-vector space of a normed vector space E, $F \neq E$, then there exists $u \in E$, $\|u\| = 1$ with $d(u,F) \geq \frac{1}{2}$.*

Proof. Let $v \in E \backslash F$ and $\delta := d(v,F)$. Let $w \in F$ with $\|v - w\| < 2\delta$. Take $u := \frac{v-w}{\|v-w\|}$. □

Proposition 4.2.1 *Let $T \in \mathcal{K}(E)$. Then*

1. *the sub-vector-space $ker(I-T)$ is of finite dimension.*
2. *the sub-vector-space $Im(I-T)$ is closed*
3. *the operator $I-T$ is invertible in $\mathcal{L}(E)$ if and only if it is injective.*

Proof. 1. Denote by $F := \ker(I-T)$. F is a closed sub-vector-space of E and $\bar{B}_F = T\bar{B}_F = T\bar{B}_E \cap F$ hence compact, so by Riesz theorem $dimF < \infty$.
2. Let $y \in \overline{\mathrm{Im}(I-T)}$ and (x_n) a sequence of E with $\lim x_n - Tx_n = y$.
<u>*First case*</u>: The sequence (x_n) is bounded. Since T is compact, by choosing a subsequence we can assume that (Tx_n) converges to $z \in E$. Then $\lim x_n = y+z$ and by continuity of T, $z = T(y+z)$ hence $y = (y+z)-T(y+z)$.
<u>*Second case*</u>: The sequence is not bounded. Set, for $n \geq 0$, $d_n := d(x_n, \ker(I-T))$. Since, by the first point, $dim\ker(I-T) < \infty$, there exists $z_n \in \ker(I-T)$ with $d_n = \|x_n - z_n\|$ (since the continuous function distance will attain its minimum on the nonempty compact $\overline{B(x_n, \|x_n\|)} \cap \ker(I-T)$). If the sequence (d_n) is bounded we can replace (x_n) by $(x_n - z_n)$ (since $Tz_n = z_n$) and apply the first case. If not, using a subsequence, we can suppose that $\lim d_n = \infty$. Since the sequence $((x_n - z_n)/d_n$ is bounded, we can assume, by use of subsequence, that $T[(x_n - z_n)/d_n]$ is convergent to some $u \in E$. We deduce that

$$\lim_{n\to\infty} \frac{x_n - z_n}{d_n} = u + \lim \frac{y}{d_n} = u,$$

which implies that $Tu = u$ and for n large, $\|x_n - z_n - d_n u\| < d_n$ which is impossible and so the sequence (d_n) is bounded and $y \in \mathrm{Im}(I-T)$.
3. Assume that $I-T$ is injective, set $E_1 := \mathrm{Im}(I-T)$ and suppose that $E_1 \neq E$. Set for all n, $E_n := \mathrm{Im}(I-T)^n$ with $E_0 := E$. Let's show by induction that for all n, E_n is closed and $E_{n+1} \subsetneq E_n$. This is true for $n = 0$. Assume it true for n. Clearly $TE_n \subset E_n$ and hence T induces $T_n \in \mathcal{L}(E_n)$. Since E_n is closed $T_n\overline{B_{E_n}} \subset \overline{T\bar{B}_E} \cap E_n$ which is compact. Hence T_n is compact on E_n. Since $E_{n+1} = (Id_{E_n} - T_n)E_n$, then by the second point, E_{n+1} is closed in E_n and hence in E. It is obvious that $E_{n+1} \subset E_{n+2}$. Now since $I-E$ is injective we get, $E_n \neq E_{n+1}$ implies that $E_{n+1} \neq E_{n+2}$ since $E_{n+1} = (I-T)(E_n)$ and $E_{n+2} = (I-T)(E_{n+1})$. To find a contradiction, by the last lemma, there is a sequence (u_n) such that for all n, $u_n \in E$, $\|u_n\| = 1$ and $d(u_n, E_{n+1}) \geq \frac{1}{2}$. Then for $n < m$, $Tu_n - Tu_m = u_n - v_{n,m}$ with $v_{n,m} = Tu_m + (I-T)u_n \in E_{n+1}$. Thus for all $n \neq m$, $\|Tu_n - Tu_m\| \geq \frac{1}{2}$. This is in contradiction with the compacity of $T\bar{B}$. Thus $I-T$ is surjective. Remainder to show continuity of $(I-T)^{-1}$. By contradiction, suppose that there is a sequence $(x_n) \not\to 0$ with $\lim x_n - Tx_n = 0$. By use of subsequence, we can assume that for all n, $\|x_n\| \geq \varepsilon$, for some $\varepsilon > 0$. Set $u_n := x_n/\|x_n\|$. Again, since T is compact, we can assume that (Tu_n) converges to some $v \in E$. But this will imply that $\lim u_n = v$ and so $\|v\| = 1$ and then by continuity $Tv = v$, so $(I-T)v = 0$ which is impossible since $I-T$ is injective.
□

Theorem 4.2.1 *Let $T \in \mathcal{K}(E)$.*

1. *If $dim E = \infty$ then $0 \in \sigma_p(T)$.*
2. *$\sigma(T)\backslash\{0\} = \sigma_p(T)$ and for all $\lambda \in \sigma_p(T)$ $dim\, E_\lambda < \infty$.*
3. *$\sigma(T)$ is countable.*

Proof. 1. If 0 is not an eigenvalue then by proposition 4.1.1 $I = TT^{-1}$ is compact and so $dim\, E < \infty$.
2. Let $\lambda \in \mathbb{K}$, $\lambda \neq 0$. $\lambda \in \sigma(T)$ if and only if $I - T/\lambda$ is not injective and $\ker(\lambda - T) = \ker(I - T/\lambda)$. On the other hand, $\lambda \in \sigma(T)$ if and only if $I - T/\lambda$ is not invertible in $\mathcal{L}(E)$. Apply then the last proposition.
3. For this it suffices to show that for all $\varepsilon > 0$ there is a finite number of $\lambda \in \sigma(T)$ with $|\lambda| \geq \varepsilon$. If not, assume that there is a sequence $(\lambda_n) \subset \sigma(T)$ of distinct elements with $|\lambda_n| \geq \varepsilon$. By the last point λ_n are eigenvalues. Let then (e_n) corresponding eigenvectors with $\|e_n\| = 1$. Thus the (e_n) is a free system. For all n, set $E_n := span\{e_0, \cdots, e_n\}$. The $(E_n$ is a sequence of strictly increasing of finite dimension spaces. From lemma 4.2.1 there exists a sequence of (u_n), $\|u_n\| = 1$ and $u_n \in E_{n+1}$ with $d(u_n, E_n) \geq \frac{1}{2}$. Set $v_n := \frac{u_n}{\lambda_{n+1}}$. This sequence is bounded by $\frac{1}{\varepsilon}$ and for $n > m$ we have $Tv_n - Tv_m = u_n - v_{n,m}$ with $v_{n,m} = Tv_m + \frac{1}{\lambda_{n+1}}(\lambda_{n+1} - T)u_n$. Since $Tv_m \in E_{m+1} \subset E_n$ and $(\lambda_{n+1} - T)E_{n+1} \subset E_n$, we get $v_{n,m} \in E_n$ and $\|Tv_n - Tv_m\| \geq \frac{1}{2}$ which is impossible since T is compact. □

4.3 Hilbert-Schmidt operators

In this section E and F are two separable Hilbert spaces (of infinite dimensions).

Lemma 4.3.1 *Let B and B' be two Hilbert bases of E and F respectively. For all $T \in \mathcal{L}(E, F)$ we have:*

$$\sum_{b \in B,\, b' \in B'} | < b', Tb > |^2 = \sum_{b \in B} \|Tb\|^2 = \sum_{b' \in B'} \|T^* b'\|^2 \leq +\infty,$$

and this value does not depends on the choice of B or B'.

Proof. For $x \in E$ and $y \in F$ we have $\|x\|^2 = \sum_{b \in B} | < x, b > |^2$ and $\|y\|^2 = \sum_{b' \in B'} | < y, b' > |^2$. Now it is clear that $\sum_{b \in B} \|Tb\|^2$ is independent of B' and $\sum_{b' \in B'} \|T^* b'\|^2$ is independent of B. □

Notation. For all $T \in \mathcal{L}(E, F)$, set

$$\|T\|_2 := \left[\sum_{b \in B} \|Tb\|^2 \right]^{\frac{1}{2}}$$

where B is any base of E. Set $\mathcal{L}^2(E,F)$ the set

$$\mathcal{L}^2(E,F) := \{T \in \mathcal{L}(E,F);\ \|T\|_2 < \infty\}.$$

Examples.

