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EXTERIOR AND SYMMETRIC ALGEBRA METHODS IN ALGEBRAIC COMBINATORICS

TIM RÖMER

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1. LECTURE: GLICCI SIMPLICIAL COMPLEXES

In this lecture we present results from [15] which is joint work with Uwe Nagel. At first we recall some definitions and notations from liaison theory. See [12] and [13] for more details and results not mentioned in the following. Liaison theory itself provides an equivalence relation among equidimensional subschemes of fixed dimension. Here we follow a purely algebraic approach to this theory and focus especially on questions related to combinatorial commutative algebra.

Let $S = K[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field K. We always assume that $|K| = \infty$. Two graded ideals $I, J \subset S$ are said to be *G-linked* (in one step) by a Gorenstein ideal $\mathfrak{c} \subset S$ if

$$\mathfrak{c} : I = J \quad \text{and} \quad \mathfrak{c} : J = I.$$

Then we write $I \sim_{\mathfrak{c}} J$. This has several consequences. For example I and J are unmixed and have the same codimension as $\mathfrak{c}$ if $I \sim_{\mathfrak{c}} J$. The concept of *Gorenstein liaison* is obtained if one takes the transitive closure of $\sim_{\mathfrak{c}}$. Thus I and J are in the same G-liaison class if and only if there are Gorenstein ideals $\mathfrak{c}_1, \dots, \mathfrak{c}_s$ such that

$$I = I_0 \sim_{\mathfrak{c}_1} I_1 \sim_{\mathfrak{c}_2} \cdots \sim_{\mathfrak{c}_s} I_s = J.$$

If we insist that all the Gorenstein ideals $\mathfrak{c}_1, \dots, \mathfrak{c}_s$ are in fact complete intersections, then we get the more classical concept of liaison. We will refer to it here as *CI-liaison*.

Of particular interest are the equivalence classes that contain a complete intersection. We say that the ideal I is *glicci* if it is in the G-liaison class of a complete intersection. It is *licci* if it is in the CI-liaison class of a complete intersection.

In codimension 2 we know a lot in CI-liaison theory. It follows from a result of Gaeta [6] that every graded codimension 2 Cohen-Macaulay ideal $I \subset S$ is licci. In codimension 3 the situation is more complicated. It is well-known that not every codimension 3 Cohen-Macaulay ideal is licci (see, e.g., Huneke-Ulrich [9]). This is one motivation to link with Gorenstein ideals instead of complete intersection ideals.

Note that glicci ideals are Cohen-Macaulay and that all complete intersections of the same codimension are in the same CI-liaison class. One of the main open problems in G-liaison theory is:

Conjecture 1.1 ([11]). *Every graded Cohen-Macaulay ideal in S is glicci.*

Several classes of Cohen-Macaulay ideals that are of interest in algebraic geometry or commutative algebra are known to be glicci. In this lecture we study 1.1 for graded Cohen-Macaulay ideals which are motivated from objects in combinatorial commutative algebra.

Recall that an (abstract) *simplicial complex* on $[n] = \{1,\dots,n\}$ is a subset Δ of the power set of $[n]$ such that $F \subseteq G$ and $G \in \Delta$ implies $F \in \Delta$. The elements of Δ are called *faces*. The maximal elements under inclusion in Δ are called *facets*. An $F \in \Delta$ with $|F| = d+1$ is called a *d-dimensional* face. Then we write $\dim F = d$. The complex Δ is called *pure* if all facets have the same dimension. If $\Delta \neq \{\}$, then the *dimension* $\dim \Delta$ is the maximum of the dimensions of the faces of Δ.

Simplicial complexes are related to algebraic objects via the following construction. For $F \subset [n]$ we write x_F for the squarefree monomial $\prod_{i \in F} x_i$. The *Stanley-Reisner ideal* of Δ is

$$I_\Delta = (x_F : F \subseteq [n],\ F \not\in \Delta)$$

and the corresponding *Stanley-Reisner ring* is $K[\Delta] = S/I_\Delta$. We will say that Δ has an algebraic property like Cohen-Macaulayness if $K[\Delta]$ has this property. For more details on this subject we refer to Bruns-Herzog [4] and Stanley [18].

It is a natural question to ask whether a Cohen-Macaulay complex Δ is glicci (i.e. if I_Δ is a glicci ideal in S). Since we are interested in squarefree monomial ideals, we study a slightly stronger property which implies being glicci, but is naturally defined in the context of simplicial complexes:

Definition 1.2. *A squarefree monomial ideal $I \subset S$ is said to be* squarefree glicci *if there is a chain of links in S*

$$I = I_0 \sim_{\mathfrak{c}_1} I_1 \sim_{\mathfrak{c}_2} \cdots \sim_{\mathfrak{c}_{2s}} I_{2s},$$

where I_j is a squarefree monomial ideal whenever j is even and I_{2s} is a complete intersection.

Let Δ be a simplicial complex with existing vertices $\{i : \{i\} \in \Delta\} = [n]$. Then we call Δ *squarefree glicci* if $I_\Delta \subset S$ has this property.

Let $I \subset S$ be an ideal and let $R = S[y]$ be the polynomial ring over S in the variable y. If I is glicci, then also the extension ideal $I \cdot R$ is glicci, since the links in S also provide links in R. This implies in particular, that if Δ is squarefree glicci, then so is any cone over Δ. We will use this fact sometimes in the following.

Next we present a method that allows us to link a given simplicial complex in two steps to a subcomplex.

Lemma 1.3. *Let $\mathfrak{c} \subset J \subset S$ be squarefree monomial ideals and let $x_k \in S$ be a variable that does not divide any minimal monomial neither in J nor in $\mathfrak{c}$. If $\mathfrak{c}$ is Cohen-Macaulay and J is unmixed such that* $\operatorname{codim} J = \operatorname{codim} \mathfrak{c} + 1$, *then $I := x_k J + \mathfrak{c}$ is a squarefree monomial ideal that is G-linked in two steps to J. We say that I is a* basic double link *of J on $\mathfrak{c}$.*

We omit the proof (see [15, Lemma and Definition 2.3] for details). Note that a basic double link is a special case of the more general concept of a basic double link for G-liaison as introduced in [11].

