

# COMPUTING IN COMMUTATIVE ALGEBRA

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# COMPUTING IN COMMUTATIVE ALGEBRA

GERHARD PFISTER

## 1. STANDARD BASES AND SINGULAR

SINGULAR is available, free of charge, as a binary programme for most common hardware and software platforms. Release versions of SINGULAR can be downloaded through ftp from our FTP site

`ftp://www.mathematik.uni-kl.de/pub/Math/Singular/`,

or, using your favourite WWW browser, from

`http://www.singular.uni-kl.de/download.html`

The basis of SINGULAR is multivariate polynomial factorization and standard bases computations.

We explain first of all the notion of a Gröbner basis (with respect to any ordering) as the basis for computations in localizations of factorings of polynomial rings. The presentation of a polynomial as a linear combination of monomials is unique only up to an order of the summands, due to the commutativity of the addition. We can make this order unique by choosing a total ordering on the set of monomials. For further applications it is necessary, however, that the ordering is compatible with the semigroup structure on  $\text{Mon}_n$ .

We give here only the important definitions, theorems and examples. Proofs can be found in [7]. The SINGULAR examples can be found on the CD in [7].

**Definition 1.1.** A monomial ordering or semigroup ordering is a total (or linear) ordering  $>$  on the set of monomials  $\text{Mon}_n = \{x^\alpha \mid \alpha \in \mathbf{N}^n\}$  in  $n$  variables satisfying

$$x^\alpha > x^\beta : \implies : x^\gamma x^\alpha > x^\gamma x^\beta$$

for all  $\alpha, \beta, \gamma \in \mathbf{N}^n$ . We say also  $>$  is a monomial ordering on  $A[x_1, \dots, x_n]$ ,  $A$  any ring, meaning that  $>$  is a monomial ordering on  $\text{Mon}_n$ .

**Definition 1.2.** Let  $>$  be a fixed monomial ordering. Write  $f \in K[x]$ ,  $f \neq 0$ , in a unique way as a sum of non-zero terms

$$f = a_\alpha x^\alpha + a_\beta x^\beta + \dots + a_\gamma x^\gamma, \quad x^\alpha > x^\beta > \dots > x^\gamma,$$

and  $a_\alpha, a_\beta, \dots, a_\gamma \in K$ . We define:

- (1)  $LM(f) := \text{leadmonom}(f) := x^\alpha$ , the leading monomial of  $f$ ,
- (2)  $LE(f) := \text{leadexp}(f) := \alpha$ , the leading exponent of  $f$ ,
- (3)  $LT(f) := \text{lead}(f) := a_\alpha x^\alpha$ , the leading term or head of  $f$ ,
- (4)  $LC(f) := \text{leadcoef}(f) := a_\alpha$ , the leading coefficient of  $f$

- (5)  $\text{tail}(f) := f - \text{lead}(f) = a_\beta x^\beta + \dots + a_\gamma x^\gamma$ , the tail.  
(6)  $\text{ecart}(f) := \deg(f) - \deg(\text{LM}(f))$ .

**SINGULAR Example 1.**

```
ring A = 0, (x,y,z), lp;
poly f = y4z3+2x2y2z2+3x5+4z4+5y2;
f; //display f in a lex-ordered way
//-> 3x5+2x2y2z2+y4z3+5y2+4z4
leadmonom(f); //leading monomial
//-> x5
leadexp(f); //leading exponent
//-> 5,0,0
lead(f); //leading term
//-> 3x5
leadcoef(f); //leading coefficient
//-> 3
f - lead(f); //tail
//-> 2x2y2z2+y4z3+5y2+4z4
```

**Definition 1.3.** Let  $>$  be a monomial ordering on  $\{x^\alpha \mid \alpha \in \mathbf{N}^n\}$ .

- (1)  $>$  is called a global ordering if  $x^\alpha > 1$  for all  $\alpha \neq (0, \dots, 0)$ ,
- (2)  $>$  is called a local ordering if  $x^\alpha < 1$  for all  $\alpha \neq (0, \dots, 0)$ ,
- (3)  $>$  is called a mixed ordering if it is neither global nor local.

**Lemma 1.4.** Let  $>$  be a monomial ordering, then the following conditions are equivalent:

- (1)  $>$  is a well-ordering.
- (2)  $x_i > 1$  for  $i = 1, \dots, n$ .
- (3)  $x^\alpha > 1$  for all  $\alpha \neq (0, \dots, 0)$ , that is,  $>$  is global.

In the following examples we fix an enumeration  $x_1, \dots, x_n$  of the variables, any other enumeration leads to a different ordering.