1. Finite dimensional case.
   If $E = F$ has finite dimension n, and (e_j) a basis formed of eigenvectors of T^*T, then

   $$\|T\|_2^2 = \sum_{k=1}^{n} < T^*Te_k, e_k >= \sum_{k=1}^{n} \lambda_k,$$

   where (λ_k) are the eigenvalues of T^*T.
   If $T^* = T$ then
   $$\|T\|_2^2 = \sum_{k=1}^{n} \beta_k^2,$$

   where (β_k) are the eigenvalues of T.

2. Let $H := L^2(0, 2\pi)$ and define the Volterra operator, for all $f \in H$ by

   $$Vf(x) := \int_0^x f(t)\, dt.$$

   By the example 4, this operator is compact. Consider the basis $e_n(t) := \frac{1}{\sqrt{2\pi}}e^{int}$, $n \in \mathbb{Z}$. It is easy to verify that $\|Ve_n\|^2 \leq \frac{2}{\pi n^2}$ and so V is a Hilbert-Schmidt operator.

Theorem 4.3.1 *Let E, F be two separable Hilbert spaces.*

1. *$\mathcal{L}^2(E,F)$ is sub-vector space of $\mathcal{L}(E,F)$.*

2. *For al $S, T \in \mathcal{L}^2(E,F)$ and all Hilbert basis B of E, $\sum_{b\in B} < Tb, Sb >$ is finite and the mapping $(S,T) \longmapsto \sum_{b\in B} < Tb, Sb >$ is a scalar product on $\mathcal{L}^2(E,F)$ (independent of the choice of B).*

3. *With this scalar product $\mathcal{L}^2(E,F)$ is a Hilbert space.*

4. *$\mathcal{L}^2(E,F) \subset \mathcal{K}(E,F)$.*

Proof. **1** Let $S, T \in \mathcal{L}(E,F)$ and B a Hilbert basis of E. For all $b \in B$ we have $| < Sb, Tb > | \leq \|Sb\|\|Tb\| \leq \frac{1}{2}[\|Sb\|^2 + \|Tb\|^2]$. We deduce that $\sum < Sb, Tb >$ is finite. Since $\|Sb+Tb\|^2 = \|Sb\|^2 + \|Tb\|^2 + 2\text{Re} < Sb, Tb >$, $S + T \in \mathcal{L}_2$ and so the first point is proved.

2. It is clear that $(S,T) \longmapsto \sum < Sb, Tb >$ is a scalar product. Now by the polarization identity (proposition 1.1.1) we have $4\sum < Sb, Tb >= \|S+T\|_2^2 - \|S-T\|_2^2 + i\|S+iT\|_2^2 - i\|S-iT\|_2^2$ we get the independence of the basis.
3. From the second point $\mathcal{L}_2$ is a pre-Hilbert space. For all $T \in \mathcal{L}_2$ and all $x \in E$, $\|x\| = 1$, by taking a Hilbert basis containing x we get that $\|T\|_2 \geq \|Tx\|$, so $\|T\|_2 \geq \|T\|$. Thus $\mathcal{L}_2$ is separate. By this inequality, if (T_n) is a Cauchy sequence in $\mathcal{L}_2$, it is also a Cauchy sequence in $\mathcal{L}$ which is complete, so (T_n) converges to an operator $T \in \mathcal{L}(E,F)$. Now for $\varepsilon > 0$ notice that the set $C_\varepsilon := \{S \in \mathcal{L}_2;\ \|S\|_2 \leq \varepsilon\}$ is the intersection on all finite subset $I \subset B$, of $\{S \in \mathcal{L}_2;\ \sum_{b\in I} \|Sb\|^2 \leq \varepsilon^2\}$, hence C_ε is a closed set in $\mathcal{L}(E,F)$ (Indeed, this last set is the inverse image of $[0,\varepsilon^2]$ by the continuous mapping which to $S \in \mathcal{L}^2$ associates $\sum_{b\in I} \|Sb\|^2$). Fix $\varepsilon > 0$. There is $N > 0$ such that for all $m,n \geq N$ we have $\|T_n - T_m\|_2 \leq \varepsilon$. Fix $n \geq N$ and since $T_m - T_n \to T - T_n$, we get that $T - T_n \in C_\varepsilon$. Thus $T \in \mathcal{L}^2$ and that $\lim \|T - T_n\|_2 = 0$.
4. Let $T \in \mathcal{L}^2$ and (e_n) a basis of E. For all k, consider the operator $T_k : E \longrightarrow F$ defined, for all $x \in E$, by $T_k x := \sum_{n\leq k} < x, e_n > Te_n$. Since T_k is of finite rank, using corollary 4.1.1, it suffices to show that $T_k \to T$. For this, we write

$$\begin{aligned}\|(T-T_k)x\| &= \|\sum_{n\geq k} < x, e_n > Te_n\| \leq \left(\sum_{n\geq k} | < x, e_n > |^2\right)^{\frac{1}{2}} \left(\sum_{n\geq k} \|Te_n\|^2\right)^{\frac{1}{2}} \\ &\leq \|x\| \left(\sum_{n\geq k} \|Te_n\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

□

Definition 4.3.1 *Operators in $\mathcal{L}^2(E,F)$ are called* **Hilbert-Schmidt operators**.

Proposition 4.3.1 *Let E, F, H Hilbert spaces. For all $S \in \mathcal{L}(E,F)$ and $T \in \mathcal{L}(F,H)$ we have:*

1. *$\|S\|_2 = \|S^*\|_2$.*

2. *If T or S is a Hilbert-Schmidt operator then TS is a Hilbert-Schmidt operator also and $\|TS\|_2 \leq \|T\|\|S\|_2$ or $\|TS\|_2 \leq \|T\|_2\|S\|$.*

Proof. **1.** Direct from the definition of $\|S\|_2$.
2. Let B be a Hilbert basis of E. For all $b \in B$ we have $\|TSb\| \leq \|T\|\,\|Sb\|$ hence $\|TS\|_2^2 = \sum_{b\in B} \|TSb\|^2 \leq \|T\|^2 \sum \|Sb\|^2 = \|T\|^2\|S\|_2^2$. The second point could be obtained substituting S and T by their adjoints.
3. Similarly as 2. □

4.4 Compact self-adjoint operators

A classic theorem of linear algebra shows that every normal matrix, i.e. a matrix that commutes with its adjoint, in a finite dimensional complex Hilbert space, is diagonalizable in an othonormal base. We will generalize this result to infinite dimensional case, but for compact self adjoint operators. Generalization to normal compact operators could be done. To omit compacity of the operator we need a very powerfull theory as spectral measures or distributions.

Assume that T is an operator of finite rank. Since $\ker T = (\mathrm{Im} T)^{\perp}$ and since $\dim \mathrm{Im} T < \infty$ we have $H = \mathrm{Im} T \oplus \ker T$. Thus T induce on the finite dimensional space $\mathrm{Im} T$ an invertible self-adjoint operator, whose eigenvalues are those ($\neq 0$) of T. Since we can diagonalize in finite dimension, we get that $\mathrm{Im} T$ is direct sum of (orthogonal) eigen-sub-spaces of T, associated to nonzero eigenvalues of T and then

$$H = \bigoplus_{\lambda \in \sigma_p(T)} \ker(\lambda - T).$$

We have proved the diagonalization of a finite rank operator. In the following we will generalize this result to the case of a compact self-adjoint operator.

In the following H is a Hilbert space, and T a compact self-adjoint operator on H (not of finite rank).