We give a combinatorial interpretation of Lemma 1.3. Let Δ be a simplicial complex on $[n]$. Recall that each $F \subseteq [n]$ induces the following simplicial subcomplexes of Δ: the *link of F*

$$\operatorname{lk} F = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\},$$

and the *deletion*

$$\Delta_{-F} = \{G \in \Delta : F \cap G = \emptyset\}.$$

Consider any $k \in [n]$. Then the cone over the link $\operatorname{lk} k$ with apex k considered as complex on $[n]$ has as Stanley-Reisner ideal $J_{\operatorname{lk} k} = I_\Delta : x_k$, and the Stanley-Reisner ideal of the deletion Δ_{-k} considered as a complex on $[n]$ is $(x_k, J_{\Delta_{-k}})$ where $J_{\Delta_{-k}} \subset S$ is the extension ideal of the Stanley-Reisner ideal of Δ_{-k} considered as a complex on $[n] \setminus \{k\}$. Note that x_k does not divide any of the minimal generators of $J_{\Delta_{-k}}$. Hence x_k is not a zerodivisor on $S/J_{\Delta_{-k}}$. Moreover, we have that

$$I_\Delta = x_k J_{\operatorname{lk} k} + J_{\Delta_{-k}}.$$

Thus we see that if Δ is pure and if the deletion Δ_{-k} is Cohen-Macaulay and has the same dimension as Δ, then Δ is a basic double link of the cone over its link $\operatorname{lk} k$ (where both are considered as complexes on $[n]$).

Example 1.4. *Let Δ be the simplicial complex on $[4]$ consisting of 4 vertices. The Stanley-Reisner ideal of Δ is*

$$I_\Delta = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4).$$

It is easy to see that I_Δ has a linear free resolution. Then it follows from [9] that it is not licci. However, I is squarefree glicci because

$$I = x_4 \cdot (x_1, x_2, x_3) + (x_1x_2, x_1x_3, x_2x_3)$$

provides that I is a basic double link of (x_1, x_2, x_3).

Next we present a class of simplicial complexes that consists of squarefree glicci complexes. Recall from [17] that a pure simplicial complex Δ is said to be *vertex-decomposable* if Δ is a simplex or equal to $\{\emptyset\}$, or there exists a vertex k such that $\operatorname{lk} k$ and Δ_{-k} are both pure and vertex-decomposable and $\dim \Delta = \dim \Delta_{-k} = \dim \operatorname{lk} k + 1$. We introduce a less restrictive concept that is defined similarly:

Definition 1.5. *Let $\Delta \neq \emptyset$ be a pure simplicial simplex on $[n]$. Then Δ is said to be* weakly vertex-decomposable *if there is some $k \in [n]$ such that Δ is a cone over the weakly vertex-decomposable deletion Δ_{-k} or there is some $k \in [n]$ such that* $\operatorname{lk} k$ *is weakly vertex-decomposable and Δ_{-k} is Cohen-Macaulay of the same dimension as Δ.*

Observe, that if Δ is not a cone over Δ_{-k}, then $\dim \Delta_{-k} = \dim \Delta$. We consider $\emptyset$ as a weakly vertex-decomposable simplicial complex.

Example 1.6.

(1) *If Δ is a simplex, then it is weakly vertex-decomposable.*

(2) *Assume that Δ is pure of dimension $n-2$. Then for any vertex $\{k\} \in \Delta$, the Stanley-Reisner ideals of Δ and the cones over* $\operatorname{lk} k$ *and Δ_{-k}, respectively, are principal ideals. Thus we see that Δ is weakly vertex-decomposable.*

The main result of this lecture is:

Theorem 1.7. *(*[15]*) Let Δ be a simplicial complex. If Δ is weakly vertex-decomposable, then Δ is squarefree glicci. In particular, Δ is Cohen-Macaulay.*

Analogously to the idea that a generic initial ideal of a given graded ideal I in S can be used to study algebraic properties of I, one can associate to every simplicial complex a shifted simplicial complex and these complexes share many combinatorial properties (see, e.g., [8] or [10] for details). Recall that a simplicial complex Δ is called *shifted* if for all $F \in \Delta$, $j \in F$ and $j < i$ such that $i \notin F$ we have $F - \{j\} \cup \{i\} \in \Delta$. Next we prove that every pure shifted simplicial complex is squarefree glicci. Hence, for Cohen-Macaulay simplicial complexes, the answer to 1.1 is affirmative up to "shifting." This result is the combinatorial counterpart of one of the main results in [14].

Corollary 1.8. *Each Cohen-Macaulay shifted complex is squarefree glicci.*

Proof. It follows from [3] that a Cohen-Macaulay shifted complex is vertex-decomposable. Then Theorem 1.7 implies that this complex is squarefree glicci. □

Recall that a complex Δ is called a *matroid* if, for all $W \subseteq [n]$, the restriction $\Delta_W = \{F \in \Delta : F \subseteq W\}$ is a pure simplicial complex. We get:

Corollary 1.9. *Each matroid is squarefree glicci.*

Proof. Let $k \in [n]$. Then $\operatorname{lk} k$ and Δ_{-k} are the corresponding link and deletion in the sense of matroid theory. In particular, they are again matroids; see [16]. It follows by induction on the number of vertices that matroids are vertex-decomposable. □

Following [1], the complex Δ is said to be 2-*CM* or doubly Cohen-Macaulay if, for each existing vertex $\{k\} \in \Delta$, the deletion Δ_{-k} is Cohen-Macaulay of the same dimension as Δ.

Corollary 1.10. *Each 2-CM complex is squarefree glicci.*

Proof. In order to see that each 2-CM complex is weakly vertex-decomposable it suffices to check that its link with respect to any vertex is again 2-CM. This has been shown in [1], [2]. □

A recent result by Casanellas-Drozd-Hartshorne [5] is that each Gorenstein ideal is glicci. The proof is non-constructive and relies on the theory developed in [5]. In the context of simplicial complexes one can prove an even stronger result.