Lemma 4.4.1 *T admits at least one eigenvalue and*

$$\|T\| = \max\{|\lambda|;\ \lambda \in \sigma_p(T)\}.$$

Proof. Clearly, if $\lambda \in \sigma_p(T)$, then $|\lambda| \leq \|T\|$. Now, by theorem 2.2.1 there is $\lambda \in \sigma(T)$ such that $|\lambda| \sup_{\|x\|=1} | < Tx, x > |$, which is equal to $\|T\|$. □

Theorem 4.4.1 *Let H be a Hilbert space and T a compact self-adjoint operator. For all $\lambda \in \sigma_p(T)$ denote by H_λ the eigen-space associated to λ. Then*

1. *$\sigma_p(T)$ is bounded, countable and infinite subset of $\mathbb{R}$, whose unique accumulation point is 0.*

2. *For all $\lambda \in \sigma_p(T)\backslash\{0\}$, $dim H_\lambda < \infty$.*

3. *For all $\lambda \neq \mu \in \sigma_p(T)$, H_λ and H_μ are orthogonal.*

4. **Spectral decomposition of the identity**.
Denote for all $\lambda \in \sigma_p(T)\backslash\{0\}$, P_λ the orthogonal projection on H_λ. Then
$$T = \sum_{\lambda \in \sigma_p(T)\backslash\{0\}} \lambda P_\lambda.$$

Proof. a. Assume that T is not of finite rank. The fact that eigenvalues of T are real and the orthogonality of eigen-spaces was shown in proposition 2.2.4

b. Let's show that $\Lambda^* := \sigma_p(T)\backslash\{0\}$ is infinite. By lemma 4.4.1, there exists $\lambda \in \sigma_p(T)$, $|\lambda| = \|T\|$. Since T is not trivial, then $\lambda \neq 0$ and so Λ^* is not empty. Assume that Λ^* is finite, $\Lambda^* = \lambda_1, \cdots \lambda_k$. Set then $G := \oplus_{j=1}^k H_{\lambda_j}$ and $F := G^\perp$. Since G is of finite dimension, $H = F \oplus G$. It is clear that $TG \subset G$ and since T is self-adjoint, $TF \subset F$. T induces then an operator T_F from F into itself, and since F is closed, T_F is compact also. If $T_F = 0$ then $\mathrm{Im}T \subset G$ and so T is of finite rank. Thus T_F is a self-adjoint non trivial operator on F. By lemma 4.4.1, T_F has a non zero eigenvalue μ. But this means that $\mu \in \sigma_p(T)\backslash\Lambda^*$, since for example there is $x \in F$, $x \neq 0$, $T_F x = Tx = \mu x$ (so $x \notin G$). This gives a contradiction and Λ^* is infinite and by theorem 4.2.1, $\sigma_p(T)$ is countable, and so 0 is the unique accumulation point.

c. Let J be a finite subset of Λ^* and $G_J := \oplus_{\lambda\in J} H_\lambda$, $F_J := G_J^\perp$. T induces on F_J a compact self-adjoint operator, whose norm $\|T_{F_J}\| = \max\{|\lambda|, \lambda \in \sigma_p(T_{F_J})\}$. But every eigenvalue λ of T_{F_J} is an eigenvalue of T does not belongs to J, since by construction $F_j \cap H_\mu = \{0\}$ for all $\mu \in J$. Therefore, $\sigma_p(T_{F_J}) \subset \sigma_p(T)\backslash J$. Conversely, if $\lambda \in \sigma_p(T)\backslash J$, then (by orthogonality), $H_\lambda \subset G_\lambda^\perp = F_J$ and hence λ is an eigenvalue of T_{F_J}. Thus $\sigma_p(T_{F_J}) = \sigma_p(T)\backslash J$ and
$$\|T_{F_J}\| = \max\{|\lambda|, \lambda \in \sigma_p(T)\backslash J\}.$$

Moreover, the orthogonal projection on G_J is $\sum_{\lambda\in J} P_\lambda$. Hence, for all $x \in E$, $x_J := x - \sum_{\lambda\in J} P_\lambda x \in F_J$ and
$$\|Tx_J\| = \|T_{F_J} x_J\| \leq \|T_{F_J}\| \, \|x\| \leq \|T_{F_J}\| \, \|x\|.$$

We deduce that,
$$\left\| T - \sum_{\lambda\in J} TP_\lambda \right\| \leq \max_{\lambda\in\sigma_p(T)\backslash J} |\lambda|,$$

and so

$$\left\| T - \sum_{\lambda \in J} \lambda P_\lambda \right\| \leq \max_{\lambda \in \sigma_p(T) \setminus J} |\lambda|.$$

Let $\varepsilon > 0$. Since 0 is an accumulation point of $\sigma_p(T)$, the set $K := \{\lambda \in \sigma_p(T),\ |\lambda| \geq \varepsilon\}$ is finite. Thus, for all finite part $J \subset \sigma_p(T)\setminus\{0\}$ that contains K, we have

$$\left\| T - \sum_{\lambda \in J} \lambda P_\lambda \right\| \leq \max_{\lambda \in \sigma_p(T) \setminus J} |\lambda| \leq \max_{\lambda \in \sigma_p(T) \setminus K} |\lambda|,$$

which terminate the proof. □

Corollary 4.4.1 *With the same notations we have*

$$\overline{ImT} = \overline{\bigoplus_{\lambda \in \sigma_p(T)\setminus\{0\}} H_\lambda}.$$

Proof. We know that, for all $x \in H$, $Tx = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda x$. Thus $\mathrm{Im}T \subset \overline{\oplus_{\lambda \in \sigma_p(T)\setminus\{0\}} H_\lambda}$. Conversely, if $\lambda \in \sigma_p(T)\setminus\{0\}$, then $H_\lambda \subset \mathrm{Im}T$. □

We can express the last theorem and corollary in the following

Corollary 4.4.2 *The space $\overline{ImT}$ admits a countable Hilbert basis $(f_n)_{n\in\mathbb{N}}$ formed of eigenvectors of T associated to nonzero eigenvalues $(\mu_n)_{n\in\mathbb{N}}$. The sequence $(\mu_n)_{n\in\mathbb{N}}$ tends to zero and, for all $x \in H$, we have*

$$Tx = \sum_{n\in\mathbb{N}} \mu_n < x, f_n > f_n.$$

Corollary 4.4.3 *For all $x \in \overline{ImT}$*

$$x = \sum_{\lambda \in \sigma_p(T)\setminus\{0\}} P_\lambda x.$$

Corollary 4.4.4 *Let P_0 be the orthogonal projection on $H_0 := kerT$. Then for all $x \in E$*

$$x = \sum_{\lambda \in \sigma_p(T)} P_\lambda x,$$

and

$$H = \overline{\bigoplus_{\lambda \in \sigma_p(T)} H_\lambda}.$$

Proof. Since T is self-adjoint, $H_0 = \ker T = \overline{\mathrm{Im} T}^{\perp}$. Hence, $H = H_0 \oplus \overline{\mathrm{Im} T}$.□

Corollary 4.4.5 *If H is a separable Hilbert space, then it admits a Hilbert basis formed of eigenvectors of T.*

Proof. By corollary 4.4.2, $\overline{\mathrm{Im} T}$ admits a countable Hilbert basis. Complete it by a basis of H_0 (formed of eigenvectors associated to 0) to get a Hilbert basis of H formed of eigenvectors. □

4.5 Fredholm equation

In this case, in the case of a compact self-adjoint operator, for all bounded function f on the set $\sigma_p(T)$, we can define the operator $f(T)$ on H as

$$f(T)x := \sum_{\lambda \in \sigma_p(T)} f(\lambda) P_\lambda x$$

for all $x \in H$. By the orthogonality of the spaces E_λ we get the following (Bessel) equalities

$$\|f(T)x\|^2 = \sum_{\lambda \in \sigma_p(T)} |f(\lambda)|^2 \|P_\lambda x\|^2,$$

$$\|x\|^2 = \sum_{\lambda \in \sigma_p(T)} \|P_\lambda x\|^2.$$

We deduce then that

$$\|f(T)\| = \sup_{\lambda \in \sigma_p(T)} |f(\lambda)|.$$

This shows that this symbolic calculus is an extension of the previous one. In particular, if $\mu \in \mathbb{K}^* \notin \sigma_p(T)$, then for all $x \in H$

$$(\mu - T)^{-1} x = \sum_{\lambda \in \sigma_p(T)} (\mu - \lambda)^{-1} P_\lambda x. \tag{4.1}$$

Now if $\mu \in \sigma_p(T)$, $\mu \neq 0$, then $\mathrm{Im}(\mu - T) = E_\mu^\perp$. Hence the operator T induces on $E_\mu^\perp$ a compact self-adjoint operator T_μ with $\sigma_p(T_\mu) = \sigma_p(T) \backslash \{\mu\}$, to T_μ we can again apply the formula (4.1) and deduce that, if $x \in E_\mu^\perp$, then for all $u \in E_\mu^\perp$ we have the equivalence

$$\mu u - Tu = x \Longleftrightarrow u = \sum_{\lambda \in \sigma_p(T) \backslash \{\mu\}} (\mu - \lambda)^{-1} P_\lambda x.$$

Now if $x \in E_\mu^\perp$ and $y \in E$, then $y = u + v$ with $u \in E_\mu^\perp$ and $v \in E_\mu$. Thus

$$\mu y - Ty = x \Longleftrightarrow \exists v \in E_\mu \text{ s.t. } y = u + \sum_{\lambda \in \sigma_p(T) \setminus \{\mu\}} (\mu - \lambda)^{-1} P_\lambda x.$$

In short, if we consider the **Fredholm equation**

$$\mu y - Ty = x, \tag{4.2}$$

with $\mu \in \mathbb{K}^*$ and $x \in E$, then we can distinguish two cases (**Fredholm alternative**):

- μ is not an eigenvalue of T. Then the equation (4.2) admits a unique solution y, given by

$$y = (\mu - T)^{-1} x = \sum_{\lambda \in \sigma_p(T)} (\mu - \lambda)^{-1} P_\lambda x.$$

- μ is an eigenvalue of T. Then the equation (4.2)
  - admits an infinite number of solutions if $x \in \ker(\mu - T)^\perp$, in this case those solutions are given by

$$y = u + \sum_{\lambda \in \sigma_p(T) \setminus \{\mu\}} (\mu - \lambda)^{-1} P_\lambda x,$$

    with $u \in \ker(\mu - T)$.
  - does not admit any solution if not, i.e. if $x \notin \ker(\mu - T)^\perp$.

Chapter 5

Unbounded self-adjoint operators

In this chapter we start by giving some properties of closed operators, then general properties of symmetric and self-adjoint operators on a Hilbert space. We terminate by defining symbolic calculus of unbounded self-adjoint operators.

In all this chapter H will be a Hilbert space and X a Banach space.

5.1 Closed operators

Definition 5.1.1 *Let $D \subset X$ be a sub-vector space. A* **linear unbounded operator** *is a linear mapping from D to X.*

Remarks.

1. An operator is always a couple (A, D). D, denoted sometimes by $D(A)$ or D_A, is called **domain of** A.

2. Changing the domain could change considerably the operator. See examples below.

3. In all this chapter we will always use **densely defined operators**, i.e. such that $\overline{D(A)} = X$.

4. Two operators (A, D_A) and (B, D_B) are equals if and only if $D_A = D_B$ and for all $x \in D_A$ we have $Ax = Bx$. And we say that B is an extension of A, $A \subset B$, if $D_A \subset D_B$ and for all $x \in D_A$ we have $Ax = Bx$.

5. The appellations "bounded", "unbounded" are due to the fact that for linear operators, continuity is equivalent to the inequality: $\|Ax\| \leq C\|x\|$ for some $C > 0$ and all $x \in X$, i.e. to boundedness on the closed unit ball.

Examples.

1. Let $X := BC^1(\mathbb{R})$ the space of continuously bounded differentiable functions and $A := \frac{d}{dx}$. Clearly A is a linear bounded operator. Observe that for all $n > 1$, the operator (A^n, D_n) defined by: $D_n := C^n(\mathbb{R})$ and $A^n f := f^{(n)}$ is an unbounded, densely defined operator.

2. Let $X := BC^\infty(\mathbb{R})$. For all $n \geq 1$, $A^n := (d/dx)^n$ is a linear bounded operator.

3. Let $X := L^2(]0,1[)$ and define the operator (A, D_A) with $D_A := \{f \in C^1([0,1]);\ f(0) = f(1) = 0\}$ and $A := d/dx$ is linear unbounded densely defined operator (since $\mathcal{D}(0,1) \subset D_A$).

4. On the same space $X := L^2(]0,1[)$ define the operator (B, D_B) with $D_B := \{f \in C^1([0,1]);\ f(0) = 0, f(1) = 1\}$ and $B := d/dx$ is linear unbounded (but not densely defined) operator.

The notion of operators whose graph is closed will play an important role:

Definition 5.1.2 *The operator (A, D_A) is called a closed operator if and only if for any $(x_n) \subset D_A$ such that $x_n \to x \in X$ and $Ax_n \to y \in X$ it follows that $x \in D_A$ and $y = Ax$.*

Remarks.

1. (A, D_A) closed is equivalent to $G(A) := \{(x, Ax);\ x \in D_A\}$ (the graph of) is closed in $X \times X$.

2. By linearity this definition is equivalent to the following: for any $(x_n) \subset D_A$ such that $x_n \to 0$ then $Ax_n \to 0$.

3. The closure of an operator (if it exists) (A, D_A) is the least closed extension of A. We say in this case that A is closable. It is denoted by $\bar{A}$. It is the operator whose graph is $\overline{G(A)}$.

4. If $D \subset D_A$ is a sub-vector space denote by $A|D$, called the part of A on D, the operator such that $A|D \subset T$ with domain $D(A|D) = \{x \in D;\ Tx \in D\}$.

If an operator (A, D_A) is injective, the operator $A^{-1}: \mathrm{Im} A \mapsto X$ is defined.

Definition 5.1.3 *Let (A, D_A) be a closed linear operator on X and $\lambda \in \mathbb{C}$. We say that $\lambda \in \rho(A)$, the resolvant of A, if $\lambda - A$ admits a bounded inverse on $Im(\lambda - A)$. We call spectrum of A, $\sigma(A)$ the complementary in $\mathbb{C}$ of $\rho(A)$: $\rho(A) = \mathbb{C} \backslash \sigma(A)$.*

Proposition 5.1.1 *The inverse of a closed injective operator is closed.*

Proof. Let $A: D_A \subset X \to Y$ be a closed injective operator, where X and Y are Banach spaces. The graph $G(A^{-1}) = \Phi(G(A))$ hence closed, where $\Phi : E \times F \to F \times E$ is the homeomorphism $\Phi(x, y) = (y, x)$. □

Remarks.

1. Let A be a closed operator on X. If $\lambda - A$ is bijective from D_A to X for some λ then $(\lambda - A)^{-1}$ is continuous from $\mathrm{Im}\lambda - A = X$ to X since closed (by the last proposition and the closed graph theorem). Hence $\lambda \in \rho(A)$.
2. The spectrum of A is union of the three disjoint following sets:
   (a) $\sigma_p(A)$ the point spectrum: the set of all eigenvalues.
   (b) $\sigma_r(A)$ the residue spectrum: the set of all λ that are not eigenvalues and such that the image of $\lambda - T$ is not dense in X.
   (c) $\sigma_c(A)$ the continuous spectrum: the complementary of $\sigma_p(A)$ and $\sigma_r(A)$ it is also the set of λ such that $\lambda - A$ is injective with dense image, but $(\lambda - A)^{-1}$ is not continuous.

Lemma 5.1.1 *Let A be an injective closed operator and $\lambda \in \rho(A)$, $\lambda \neq 0$. Then $1/\lambda \in \rho(A^{-1})$ and*

$$(\lambda^{-1} - A^{-1})^{-1} = \lambda A(\lambda - A)^{-1} = -\lambda - \lambda^2(\lambda - A)^{-1}.$$

Proof. $\lambda^{-1} - A^{-1} = -\lambda^{-1}(\lambda - A)A^{-1}$ (they have the same domain $\mathrm{Im}A$). Thus $\lambda^{-1} - A^{-1}$ is bijective from $D(A^{-1})$ onto X and its inverse is $-\lambda A R(\lambda, A)$. But $AR(\lambda, A) - \lambda R(\lambda, A) = Id$, so we get the result. □

Proposition 5.1.2 *Let A be a closed operator.*

1. *The spectrum $\sigma(A)$ is a closed set of $\mathbb{C}$.*

2. *The mapping $\lambda \in \rho(A) \longmapsto R(\lambda, A) \in \mathcal{L}(X)$ is analytic.*

Proof. If $\sigma(A) = \mathbb{C}$ there is nothing to show. Otherwise, rescalling A by some $\lambda \in \rho(A)$ we can assume that $0 \in \rho(A)$. Set $B := A^{-1}$.
1. By lemma 5.1.1, $\sigma(A) = \{\lambda \neq 0;\ \lambda^{-1} \in \sigma(B)\}$ and since $\sigma(B)$ is compact, $\sigma(A)$ is closed.
2. By lemma 5.1.1, $R(\lambda, A) = -\lambda^{-1}BR(\lambda^{-1}, B)$, so the mapping $\lambda \longmapsto R(\lambda, A)$ is analytic on $\rho(A)\backslash\{0\}$. Since $\sigma(A)$ is closed, there is $\lambda_0 \in \rho(A)$, $\lambda_0 \neq 0$. Rescalling we get that the mapping $\lambda \longmapsto R(\lambda, A)$ is analytic on $\rho(A)\backslash\{\lambda_0\}$. $\square$

Remark. For all nonempty closed set S of $\mathbb{C}$ we can construct a closed operator whose spectrum is S:
Since S is not empty, let (λ_n) be a dense sequence in S. Consider the operator A on $H := \ell^2(\mathbb{N})$, with domain the set of sequences $(x_n) \in H$ such that $(\lambda_n x_n) \in H$, and $A(x_n) = (\lambda_n x_n)$ (A is called the multiplication operator see the section forthcoming). It is not difficult to verify that A is closed, densely defined, and $\sigma(A) = \sigma_p(A) = S$.