Corollary 1.11. *Each simplicial homology sphere is squarefree glicci.*

Proof. Note that the Stanley-Reisner ring of a homology sphere is Gorenstein. Furthermore, Hochster's Tor formula provides that each Gorenstein ideal is 2-CM; see [1]. □

In the remaining part of this lecture we present two examples.

Example 1.12. *Let $S = K[x_1, \dots, x_6]$. Using the notation from* [4, p. 236]*, the Stanley-Reisner ideal of the triangulation of the real projective plane $\mathbb{P}^2$ is given by*

$$I_\Delta = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6).$$

If $\operatorname{char} K \neq 2$ *this is a* 2*-dimensional Cohen-Macaulay complex. For* $\operatorname{char} K = 2$ *this complex is not Cohen-Macaulay. Assume now that $K = \mathbb{Q}$. We used Macaulay* 2 [7] *to check that Δ is not weakly vertex-decomposable.*

One of the main open questions in liaison theory is whether every Cohen-Macaulay ideal is glicci. In view of the above dependence of the Cohen-Macaulayness on the characteristic, we propose the following:

Problem 1.13. *Decide whether the Stanley-Reisner ideal of the above triangulation of $\mathbb{P}^2_{\mathbb{R}}$ is glicci.*

It would also be interesting to know if so-called shellable simplicial complexes are glicci. The second example shows that the two properties being weakly vertex-decomposable and being squarefree glicci depend on the characteristic of K.

Example 1.14. *Let $S = K[x_1, \dots, x_7]$. We consider the ideals*

$$\begin{aligned} \mathfrak{c} &= (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6), \\ J &= (x_1, \dots, x_4), \text{ and} \\ I &= x_7J + \mathfrak{c}. \end{aligned}$$

Notice that $\mathfrak{c}$ is the extension of the Stanley-Reisner ideal of the triangulation of the real projective plane $\mathbb{P}^2$ in 6 *variables. Hence, $S/\mathfrak{c}$ is Cohen-Macaulay if and only if* $\operatorname{char} K \neq 2$*. Therefore I is a basic double link of the complete intersection J if* $\operatorname{char} K \neq 2$*. It follows that in this case I is squarefree glicci and that the induced simplicial complex Δ is weakly vertex-decomposable.*

Next assume that the characteristic of K is 2. Using the exact sequence

$$0 \to \mathfrak{c}(-\deg x_7) \to \mathfrak{c} \oplus J(-\deg x_7) \to I \to 0,$$

it is not too difficult to check that S/I has depth $2 < \dim S/I = 3$*, thus S/I is not Cohen-Macaulay. It follows that Δ is neither (squarefree) glicci nor weakly vertex decomposable if* $\operatorname{char} K = 2$*.*

REFERENCES

[1] K. Baclawski, *Cohen-Macaulay connectivity and geometric lattices*. European J. Combin. **3** (1982), no. 4, 293–305.

[2] K. Baclawski, *Canonical modules of partially ordered sets*. J. Algebra **83** (1983), no. 1, 1–5.

[3] A. Björner and G. Kalai, *On f-vectors and homology*. In Combinatorial Mathematics: Proceedings of the Third International Conference (New York, 1985), 63–80, Ann. New York Acad. Sci. **555**, New York Acad. Sci. 1989.

[4] W. Bruns and J. Herzog, *Cohen-Macaulay rings. Rev. ed.*. Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press 1998.

[5] M. Casanellas, E. Drozd, and R. Hartshorne, *Gorenstein liaison and ACM sheaves*. J. Reine Angew. Math. **584** (2005), 149–171.

[6] F. Gaeta, *Nuove ricerche sulle curve sghembe algebriche di risiduale finito e sui gruppi di punti del piano*. An. Mat. Pura Appl. (4) **31** (1950), 1–64.

[7] D. R. Grayson and M. E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*. Available at http://www.math.uiuc.edu/Macaulay2/.

[8] J. Herzog, *Generic initial ideals and graded Betti numbers*. In Computational commutative algebra and combinatorics (Osaka, 1999), 75–120, Adv. Stud. Pure Math. **33**, Math. Soc. Japan 2002.

[9] C. Huneke and B. Ulrich, *The structure of linkage*. Ann. Math. (2) **126** (1987), no. 2, 277–334.

[10] G. Kalai, *Algebraic shifting*. In Computational commutative algebra and combinatorics (Osaka, 1999), 121–163, Adv. Stud. Pure Math. **33**, Math. Soc. Japan 2002.

[11] J. Kleppe, J. Migliore, R. M. Miró-Roig, U. Nagel, and C. Peterson, *Gorenstein liaison, complete intersection liaison invariants and unobstructedness*. Mem. Amer. Math. Soc. **154** (2001), no. 732.

[12] J. Migliore, *Introduction to Liaison Theory and Deficiency Modules*. Progress in Mathematics **165**, Birkhäuser 1998.

[13] J. Migliore and U. Nagel, *Liaison and related topics: Notes from the Torino Workshop/School*. Rend. Sem. Mat. Univ. Politec. Torino **59** (2003), no. 2, 59–126.

[14] J. Migliore and U. Nagel, *Monomial ideals and the Gorenstein liaison class of a complete intersection*. Compositio Math. **133** (2002), no. 1, 25–36.

[15] U. Nagel and T. Römer, *Glicci simplicial complexes*. J. Pure Appl. Algebra **212** (2008), no. 10, 2250–2258.

[16] J. G. Oxley, *Matroid Theory*. Oxford University Press 1992.

[17] J. S. Provan and L. J. Billera, *Decompositions of simplicial complexes related to diameters of convex polyhedra*. Math. Oper. Res. **5** (1980), no. 4, 576–594.

[18] R. P. Stanley, *Combinatorics and commutative algebra. Second edition*. Progress in Mathematics **41**, Birkhäuser Boston 1996.

2. LECTURE: H-VECTORS OF GORENSTEIN POLYTOPES

In this lecture we an overview on results from [4] which is joint work with Winfried Bruns. Let $P \subseteq \mathbb{R}^{n-1}$ be an integral convex polytope; see [12] for details on convex geometry. In algebraic combinatorics one considers the *Ehrhart function* of P which is defined by

$$E(P,m) \begin{cases} |\{z \in \mathbb{Z}^{n-1} : \frac{z}{m} \in P\}| & \text{if } m > 0, \\ E(P,0) = 1 & \text{if } m = 0. \end{cases}$$

It is a classical result that $E(P,m)$ is a polynomial in m of degree $\dim(P)$ and the corresponding *Ehrhart series* $E_P(t) = \sum_{m \in \mathbb{N}} E(P,m)t^m$ is a rational function

$$E_P(t) = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^{\dim(P)+1}}.$$

(See also below for a sketch of a proof.) The vector $h(P) = (h_0, \dots, h_d)$ (where $h_d \neq 0$) is called the *(Ehrhart) h-vector* of P. See, e.g., [3] or [10] for results related to this vector. Here we are interested in the following two questions:

(1) For which polytopes is $h(P)$ *symmetric*, i. e. $h_i = h_{d-i}$ for all i?
(2) For which polytopes is $h(P)$ *unimodal*, i. e. there exists a natural number t such that $h_0 \leq h_1 \leq \cdots \leq h_t \geq h_{t+1} \geq \cdots \geq h_d$?

The classical approach to this problem using methods from commutative algebra is due to Stanley; see, e.g., [3] or [10]. We sketch the arguments.

At first one considers the cone $C(P)$ defined as

$$C(P) = \operatorname{cone}((a,1) : a \in P) \subseteq \mathbb{R}^n.$$

Then we set

$$E(P) = C(P) \cap \mathbb{Z}^n.$$

Note that $E(P)$ is an affine monoid and we can consider its affine monoid algebra $K[E(P)]$ where K is a field. The monomial in $K[E(P)]$ corresponding to the lattice point a is denoted by X^a where X represents a family of n indeterminates. The algebra $K[E(P)]$ is graded in such a way that the degree of X^a (or of a) is the last coordinate of a. It is an easy exercise to show that the Hilbert function of $K[E(P)]$ coincides with the Ehrhart function of P. In particular, using standard results about Hilbert-functions we see that $E(P,m)$ is indeed a polynomial in m, and that $E_P(t)$ is a rational function. Using the fact that the Krull dimension of $K[E(P)]$ is $\dim(P)+1$ concludes the proof of the results mentioned above about Ehrhart functions.

Since P is integral one proves that $K[E(P)]$ is a finite module over its subalgebra generated by the degree 1 elements. But usually $K[E(P)]$ is not generated by its degree 1 elements. If this is the case, we say that P is *integrally closed*. Then we simplify our notation and write $K[P] = K[E(P)]$.

Recall that a *unimodular triangulation* of P is a triangulation into simplices

$$\operatorname{conv}(s_0, \dots, s_r) \text{ for } s_0, \dots, s_r \in \mathbb{Z}^{n-1}$$

such that

$$s_1 - s_0, \dots, s_r - s_0 \text{ generate a direct summand of } \mathbb{Z}^{n-1}.$$

If $\dim P \geq 3$, P need not have a unimodular triangulation. However, if a unimodular triangulation of P exists, then P is integrally closed. This follows easily from the fact that a unimodular simplex is integrally closed; see, e.g., [2] for details.

The monoid $E(P)$ is always *normal*, i.e. an element a of the subgroup of $\mathbb{Z}^n$ generated by $E(P)$ such that $ka \in E(P)$ for some $k \in \mathbb{N}$, $k \geq 1$, belongs itself to $E(P)$. It follows from a famous theorem of Hochster, that $K[E(P)]$ is Cohen-Macaulay. For example this implies by standard arguments that

$$h_i \geq 0 \text{ for all } i = 1, \ldots, d.$$

Stanley characterized Gorenstein rings among the Cohen-Macaulay domains in terms of the Hilbert series. This implies that $h(P)$ is symmetric if and only if $K[E(P)]$ is a Gorenstein ring. For the monoid $E(P)$, the Gorenstein property has a simple interpretation: it holds if and only if $E(P) \cap \operatorname{relint} C(P)$ is of the form $y + E(P)$ for some $y \in E(P)$. This follows from the description of the canonical modules of normal affine monoid algebras by Danilov and Stanley. See [3] and [10] for terminology and results from commutative algebra.

It was conjectured by Stanley that question (2) has a positive answer for the *Birkhoff polytope P*, whose points are the real doubly stochastic $n \times n$ matrices and for which $E(P)$ encodes the *magic squares*. This long standing conjecture was proved by Athanasiadis [1]. (That P is integrally closed and $K[P]$ is Gorenstein in this case is easy to see.)

Questions (1) and (2) can be asked similarly for the combinatorial h-vector $h(\Delta(Q))$ of the boundary complex $\Delta(Q)$ of a simplicial polytope Q (derived from the f-vector of $\Delta(Q)$). Here the following answers are known. The Dehn–Sommerville equations express the symmetry, while unimodality follows from McMullen's famous g-theorem (proved by Stanley [9]): the vector $(1, h_1 - h_0, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an M-sequence, i. e. it represents the Hilbert function of a graded artinian K-algebra that is generated by its degree 1 elements. In particular, its entries are nonnegative, and so the h-vector is unimodal.