5.2 Adjoint of an operator

In this section H will denote a Hilbert space and $< \cdot, \cdot >$ its scalar product.

Lemma 5.2.1 *Let (A, D_A) be a linear densely defined operator on H. Let $y \in H$, and assume that there exists $y^* \in H$ such that for every $x \in D_A$*

$$< Ax, y > = < x, y^* > . \tag{5.1}$$

Then y^* is unique.

Proof. If there is $z \in H$ s.t. $< Ax, y > = < x, y^* > = < x, z >$ for all $x \in D_A$, we get that $z - y^* \in \overline{D_A}^{\perp}$ which is trivial since $\overline{D(A)} = H$. $\square$

Definition 5.2.1 *Let (A, D_A) be a linear densely defined operator on H. Define the (unbounded) operator A^*,* **adjoint of** *A by*

$$D(A^*) := \{y \in H,\ \textit{so that}\ \exists y^* \in H\ \textit{s.t.}\ (5.1)\ \textit{is verified}\}$$

and

$$A^* y = y^*.$$

The adjoint could be characterized by

$$\text{for all } x \in D_A \text{ and all } y \in D_{A^*} \quad < Ax, y >=< x, A^*y > . \tag{5.2}$$

Definition 5.2.2 *We say that A is a symmetric operator if $\overline{D_A} = H$ and for every $x, y \in D_A$ we have*

$$< Ax, y >=< x, Ay > .$$

We say that A is self-adjoint if $A = A^*$.

In the following we give direct properties:

Proposition 5.2.1 *Let (A, D_A) be a linear densely defined operator on H. Then*

1. *The adjoint of A is always closed.*
2. *If B is an extension of A, $A \subset B$ then $B^* \subset A^*$.*
3. *If A is closable, then $(\bar{A})^* = A^*$.*
4. *If $\overline{D(A^*)} = H$ then A is closable and*
$$\bar{A} \subset A^{**}.$$
5. *If A is a symmetric operator then every symmetric extension B of A verify: $A \subset B \subset A^*$.*
6. *$ImA^{\perp} = kerA^*$.*

Proof. 1. If $y_n \to y$, $y_n^* \to y^*$ and $< Ax, y_n >=< x, y_n^* >$ for every $x \in D_A$ then $< Ax, y >=< x, y^* >$ so $y \in D_{A^*}$ and $A^*y = y^*$.
Points 2, 3, 4 and 5 are obvious.
6. $y \perp \overline{\mathrm{Im}A}$ means that $< Ax, y >=< x, A^*y >= 0$ for all $x \in D_A$, and so $A^*y = 0$. □

Theorem 5.2.1 *let (A, D_A) be a symmetric operator on a Hilbert space H. If $D_A = H$ then A is bounded.*

Proof. For all $x, y \in H$ we have $| < Ax, y > | = | < x, Ay > | \leq \|x\| \, \|Ay\|$. So by The Banach- Steinhauss theorem A is bounded. □

Let's see some examples:

Examples.

1. Let $H := L^2(0,1)$ and define the operator (A, D) by

$$D(A) := \{f \in H; \quad f \in C^1 \text{ and } f(0) = f(1) = 0\},$$

$$Af(t) = if'(t),$$

in other terms $A = i\frac{d}{dx}$. This operator is symmetric and $A \subset A^*$: Indeed, integrating by parts ($G(t) := \int_0^t g(s)\,ds$) we get

$$\begin{aligned} < Af, g > &= < g, g^* >= \int_0^1 f\bar{g^*}\,dt \\ &= f(1)\bar{G}(1) - f(0)\bar{G}(0) - \int_0^1 f'(t)\bar{G}(t)\,dt \\ &= \int_0^1 (if')(t)(\overline{-iG})(t)\,dt. \end{aligned}$$

Since ImA is dense in E, we get $g = -iG$. Therefore there is $y' \in L^2$ with $-y' = iy^*$. Hence $D(A^*) = \{y \in H;\, y' \in L^2\}$ and $y^* = A^*y = iy'$. Thus A is symmetric and $A \subset A^*$. A is not closed (because of the boundary conditions) but is closable, it is closure $A_1 := \bar{A}$ is defined by

$$D(A_1) := \{f \in H; \quad f' \in L^2 \text{ and } f(0) = f(1) = 0\},$$

$$A_1 f(t) = if'(t).$$

2. Define A_2 on the same space H by

$$D(A_2) := \{f \in H; \quad f' \in L^2 \text{ and } f(0) = f(1)\},$$

$$A_2 f(t) = if'(t).$$

Thus $A_1 \subset A_2$ and then $A_2^* \subset A_1^*$. Let's show that A_2 is self adjoint: For this let's calculate

$$\begin{aligned} < A_2 f, g > &= \int_0^1 (if')\bar{g}\,dt \\ &= i[f(1)\bar{g}(1) - f(0)\bar{g}(0)] + \int_0^1 f(t)\overline{ig'}(t)\,dt \\ &= if(1)[\bar{g}(1) - \bar{g}(0)] + \int_0^1 f(t)\overline{ig'}(t)\,dt. \end{aligned}$$

Since $A_2^* \subset A_1^*$ then $< A_2 f, g >=< f, A_1^* g >=< f, iy' >$ so $if(1)[\bar{g}(1) - \bar{g}(0)] = 0$. If $g(1) \neq g(0)$ then choosing a sequence $f_n \to 0$ with $f_n(0) = f_n(1) = 1$ we get that $< A_2 f_n, g > \not\to 0$ but $< f_n, A_2^* g > \to 0$. So $g(1) = g(0)$ and then $A_2^* = A_2$.

3. Define, on the space $H := L^2[0,\infty[$, the operator

$$D(A_1) := \{f \in E; \quad f'' \in L^2 \text{ and } f(0) = f'(0) = 0\},$$

$$A_1 f(t) = -f''(t).$$

By the same way as in the first example, $y \in D(A^*)$ implies that $y'' \in L^2$, then

$$\begin{aligned} < A_1 f, g > &= \int_0^\infty (-f'')\bar{g}\, dt \\ &= -[f'\bar{g}]_0^\infty + \int_0^\infty f'(t)\bar{g}'(t)\, dt \\ &= [-f'\bar{g} + f\bar{g'}]_0^\infty + < f, -g'' > . \end{aligned}$$

Therefore we see that $< A_1 f, f > \geq 0$, and that $y^* = A^* y = -y''$ and $D(A^*) = \{y \in L^2; \quad y'' \in L^2\}$. So $A_1 \subset A_1^*$ and A_1 is symmetric but not self-adjoint.

4. Consider the same operation on the same space, $L^2[0,\infty[$,

$$D(A_2) := \{f \in H; \quad f'' \in L^2 \text{ and } f(0) = 0\},$$

$$A_2 f(t) = -f''(t).$$

Obviously $A_1 \subset A_2$ and then $A_2^* \subset A_1^*$. Repeating the same calculation as above we get

$$< Af, g > = < f, g^* > = -f'(0)\overline{g(0)} + < f, -g'' >,$$

and necessarily $g(0) = 0$ (otherwise consider a sequence $f_n \to 0$ in L^2 with $f_n'(0) = 1$ to get a contradiction). This shows that A_2 is self-adjoint (extension of A_1).