The key idea of Athanasiadis proof of Stanley's conjecture for the Birkhoff polytope P mentioned above is to show that there exists a simplicial polytope P' with

$$h(\Delta(P')) = h(P).$$

More generally, the result of Athanasiadis applies to compressed polytopes (i. e. integer polytopes all of whose pulling triangulations are unimodular). (The Birkhoff polytope is compressed; see [8] and [10].) The main result of [4] is:

Theorem 2.1. *Let P be an integral polytope such that P has a regular unimodular triangulation and $K[P]$ is Gorenstein. Then the h-vector of P satisfies the inequalities $1 = h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$. More precisely, the vector $(1, h_1 - h_0, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an M-sequence.*

(See [2] or [11] for a discussion of regular subdivisions and triangulations.) The proof of this theorem is a little bit technical.

The strategy of the proof is to consider the algebra $K[M]$ of a normal affine monoid M for which $K[M]$ is Gorenstein. The Hilbert series of $K[M]$ is related to the one of a simpler affine monoid algebra $K[N]$ which ones gets by factoring out a suitable regular sequence of $K[M]$. In the situation of an algebra $K[P]$ for an integrally closed polytope P, the regular sequence is of degree 1, and one obtains an integrally closed and, up to a translation, reflexive polytope such that $h(P) = h(Q)$. Note that Mustaţă and Payne [7] have given an example of a reflexive polytope which is not integrally closed and has a nonunimodal h-vector. If P has even a regular unimodular triangulation, then there exists a simplicial polytope P' such that the combinatorial h-vector $h(\Delta(P'))$ of the boundary complex of P' coincides with $h(P)$, that is

$$h(P) = h(\Delta(P'))$$

Then it only remains to apply the g-theorem to P'. Note that if the g-theorem could be generalized from polytopes to simplicial spheres, then the theorem would hold for all polytopes with a unimodular triangulation.

As a side effect of the proof one gets that the toric ideal of a Gorenstein polytope with a square-free initial ideal has also a Gorenstein square-free initial ideal. More precisely, it is possible to prove:

Corollary 2.2. *Let P be an integer Gorenstein polytope such that the toric ideal I_P has a squarefree initial ideal. Then it also has a square-free initial ideal that is the Stanley-Reisner ideal of the join of a boundary of a simplicial polytope and a simplex, and thus defines a Gorenstein ring.*

The corollary answers a question of Conca and Welker; see [5, Question 6] and [6] for more details related to this result.

References

[1] C. A. Athanasiadis, *Ehrhart polynomials, simplicial polytopes, magic squares and a conjecture of Stanley*. J. Reine Angew. Math. **583** (2005), 163–174.

[2] W. Bruns and J. Gubeladze, *Polytopes, rings, and K-theory*. Preliminary version at `http://www.math.uos.de/staff/phpages/brunsw/preprints.htm`

[3] W. Bruns and J. Herzog, *Cohen–Macaulay rings. Rev. ed.* Cambridge Studies in Advanced Mathematics **39**, Cambridge, Cambridge University Press (1998).

[4] W. Bruns and T. Römer, *h-vectors of Gorenstein polytopes*. J. Comb. Theory, Ser. A **114** (2007), No. 1, 65–76.

[5] A. Conca, *Betti numbers and initial ideals*. Oberwolfach Reports **1**(2004), 1710–1712.

[6] A. Conca, S. Hosten and R. Thomas, *Nice initial complexes for some classical ideals*. In: C.A. Athanasiadis (ed.) et al., Algebraic and geometric combinatorics, American Mathematical Society (AMS), Contemporary Mathematics **423** (2006), 11–42.

[7] M. Mustaţă and S. Payne, *Ehrhart polynomials and stringy Betti numbers*. Math. Ann. **333** (2005), 787–795.

[8] R. P. Stanley, *Decompositions of rational convex polytopes*. Annals of Discrete Math. **6** (1980), 333–342 .

[9] R. P. Stanley, *The number of faces of a simplicial convex polytope*. Adv. in Math. **35** (1980), 236–238.

[10] R. P. Stanley, *Combinatorics and commutative algebra. 2nd ed*. Progress in Mathematics **41**, Basel, Birkhäuser (1996).

[11] B. Sturmfels, *Gröbner Bases and Convex Polytopes*. American Mathematical Society, Univ. Lectures Series **8**, Providence, Rhode Island (1996).

[12] G. M. Ziegler, *Lectures on polytopes*. Graduate Texts in Mathematics **152**, Berlin, Springer (1995).

3. LECTURE: KOSZUL HOMOLOGY AND SYZYGIES OF VERONESE SUBALGEBRAS

In this lecture we give an overview about recent results from [2] which is joint work with Winfried Bruns and Aldo Conca. Let K be a field. Recall the following definitions of Green and Lazarsfeld; see [5] and [6]. A finitely generated positively $\mathbb{Z}$-graded K-algebra $R = \oplus_{i\in\mathbb{Z}} R_i$ satisfies property N_0 if R is generated in degree 1. In the following we will always assume that this is the case. Then we can present R as a quotient $R = S/I$ where S is a standard graded polynomial ring and $I \subset S$ is a graded ideal. We are interested in the following property:

Definition 3.1. *The K-algebra R satisfies* property N_p *for some $p > 0$ if $\beta^S_{i,j}(R) = 0$ for $j > i+1$ and $1 \leq i \leq p$.*

Here $\beta^S_{i,j}(R) = \dim_K \operatorname{Tor}^S_i(R,K)$ are the graded Betti numbers of R as an S-module.

Example 3.2.

(1) *The property N_1 means just that R is defined by quadrics, i.e. if we write $R = S/I$ for a graded ideal I containing no linear forms, then I is generated by homogeneous polynomials of degree 2.*

(2) *The property N_p for $p > 1$ means that $R = S/I$ is defined by quadrics and that the minimal graded free resolutions of R is of the form*

$$\cdots \to F_{p+1} \to S(-p+1)^{\beta_p} \to \cdots \to S(-2)^{\beta_1} \to S \to R \to 0.$$

If R satisfies N_p for some $p > 1$, then R satisfies $N_{p'}$ for every $1 \leq p' \leq p$. This motivates the following definition.

Definition 3.3. *Let R be a standard graded K-algebra. If R satisfies N_p for every $p \geq 1$, then we set* $\operatorname{index}(R) = \infty$. *Otherwise we define* $\operatorname{index}(R)$ *to be the largest integer $p \geq 1$ such that R has N_p. We call* $\operatorname{index}(R)$ *the* Green-Lazarsfeld index *of R.*

Determining $\operatorname{index}(R)$ in general seems to be a difficult problem. Here we focus on the case of a Veronese subring $R^{(c)} = \bigoplus_{i\in\mathbb{N}} R_{ic}$ ($d \geq 1$) of a standard graded K-algebra R. Observe that we consider $R^{(c)}$ as a standard graded K-algebra with homogeneous component of degree i equal to R_{ic}.

Already the case of a polynomial ring $S = K[X_1,\dots,X_n]$ is interesting.

Example 3.4. *If $n \leq 2$ or $c \leq 2$, then $S^{(c)}$ is a determinantal ring. In this case the minimal free resolution of $S^{(c)}$ is well-known and one is able to determine the Green-Lazarsfeld index.*

At first assume that $n = 2$. Then the minimal free graded resolution of $S^{(c)}$ is given by the Eagon-Northcott complex which implies that $\operatorname{index}(S^{(c)}) = \infty$.

Next we consider the case $c = 2$. The resolution of $S^{(2)}$ in characteristic 0 *is known by work of Jozefiak, Pragacz and Weyman in* [7]. *We get that* $\operatorname{index}(S^{(2)}) = 5$ *if $n > 3$ and* $\operatorname{index}(S^{(2)}) = \infty$ *if $n \leq 3$.*

For $n \leq 6$ we get from results of Andersen [1] *that* $\operatorname{index}(S^{(2)})$ *is independent on* $\operatorname{char} K$. *For $n > 6$ and* $\operatorname{char} K = 5$ *she showed that* $\operatorname{index}(S^{(2)}) = 4$.

For $n > 2$ and $c > 2$ the following is known:

$$c \leq \operatorname{index}(S^{(c)}) \leq 3c-3. \tag{1}$$

The lower bound follows from Green [4] for any c and n. Ottaviani and Paoletti [8] proved the upper bound in characteristic 0. They also showed that $\operatorname{index}(S^{(c)}) = 3c-3$ for $n = 3$. Motivated by these results they conjectured:

Conjecture 3.5. *We have* $\operatorname{index}(S^{(c)}) = 3c-3$ *for every* $n \geq 3$ *and* $c \geq 3$.

For $n = 4$ and $c = 3$ the conjecture is true by [8, Lemma 3.3]. See also Eisenbud, Green, Hulek and Popescu [3] for related results.

Rubei [9] proved that $\operatorname{index}(S^{(3)}) \geq 4$ if $\operatorname{char} K = 0$. One of the main results of [2] is the following improvement:

Theorem 3.6. *We have:*

(1) $c+1 \leq \operatorname{index}(S^{(c)})$ *if* $\operatorname{char} K = 0$ *or* $> c+1$.

(2) *If* $R = S/I$ *for a graded ideal* $I \subset S$*, then*

$$\operatorname{index}(R^{(c)}) \geq \operatorname{index}(S^{(c)}) \text{ for every } c \geq \operatorname{rate}_S(R).$$

In particular, if R *is Koszul then* $\operatorname{index}(R^{(c)}) \geq \operatorname{index}(S^{(c)})$ *for every* $c \geq 2$,

Using our methods one can also prove characteristic free the bounds (1) and of the equality for $n = 3$.

We do not present a proof here and refer to [2] for details. The idea of the proof is to study the Koszul complex associated to the c-th power of the maximal ideal of S which is closely related to the problems described so far.

Indeed let $\mathfrak{m}$ the maximal graded ideal of S. Let $K(\mathfrak{m}^c)$ denote the Koszul complex associated to $\mathfrak{m}^c$, $Z_t(\mathfrak{m}^c)$ the module of cycles of homological degree t and $H_t(\mathfrak{m}^c)$ the corresponding homology module. Let T be the symmetric algebra on vector space S_c. Then it is easy to see that:

Lemma 3.7. *For* $i \in \mathbb{N}$, $j \in \mathbb{Z}$ *and* $0 \leq k < c$ *we have*

$$\beta^T_{i,j}(S^{(c)}) = \dim_K H_i(\mathfrak{m}^c)_{jc}.$$

Thus studying the N_p-property of $S^{(c)}$ is equivalent to study vanishing theorems of $H_i(\mathfrak{m}^c, S)$. Studying $Z_t(\mathfrak{m}^c)$ carefully allows to prove a result of Green [4, Theorem 2.2]:

Theorem 3.8. *We have:*

$$H_i(\mathfrak{m}^c)_j = 0 \text{ for every } j \geq ic+i+c.$$

More precisely, a generalizations of this theorem is proved in [2] since one can give upper bounds on the degrees of a minimal system of generators of $Z_t(\mathfrak{m}^c)$. Using the last two results and some further arguments allows us to give a proof of Theorem 3.6; see [2] for further details and more general statements.

Using an Avramov-Golod type of duality one can get Ottaviani and Paoletti's upper bound $\operatorname{index}(S^{(c)}) \leq 3c-3$ in arbitrary characteristic. It is also not difficult to prove that for $n = 3$ one has $\operatorname{index}(S^{(c)}) = 3c-3$ independently of the characteristic. Again we refer to [2] for details.

REFERENCES

[1] J. L. Andersen, *Determinantal rings associated with symmetric matrices: a counterexample.* Ph.D. Thesis, University of Minnesota (1992).

[2] W. Bruns, A. Conca and T. Römer, *Koszul homology and syzygies of Veronese subalgebras*. Preprint 2009, arXiv:0902.2431.

[3] D. Eisenbud, M. Green, K. Hulek and S. Popescu, *Restricting linear syzygies: algebra and geometry.* Compos. Math. **141** (2005), 1460–1478.

[4] M. L. Green, *Koszul cohomology and the geometry of projective varieties. II.* J. Differ. Geom. **20** (1984), 279–289.