Theorem 5.2.2 *If A is a symmetric operator and $ImA = H$ then A is self-adjoint.*

Proof. We know that $A \subset A^*$. Now let $y \in D(A^*)$ and set $y^* = A^* y$. Since $\mathrm{Im}A = H$ there is $x \in D(A)$ such that $A^* y = y^* = Ax$. For every $z \in D(A)$ we have

$$< Az, y > = < z, A^* y > = < z, y^* > = < z, Ax > = < Az, x >,$$

thus $y = x$ and so $A = A^*$. $\square$

Theorem 5.2.3 *Let A be a bounded self-adjoint operator. Assume that $kerA = \{0\}$ then A^{-1} is also self adjoint.*

Proof. First let's show that A^{-1} is densely defined: If not, $\overline{\mathrm{D}(A^{-1})} = \overline{\mathrm{Im}A} \neq H$, then by proposition 5.2.1.6, there is $y_0 \in \ker A^*$, $y_0 \neq 0$, which is not possible since $A = A^*$. Now $< A^{-1}x, y >=< x, A^{-1*}y >$, setting $z = A^{-1}x$, $x = Az$ we get $< z, y >=< x, y^* >=< Az, y^* >=< z, A^*y^* >$ and so $y = A^*y^* = Ay^*$ or $y^* = A^{-1}y$, but $y^* = A^{-1*}y$. Thus $y \in \mathrm{Im}A = D(A^{-1})$, and $A^{-1}y = A^{-1*}y$. Thus A^{-1} is self-adjoint. $\square$

Remark. This theorem 5.2.3 gives us many examples of unbounded self-adjoint operators. Start with any self-adjoint compact operator A with $\ker A = \{0\}$. Then A^{-1} is an unbounded self-adjoint operator.

5.3 The L^∞-spectral theorem

In this section we will show a theorem, known as spectral theorem, for bounded self-adjoint or unitary operators, stating that each self-adjoint or unitary operator is unitary equivalent to a real multiplication operator. Thus, self-adjoint and real multiplication operators are effectively the same things. It is frequent to regard an arbitrary self-adjoint operator as being a real multiplication operator.

We will start by defining multiplication operator:

Let (X, μ) a measured space and $f \in L^\infty(X, \mu)$.

Definition 5.3.1 *The multiplication operator $A_f\colon L^2(X, \mu) \longrightarrow L^2(X, \mu)$ is defined by $A_f(g) := fg$.*

It is easy to see that A_f is a linear bounded operator on L^2, and $\|A_f\| \leq \|f\|_\infty$. Moreover, $A_f^* = A_{\bar{f}}$, hence A_f is normal. If in addition f is real-valued then A_f is self-adjoint and if $|f| = 1$ a.e. then $A_f^* A_f = A_{|f|^2} = Id_{L^2}$.

Proposition 5.3.1 $\sigma(A_f) = R_{ess}(f) = \{\lambda \in \mathbb{C}; \forall\varepsilon > 0,$ *the set of* $x \in X$, $|f(x) - \lambda| < \varepsilon$ *is not* μ*-negligeable*$\}$.

Proof. Let $\lambda \in \mathbb{C}\backslash R_{ess}(f)$. If there is $\varepsilon > 0$, such that the set of $x \in X$, $|f(x)-\lambda| < \varepsilon$ is μ-negligeable, denoting by h the function $h(x := (f(x)-\lambda)^{-1}$ for $f(x) \neq \lambda$ and 0 if not. $|h(x)| < \varepsilon^{-1}$ for μ-a.e. x and $h(x)(\lambda - f(x)) = 1$. Thus $h \in L^\infty(X, \mu)$ and $A_h(A_f - \lambda) = (A_f - \lambda)A_h = Id_{L^2}$.
Now if $\lambda \in R_{ess}(f)$ then for every ε, the set $A_\varepsilon := \{x \in X, |f(x) - \lambda| < \varepsilon\}$

is not μ-negligeable, consider a function $\chi \in L^2(X,\mu)$, $\|\chi\|_2 = 1$, $\chi = 0$ outside A_ε. Then $|(A_f - \lambda)\chi| \leq \varepsilon|\chi|$, hence $\|(A_f - \lambda)\chi\| \leq \varepsilon$. Thus $A_f - \lambda$ is not bijective. $\square$

Proposition 5.3.2 *For all $g \in R_{A_f}$ we have $g(A_f) = A_{g(f)}$. If A_f is self-adjoint or unitary, then for all $g \in C(\sigma(A_f)$ then $g(A_f) = A_{g(f)}$.*

Proof. The mapping $g \longmapsto A_{g(f)}$ is linear morphism of ring, so we get the first point using uniqueness in proposition 3.1.1. The second point could be obtained by applying theorem **??**. $\square$

In the following, H is a Hilbert space and T a bounded self-adjoint or unitary operator.

Lemma 5.3.1 *Let $x \in H$.*

1. *There exists a finite measure μ_x on $\sigma(T)$ such that, for all $f \in C(\sigma(T))$ we have $< f(T)x, x >= \int_{\sigma(T)} f(t)\, d\mu_x(t)$.*

2. *Denote by $\phi_x: C(\sigma(T)) \longrightarrow H$ the linear mapping defined by $\phi_x(f) := f(T)x$, and $w: C(\sigma(T)) \longrightarrow L^2(\sigma(T), \mu_x)$ the mapping that to a continuous function associate its class in L^2. There exists an isometry $\psi_x: L^2(\sigma(T), \mu_x) \longrightarrow H$ such that $\psi_x \circ w = \phi_x$. Moreover, $\psi_x(1) = x$, $\psi_x(A_z) = T\phi_x$ and $\psi_x(A_{\bar{z}}) = T^*\phi_x$.*

Proof. 1. The linear form $\Phi_x: f \longmapsto < f(T)x, x >$ is positive on $C(\sigma(T))$: Indeed, if f is positive, then $f(T)$ is a positive operator by theorems **??** and 3.2.3 ($f(T)$ is self-adjoint and $\sigma(f(T)) = f(\sigma(T)) \subset \mathbb{R}^+$). There exists then a unique measure μ_x on $\sigma(T)$ such that $< f(T)x, x >= \Phi_x(f) = \int_{\sigma(A)} f(t)\, d\mu_x(t)$.
2. Let $f \in C(\sigma(T))$. We have $\|\phi_x(f)\|^2 =< f(T)x, f(T)x >=< \bar{f}(T)f(T)x, x >$

$=< [f\bar{f}](T)x, x >= \int_\sigma |f(t)|^2\, d\mu(t) = \|w(f)\|^2$. On the space $C(\sigma(T))$ endowed with semi-norm $\|\phi_x(f)\|$, w is a linear isometric of dense image, there exists then $\psi_x \in \mathcal{L}(L^2(\sigma(T), \mu_x), H)$ such that $\phi_x = \psi_x \circ w$. For all $f \in C(\sigma(T))$ we have $\|\psi_x(w(f))\| = \|\phi_x(f)\| = \|w(f)\|$, then by density of the image of w we have $\|\psi_x(g)\| = \|g\|$ for all $g \in L^2(\sigma(T), \mu_x)$: ψ_x is isometric. $\psi_x(1) = \phi_x(1) = Ix = x$.
Finally, for all $f, g \in C(\sigma(T))$, we have $\psi_x(A_f(w(g)))\psi_x(w(fg)) = \phi_x(fg) = f(T)g(T)x = f(T)\phi_x(g) = f(T)\psi_x(w(g))$. Again, by the density of the image of w, we get for all $f \in C(\sigma(T))$ and all $g \in L^2(\sigma(T), \mu_x)$, we have $\psi_x(A_f(g)) = f(T)\psi_x(g)$. Take $f = z$ and $f = \bar{z}$ to terminate. $\square$

Lemma 5.3.2 *Let $x \in H$ and denote by E_x the image of ψ_x.*

1. *If $y \in E_x^\perp$ then $E_y \subset E_x^\perp$.*
2. *There exists a subset $D \subset H$ such that for all $x, y \in D$, $x \neq y$, $E_x \perp E_y$ and $\overline{\oplus_{x\in D} E_x} = H$.*