[5] M. L. Green and R. Lazarsfeld, *On the projective normality of complete linear series on an algebraic curve.* Invent. Math. **83** (1986), 73–90.

[6] M. L. Green and R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves.* Compos. Math. **67** (1988), 301–314.

[7] T. Jozefiak, P. Pragacz and J. Weyman, *Resolutions of determinantal varieties and tensor complexes associated with symmetric and antisymmetric matrices.* Astérisque **87-88**, 109–189 (1981).

[8] G. Ottaviani and R. Paoletti, *Syzygies of Veronese embeddings.* Compos. Math. **125** (2001), 31–37.

[9] E. Rubei, *A result on resolutions of Veronese embeddings.* Ann. Univ. Ferrara, Nuova Ser., Sez. VII **50** (2004), 151–165.

4. Lecture: Homological properties of Orlik-Solomon algebras

The content of this lecture is based on the article [7] which is joint work with Gesa Kämpf. We beginn by reviewing notions from matroid theory that will be used in the following. More details can be found, e.g., in the books [9] or [10].

Let M be a non-empty matroid over $[n] = \{1,\dots,n\}$, i.e. M is a collection of subsets of $[n]$, called *independent sets*, satisfying the following conditions:

(1) $\emptyset \in M$.
(2) If $F \in M$ and $G \subseteq F$, then $G \in M$.
(3) If $F, G \in M$ and $|G| < |F|$, then there exists an element $i \in F \setminus G$ such that $G \cup \{i\} \in M$.

The subsets of $[n]$ that are not in M are called *dependent*, minimal dependent sets are called *circuits*. The cardinality of maximal independent sets (called *bases*) is constant and denoted by $r(M)$, the *rank* of M. In the following the letter "M" denotes always a matroid and never a module.

Conditions (1) and (2) are just saying that a matroid is a simplicial complex. In an earlier lecture we gave already a definition of a matroid, but this one is equivalent to the one we consider here. There are two examples of classes matroids which motivate large part of the theory:

Example 4.1.

(1) *Let K be field and $v_1,\dots,v_n \in K^m$. Let M be the collection of sets $\{i_1,\dots,i_r\}$ such that $v_{i_1},\dots,v_{i_t}$ are K-linearly independent. Then it follows from basic results in linear algebra that M is a matroid. Such matroids are called* representable.
(2) *Let G be a finite Graph with edges $e_1,\dots,e_n$. Now define $M(G)$ as the collection of sets $\{i_1,\dots,i_r\}$ such that $\{e_{i_1},\dots,e_{i_t}\}$ contains no cycle. Then it follows that $M(G)$ is also a matroid. $M(G)$ is called a* graphic *matroid.*

Next we introduce algebraic objects associated to matroids. Let $E = K\langle e_1,\dots,e_n\rangle$ be the standard graded exterior algebra over K where $\deg e_i = 1$ for $i = 1,\dots,n$ and $\mathfrak{m} = (e_1,\dots,e_n)$. For $F = \{j_1,\dots,j_t\} \subseteq [n] = \{1,\dots,n\}$ we set $e_F = e_{j_1} \wedge \dots \wedge e_{j_t}$. Usually we assume that $1 \le j_1 < \dots < j_t \le n$. The elements e_F are called *monomials* in E. Now we define:

Definition 4.2. *Let M be a matroid in $[n]$. Then we define the* Orlik-Solomon ideal $J(M)$ *of M to be the ideal which is generated by all*

$$\partial e_F = \sum_{i=1}^{t} (-1)^{i-1} e_{j_1} \wedge \dots \wedge \widehat{e_{j_i}} \wedge \dots \wedge e_{j_t} \text{ for } F = \{j_1,\dots,j_t\} \subseteq [n] \tag{2}$$

where $\{j_1,\dots,j_t\}$ is a dependent set of M. The algebra $R(M) = E/J(M)$ is called the Orlik-Solomon algebra *of M.*

The motivating example for this definition is:

Example 4.3. *Let $\mathscr{A} = \{H_1,\dots,H_n\}$ be an essential central affine hyperplane arrangement in $\mathbb{C}^m$, X its complement and K a field. We choose linear forms $\alpha_i \in (\mathbb{C}^m)^*$ such that* $\operatorname{Ker}\alpha_i = H_i$ *for $i = 1,\dots,n$. Let $M(\mathscr{A})$ be the collection of sets $\{i_1,\dots,i_r\}$ such* *that $\alpha_{i_1},\dots,\alpha_{i_t}$ are K-linearly independent. It is well-known that the singular cohomology $H^\bullet(X;K)$ of X with coefficients in K is isomorphic to the Orlik-Solomon algebra of $M(\mathscr{A})$.*

In the last years many researchers have studied the relationship between ring properties of $R(M(\mathscr{A}))$ and properties of $\mathscr{A}$. See, e.g., the book of Orlik-Terao [8] and the survey of Yuzvinsky [11] for more information.

In the following we study the Orlik-Solomon algebra $R(M)$ of an arbitrary matroid M. For this we consider module theory over the exterior algebra. Let N be a graded left and right E-module satisfying $am = (-1)^{\deg a \deg m} ma$ for homogeneous elements $a \in E$, $m \in N$. For example, if $J \subseteq E$ is a graded ideal, then E/J is such a module. Following [3] we define:

Definition 4.4.

(1) *We call an element $v \in E_1$* regular *on N (or N-*regular*) if the annihilator $0 :_N v$ of v in N is vN.*
(2) *An N-*regular sequence *is a sequence $v_1,\dots,v_s$ in E_1 such that the element v_i is $N/(v_1,\dots,v_{i-1})N$-regular for $i = 1,\dots,s$ and $N/(v_1,\dots,v_s)N \neq 0$.*
(3) *This maximal length of an N-regular sequence is called the* depth *of N over E and is denoted by* $\operatorname{depth} N$.

Note that every N-regular sequence can be extended to a maximal one and all maximal regular sequences have the same length.

Projective dimension is a meaningless concept for E-modules. Instead we consider the following definition. For $i \in \mathbb{N}$ and $j \in \mathbb{Z}$ we call $\beta_{i,j}(N) = \dim_K \operatorname{Tor}_i^E(K,N)_j$ the *graded Betti numbers* and $\mu_{i,j}(N) = \dim_K \operatorname{Ext}_E^i(K,N)_j$ the *graded Bass numbers* of N.

The complexity of N measures the growth rate of the Betti numbers of N:

Definition 4.5. *We call*

$$\operatorname{cx} N = \inf\{c \in \mathbb{N} : \beta_i(N) \leq \alpha i^{c-1} \text{ for all } i \geq 1, \alpha \in \mathbb{R}\}$$

complexity *of N.*

Here $\beta_i(N) = \sum_{j\in\mathbb{Z}} \beta_{i,j}(N)$ is the i-th total *Betti number* of N. A result of Aramova, Avramov and Herzog [1, Theorem 3.2] states that

$$\operatorname{cx} M + \operatorname{depth} M = n.$$

In this sense the complexity plays the role projective dimension has over polynomial rings.

Recall that a module N has a *d-linear (projective) resolution* if $\beta_{i,i+j}(N) = 0$ for all i and $j \neq d$. Since E is injective, injective resolutions are much simpler than over arbitrary rings. We say that N has a *d-linear injective resolution* if $\mu_{i,j-i}(N) = 0$ for all i and $j \neq d$.

Example 4.6. *Let Δ be a simplicial complex on $[n]$. Then Δ is Cohen-Macaulay if and only if the face ideal $J_{\Delta^*} = (e_F : F \notin \Delta^*)$ of the Alexander dual $\Delta^* = \{F \subseteq [n] : F^c \notin \Delta\}$ (here F^c denotes the complement of F in $[n]$) has a linear projective resolution as was shown in* [2, Corollary 7.6].

This is equivalent to say that the face ring $K\{\Delta\} = E/J_\Delta$ has a linear injective resolution as it is the dual $(J_{\Delta^*})^* = \operatorname{Hom}_E(J_{\Delta^*},E) \cong E/(E/J_{\Delta^*})^* \cong E/0 :_E J_{\Delta^*} \cong E/J_\Delta$ of J_{Δ^*}.

The second motivating example of modules with linear injective resolutions are Orlik-Solomon algebras. This fact was first observed by Eisenbud, Popescu and Yuzvinsky in [6] for Orlik-Solomon algebras defined by hyperplane arrangements, although their proof works for arbitrary Orlik-Solomon algebras as well. A variation of the proof as it is presented in [7].

Theorem 4.7. [6, Theorem 1.1] *Let $l = r(M)$ be the rank of the matroid M. Then the Orlik-Solomon algebra $E/J(M)$ of M has an l-linear injective resolution.*

Module with linear injective resolutions have very good algebraic properties. We collect some results from [7]:

Theorem 4.8. *Let E/J has a d-linear injective resolution. Then:*

(1) $\operatorname{reg} E/J + \operatorname{depth} E/J = d$.

(2) *There exists a polynomial $Q(t) \in \mathbb{Z}[t]$ with non-negative coefficients such that*

$$H(E/J,t) := \sum_{i\in\mathbb{Z}} \dim_K (E/J)_i t^i = Q(t) \cdot (1+t)^{\operatorname{depth} E/J} \textit{ and } Q(-1) \neq 0.$$

Here $\operatorname{reg} N = \max\{j - i : \beta_{i,j}(N) \neq 0\}$ is the *regularity* of N. The proof of this theorem is based on techniques from Gröbner-basis theory which can be developed similar to the polynomial ring case. Specializing the results to matroids one gets for example:

Theorem 4.9. *([7]) Let $J \subseteq E$ be the Orlik-Solomon ideal of a loopless matroid M on $[n]$ of rank l with k components, then*

$$\operatorname{depth} E/J = k \textit{ and } \operatorname{reg} E/J = l - k.$$

In [7] also a characterization is given of those matroids whose Orlik-Solomon ideal has a linear projective resolution. We refer to that paper for details.

REFERENCES

[1] A. Aramova, L. L. Avramov and J. Herzog, *Resolutions of monomial ideals and cohomology over exterior algebras*. Trans. Am. Math. Soc. **352** (2000), No.2, 579–594.

[2] A. Aramova and J. Herzog, *Almost regular sequences and Betti numbers*. Am. J. Math. **122** (2000), No.4, 689–719.

[3] A. Aramova, J. Herzog and T. Hibi, *Gotzmann Theorems for Exterior Algebras and Combinatorics*. J. Algebra **191** (1997), No.1, 174–211.

[4] A. Björner, *The homology and shellability of matroids and geometric lattices*. In: Matroid applications, Encycl. Math. Appl. **40**, 226–283, Cambridge University Press (1992).

[5] G. Denham and S. Yuzvinsky, *Annihilators of Orlik-Solomon relations*. Adv. Appl. Math. **28** (2002), No.2, 231–249.

[6] D. Eisenbud, S. Popescu and S. Yuzvinsky, *Hyperplane Arrangement Cohomology and Monomials in the Exterior Algebra*. Trans. Am. Math. Soc. **355** (2003), No. 11, 4365–4383.

[7] G. Kämpf and T. Römer, *Homological properties of Orlik-Solomon algebras*. To appear in Manuscr. Math.

[8] P. Orlik and H. Terao, *Arrangements of hyperplanes*. Grundlehren der Mathematischen Wissenschaften **300**, Springer-Verlag (1992).

[9] J. G. Oxley, *Matroid Theory*. Oxford Graduate Texts in Mathematics **3**, Oxford University Press (2006).

[10] D. Welsh, *Matroid Theory*. L.M.S. Monographs **8**, Academic Press (1976).

[11] S. Yuzvinsky, *Orlik-Solomon algebras in algebra and topology*. Russ. Math. Surv. **56** (2001), No. 2, 293–364.