Proof. **1.** If $y \in E_x^\perp$, then for all $f \in \sigma(T)$ and all $g \in L^2(\sigma(T), \mu)$, we have $< \psi_x(g), f(T)y >=< \bar{f}(T)\psi_x(g), y >=< \psi_x(A_{\bar{f}}g), y >= 0$. Since $\{f(T)y;\ f \in C(\sigma(T))\}$ is dense in E_y, we get the result.
2. Denote by G the set of subsets D of $H\backslash\{0\}$ such that for all $x, y \in D$, $E_x \perp E_y$. Endowed with inclusion G is inductive. It is easy to show that G admits a maximal element D that is our candidate. □

Theorem 5.3.1 *Let H be a separable Hilbert space and $T \in \mathcal{L}(H)$ a self-adjoint or unitary bounded operator. There exists a measured space (X, μ), a function $f \in L^\infty(X, \mu)$ and an isomorphism $\psi: L^2(X, \mu) \longrightarrow H$ such that $T = \psi A_f \psi^*$*

Proof. Let $x \in H$ and D as in the last lemma. For all $y \in H$, we have $y \in E_y$ and since H is separable, D is countable. Rearrange D to be a discrete set and set $X := \sigma(T) \times D$. If g is a function on X, denote for $y \in D$, g_y the function $t \longmapsto g(t, y)$. Denote by $C_c(X)$ the set of continuous functions of compact support on X, i.e. $g \in C_c(X)$ if all except finite number of the functions g_y are null. Denote by

$$\Phi(g) := \sum_{y\in D} \int_{\sigma(T)} g_y(t)\, d\mu_y(t).$$

Since Φ is a positive linear form on $C_c(X)$, there exists a unique measure μ on X such that, for all l$g \in C_c(X)$ we have

$$\int_X g(x)\, d\mu(x) = \Phi(g) = \sum_{y\in D} \int_{\sigma(T)} g_y(t)\, d\mu_y(t).$$

Denote by $\phi: C_c(X) \longrightarrow H$ the mapping defined by $\phi(g) := \sum_{y\in D} g_y(T)y$ and $w: C_c(X) \longrightarrow L^2(X, \mu)$ the function class. For $g \in C_c(X)$ we have $g_y(T)y \in E_y$. by orthogonality we get $\|\phi(g)\|^2 = \sum_{y\in D} < g_y(T)y, g_y(T)y >$ $=< \bar{g}_y(T)g_y(T)y, y >= \sum_{y\in D} \int_{\sigma(T)} |g_y(t)|^2\, d\mu_y(t) = \int_X |g(x)|^2\, d\mu(x) = \|w(g)\|^2$. On the space $C_c(X)$ endowed with semi-norm $\|\phi(f)\|$, w is a linear isometric of dense image, there exists then $\psi \in \mathcal{L}(L^2(X, \mu), H)$ such that $\phi = \psi \circ w$. For all $f \in C(\sigma(T))$ we have $\|\psi(w(f))\| = \|\phi(f)\| = \|w(f)\|$, then by density of the image of w we have $\|\psi(g)\| = \|g\|$ for all $g \in L^2(X, \mu)$: ψ is isometric. Let's show that ψ is surjective. Let $y \in D$ and $h \in C(\sigma(T))$. Set $g(t, y) :=$

$h(t)$ and $g(t,x) = 0$ for $x \in D$, $x \neq y$. We have $\psi(w(g)) = \phi(g) = h(T)y$. Hence the image of ψ contains all $h(T)y$, $y \in D$, $h \in C(\sigma(T))$. Since ψ is isometric, its image is closed, hence contains all E_y. Thus ψ is surjective.
Denote by P_1 the first projection on X. For all $g \in C_c(X)$ and all $y \in D$ we have $(P_1 g)_y = z g_y$, hence $(P_1 g)_y(T) = T g_y(T)$. So $\psi(A_{P_1} w(g)) = \psi(w(P_1 g)) = \phi(P_1 g) = \sum_{y \in D} (P_1 g)_y(T) y = T\phi(g) = T\psi(w(g))$. By density of the image of w, we deduce that for all $g \in L^2(X,\mu)$, $\psi(A_f(g)) = T\psi(g)$. This implies that $\psi A_f = T\psi$ hence $T = \psi A_f \psi^{-1} = \psi A_f \psi^*$. □

Using this theorem, one can define a symbolic calculus from $C(\sigma(T))$ into $\mathcal{L}(H)$: for $g \in C(\sigma(T))$, wet $g(T) := \psi A_{g \circ f} \psi^*$. Notice that this symbolic calculus could be extended to the $\mathcal{B}(\sigma(T))$ the vector space of bounded borelean functions on $\sigma(T)$.

5.4 The L^2-spectral theorem

In this section we consider a particular self-adjoint operator which appears to be a very particular (and simple) example, but which will be central to the description and application of the spectral theorem. This will be seen by the main theorem of the next section.
In the last section we have defined multiplication operator for a bounded function, which gives a bounded operator. In this section we will define the multiplication operator for L^2-functions, which gives unbounded operator. The proofs are roughly the same, so they are omitted.

Let (X,μ) be a measured space. Define $H := L^2(X,\mu)$ the space of all measurable functions of square integrable, with the classical identification between two functions if ever they are equal almost everywhere.

Fix a measurable real-valued function a that is bounded on every bounded subset of X. Let D be the set of all functions $f \in H$ such that

$$\int_X [1 + a(x)^2] |f(x)|^2 \, d\mu < \infty,$$

and define the operator A_a with domain D by

$$A_a f(x) := a(x) f(x),$$

the multiplication operator.

Lemma 5.4.1 *The operator (A_a, D) is self-adjoint.*

Define the essential range of a, $R_{ess}(a)$, the set of all $\lambda \in \mathbb{R}$ such that for all $\varepsilon > 0$ the measure of the set $\{x \in X;\ |a(x) - \lambda| < \varepsilon\}$ is zero.

Lemma 5.4.2 *$\sigma(A_a) = R_{ess}(a)$, and if $\lambda \notin \sigma(A)$ then*

$$\left[(\lambda - A_a)^{-1} f\right](x) = [\lambda - a(x)]^{-1} f(x)$$

for all $f \in H$ and all $x \in X$, and

$$\|(\lambda - A_a)^{-1}\| = \frac{1}{dist(\lambda, \sigma(A_a))}.$$

Now we can generalize the results of the last section to the case of unbounded self-adjoint operators.

Theorem 5.4.1 (Spectral theorem) *Let H be a Hilbert space and T a densely defined self-adjoint operator on H. Then*

1. *$\sigma(T) \subset \mathbb{R}$.*
2. *The operator $U := (i - T)(i + T)^{-1}$ is a unitary operator in $\mathcal{L}(H)$.*
3. *There exists a measured space (X, μ) a measurable function $f\colon X \to \mathbb{R}$ and an isomorphism $\psi\colon L^2(X, \mu) \to H$ of Hilbert spaces such that $T = \psi A_a \psi^*$.*

Proof. 1. Let $\lambda \in \mathbb{C} \backslash \mathbb{R}$. Denote by b its imaginary part. For all $x \in D(T)$ we have $< Tx, x >=< x, Tx >$ hence $< Tx, x > \in \mathbb{R}$ and the imaginary part of $< (\lambda - T)x, x >$ is then $b\|x\|^2$. Thus $|b|\|x\|^2 \leq |< (\lambda - T)x, x >| \leq \|(\lambda - T)x\|\|x\|$ and so $\|(\lambda - T)x\| \geq |b|\|x\|$. Thus, for all $(x, y) \in G(\lambda - T)$, we have $\|y\| \geq |b|\|x\|$, hence $(1 + b^2)\|y\|^2 \geq b^2(\|x\|^2 + \|y\|^2)$. By proposition , the mapping $(x, y) \longmapsto y$ from $G(\lambda - T)$ into H is injective of closed image. Since $(\lambda - T)^* = \bar{\lambda} - T$ is also injective, we deduce, by proposition 5.2.1.6, that the image of $\lambda - T$ is dense.
2. Since $\mathrm{Im}((i + T)^{-1}) = D(i - T)$, $D(U) = H$ and U is bijective by 1. Now for $x \in D(T)$, we have $\|(i - T)x\|^2 = \|Tx\|^2 + \|x\|^2 - i < x, Tx > + i < Tx, x >= \|Tx\|^2 + \|x\|^2 = \|(i + T)x\|^2$. For $y \in H$, set $x = (i + T)^{-1}y$, we have $\|Uy\| = \|(i - T)x\| = \|(i + T)x\| = \|y\|$. Thus U is isometric.
3. Let $y \in H$ and set $x := (i + T)^{-1}y$, we have $Uy = (i - T)x = 2ix - (i + T)x = 2ix - y$. Thus $x = 2(Uy + y)/i$ and then $(i + T)^{-1} = 2(U + Id)/i$ and $T = i(U + Id)^{-1} - i$.
By theorem 5.3.1, there exists a measured space (X, μ), a function $g\colon X \longrightarrow \mathbb{C}$, $|g| = 1$ measurable and an isomorphism $\psi\colon L^2(X, \mu) \longrightarrow H$ such that $U = \psi A_g \psi^*$. Since $U - Id$ is injective, $A_{g^{-1}}$ is injective, and so the set $\{x \in X; \, g(x) = 1\}$ is μ-negligeable. Then $(U - Id)^{-1} = (\psi A_{g^{-1}} \psi^*)^{-1} = (\psi^*)^{-1} A_g \psi^{-1} \psi A_g \psi^* = U$. Thus $T = \psi A_f \psi^*$ where $f := 2(g - 1)^{-1}/i - i = -i(g + 1)(g - 1)^{-1}$. □

As in the bounded case, one can define a symbolic calculus on $\mathcal{B}(\sigma(T))$ the space of bounded borelean functions on $\sigma(T)$.

5.5 Stone's theorem

Definition 5.5.1 *Let E be a Banach space. We call a one parameter C_0-group any family of linear bounded operators $(G(t))_{t\in\mathbb{R}} \subset \mathcal{L}(E)$ verifying*

1. *$G(0) = Id_E$.*
2. *$G(t+s) = G(t)G(s)$, for all $t, s \in \mathbb{R}$.*
3. *For all $x \in E$, the mapping $t \longmapsto G(t)x$ is continuous.*

The operator defined by

$$D(A) := \{x \in E;\ \lim_{t\to 0} \frac{G(t)x - x}{t} \text{ exists}\},$$

$$Ax := \lim_{t\to 0} \frac{G(t)x - x}{t}$$

is called generator of the C_0-group.

Let H be a Hilbert space. A C_0-group is called unitary C_0-group if each operator is unitary.

Theorem 5.5.1 *Let H be a separable Hilbert space. Let $(A, D(A))$ be a densely defined operator. The following are equivalent:*

(i) iA generates a unitary C_0-group.

(ii) A is self-adjoint.

Proof. (i)$\Longrightarrow$(ii). We have $G^*(t) = G(t)^{-1} = G(-t)$. Let's show that $A \subset A^*$. Indeed, let $x, y \in D(A)$, we have

$$\begin{aligned} < Ax, y > &= -i \lim_{t\to 0} \left\langle \frac{G(t)x - x}{t}, y \right\rangle = -i \lim_{t\to 0} \left\langle x, \frac{G^*(t)y - y}{t} \right\rangle \\ &= -i \lim_{t\to 0} \left\langle x, \frac{G^{-1}(t)y - y}{t} \right\rangle \\ &= -i \lim_{t\to 0} \left\langle x, \frac{G(-t)y - y}{t} \right\rangle = -i < x, -iAy >, \end{aligned}$$

thus $x \in D(A^*)$ and $< Ax, y >=< A^*x, y >$.
$A = A^*$: Let $x \in D(A)$, $y \in D(A^*)$ we have

$$\begin{aligned} < x, A^*y > &= < Ax, y >= -i \lim_{t\to 0} \left\langle \frac{G(t)x - x}{t}, y \right\rangle = -i \lim_{t\to 0} \left\langle x, \frac{G^*(t)y - y}{t} \right\rangle \\ &= -i \lim_{t\to 0} \left\langle x, \frac{G^{-1}(t)y - y}{t} \right\rangle \\ &= -i \lim_{t\to 0} \left\langle x, \frac{G(-t)y - y}{t} \right\rangle =< x, Ay >, \end{aligned}$$

Therefore $y \in D(A)$ and hence $A = A^*$.
(ii)$\Longrightarrow$(i). Since A admits a $L^\infty(\sigma(B))$ symbolic calculus. Denote by Φ this symbolic calculus and define, for all $t \in \mathbb{R}$, $G(t) := \Phi(e_t) = e_t(A)$, where $e_t(s) := exp(ist)$. Since $\sigma(A) \subset \mathbb{R}$, e_t is bounded. Using properties of the symbolic calculus, it is easy to verify that $(G(t))$ is a unitary group generated iA. □

5.6 Laplace operator on bounded open domain of $\mathbb{R}^N$

Let Ω be an open of $\mathbb{R}^N$ and $H = L^2(\Omega)$ as a real Hilbert space. Define the operator Δ_0 on H by

$$D(\Delta_0) := \{u \in H_0^1(\Omega),\, \Delta u \in L^2(\Omega)\},$$

$$\Delta_0 u = \Delta u, \qquad u \in D(\Delta_0).$$

Then we have

Proposition 5.6.1 $(\Delta_0, D(\Delta_0))$ *is negative self adjoint operator.*

Proof. Since $\mathcal{D}(\Omega) \subset D(\Delta_0)$, $D(\Delta_0)$ is dense in H. Let $u \in D(\Delta_0) \subset H_0^1(\Omega)$, by Green's formula, we have

$$< \Delta_0 u, u >= \int_\Omega \Delta u \cdot u \, dx = -\int_\Omega |\nabla u|^2 \, dx$$

so Δ_0 is negative. By a similar calculation, one can see that Δ_0 is symmetric. In order to use theorem 5.2.2, let's show that $Im\Delta_0 = H$. In fact we will show that $0 \notin \sigma(\Delta_0)$. For this, and using Lax-Milgram lemma, for all $f \in H$, there exists $u \in H_0^1(\Omega)$ such that, for all $v \in H_0^1(\Omega)$

$$\int (\lambda uv + \nabla u \cdot \nabla v) \, dx = \int fv,$$

for all $\lambda > -\lambda_0$, λ_0 being one over the Poincar constant. Which gives (by Green) that in the distribution sens

$$\lambda u - \Delta u = f.$$

Thus $(u \in H_0^1)$ $\Delta u = u - f \in L^2$, i.e. $u \in D(\Delta_0)$. In other terms $\sigma(\Delta_0) \subset]-\infty, -\lambda_0[$. □

Corollary 5.6.1 $i\Delta_0$ *generates a unitary* C_0*-group.*

Remark 5.6.1 *If the boundary of* Ω *is bounded and is of class* C^2, *then* $D(\Delta_0) = H^2(\Omega) \cap H_0^1(\Omega)$ *with equivalent norm.*

In order to determine the eigenvalues of Laplace operator, notice first that if u is an eigenvalue then there is λ $(\leq -\lambda_0)$ such that

$$\Delta_0 u = \lambda u. \tag{5.3}$$

This means that $u \in D(\Delta_0)$. But asking u to be in H_0^1 is sufficient, since in this case, $\Delta u \in L^2$. Therefore it is sufficient to solve (5.3) in H_0^1. We start by the following direct application of theorem 5.2.3.

Corollary 5.6.2 $(-\Delta_0)^{-1}\colon L^2(\Omega) \longmapsto L^2(\Omega)$ *is a positive bounded self-adjoint operator.*

Corollary 5.6.3 $(-\Delta_0)^{-1}\colon H_0^1(\Omega) \longmapsto H_0^1(\Omega)$ *is a positive compact self-adjoint operator.*

Proof. Remainder to show that this operator is compact. Denoting by A this operator then $A = -\Delta_0^{-1} \circ J$, where $J\colon u \longmapsto u$ is the canonical injection from H_0^1 into L^2. Since J is compact (Rellich theorem) and using proposition 4.1.1 A is compact. □

We terminate by

Theorem 5.6.1 *The set of eigenvalues of Laplace operator with Dirichlet condition* Δ_0 *on* Ω *is a strictly decreasing sequence that tends to* $-\infty$.
Each eigen-space is of finite dimension.
Denote by (μ_n) *the sequence of eigenvalues of* $-\Delta_0$, *repeated each with its multiplicity. Then there exists a Hilbert basis* (u_n) *of* $H_0^1(\Omega)$ *such that for all* n, *we have* $\Delta_0 u_n = \mu_n u_n$

Remark 5.6.2 *By the same argument above each* $u_n \in H_0^\infty(\Omega)$, *hence* C^∞ *and so* u_n *is an ordinary solution of the equation* $\Delta u_n = \mu_n u_n$.