*%beginenumerate GLOBAL ORDERINGS*

(i) *Lexicographical ordering  $>_{lp}$*  (also denoted by *lex*):

$$x^\alpha >_{lp} x^\beta : \iff \exists 1 \leq i \leq n : \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i.$$

(ii) *Degree reverse lexicographical ordering  $>_{dp}$*  (denoted by *degrevlex*):

$$x^\alpha >_{dp} x^\beta : \iff : \deg x^\alpha > \deg x^\beta$$

$$\text{or} : (\deg x^\alpha = \deg x^\beta \text{ and } \exists 1 \leq i \leq n :$$

$$\alpha_n = \beta_n, \dots, \alpha_{i+1} = \beta_{i+1}, \alpha_i < \beta_i),$$

where  $\deg x^\alpha = \alpha_1 + \dots + \alpha_n$ .

*%le i*

*LOCAL ORDERINGS*

(i) *Negative lexicographical ordering*  $>_{ls}$ :

$$x^\alpha >_{ls} x^\beta : \iff \exists 1 \leq i \leq n, \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i.$$

(ii) *Negative degree reverse lexicographical ordering*:

$$\begin{aligned} x^\alpha >_{ds} x^\beta &: \iff: \quad \deg x^\alpha < \deg x^\beta, \text{ where } \deg x^\alpha = \alpha_1 + \dots + \alpha_n, \\ &\text{or : } (\deg x^\alpha = \deg x^\beta \text{ and } \exists 1 \leq i \leq n : \\ &\quad \alpha_n = \beta_n, \dots, \alpha_{i+1} = \beta_{i+1}, \alpha_i < \beta_i). \end{aligned}$$

Let  $>$  be a monomial ordering on the set of monomials  $\text{Mon}(x_1, \dots, x_n) = \{x^\alpha \mid \alpha \in \mathbf{N}^n\}$ , and  $K[x] = K[x_1, \dots, x_n]$  the polynomial ring in  $n$  variables over a field  $K$ . Then the leading monomial function  $\text{LM}$  has the following properties for polynomials  $f, g \in K[x] \setminus \{0\}$ :

- (1)  $\text{LM}(gf) = \text{LM}(g)\text{LM}(f)$ .
- (2)  $\text{LM}(g+f) \leq \max\{\text{LM}(g), \text{LM}(f)\}$  with equality if and only if the leading terms of  $f$  and  $g$  do not cancel.

In particular, it follows that

$$S_{>} := \{u \in K[x] \setminus \{0\} \mid \text{LM}(u) = 1\}$$

is a multiplicatively closed set.

**Definition 1.5.** For any monomial ordering  $>$  on  $\text{Mon}(x_1, \dots, x_n)$ , we define

$$K[x]_{>} := S_{>}^{-1}K[x] = \left\{ \frac{f}{u} \mid f, u \in K[x], \text{LM}(u) = 1 \right\},$$

the localization of  $K[x]$  with respect to  $S_{>}$  and call  $K[x]_{>}$  the ring associated to  $K[x]$  and  $>$ .

Note that  $S_{>} = K^*$  if and only if  $>$  is global and  $S_{>} = K[x] \setminus \langle x_1, \dots, x_n \rangle$  if and only if  $>$  is local.

**Definition 1.6.** Let  $>$  be any monomial ordering:

- (1) For  $f \in K[x]_{>}$  choose  $u \in K[x]$  such that  $\text{LT}(u) = 1$  and  $uf \in K[x]$ . We define

$$\begin{aligned} \text{LM}(f) &:= \text{LM}(uf), \\ \text{LC}(f) &:= \text{LC}(uf), \\ \text{LT}(f) &:= \text{LT}(uf), \\ \text{LE}(f) &:= \text{LE}(uf), \end{aligned}$$

and  $\text{tail}(f) = f - \text{LT}(f)$ .

- (2) For any subset  $G \subset K[x]_{>}$  define the ideal

$$L_{>}(G) := L(G) := \langle \text{LM}(g) \mid g \in G \setminus \{0\} \rangle_{K[x]}.$$

$L(G) \subset K[x]$  is called the leading ideal of  $G$ .

**Definition 1.7.** Let  $I \subset R = K[x]_{>}$  be an ideal.

(1) A finite set  $G \subset R$  is called a standard basis of  $I$  if

$$G \subset I, \text{ and } L(I) = L(G).$$

That is,  $G$  is a standard basis, if the leading monomials of the elements of  $G$  generate the leading ideal of  $I$ , or, in other words, if for any  $f \in I \setminus \{0\}$  there exists a  $g \in G$  satisfying  $LM(g) \mid LM(f)$ .

(2) If  $>$  is global, a standard basis is also called a Gröbner basis.

(3) If we just say that  $G$  is a standard basis, we mean that  $G$  is a standard basis of the ideal  $\langle G \rangle_R$  generated by  $G$ .

Standard bases can be characterized using the notion of the normal form. We need the following definitions:

**Definition 1.8.** Let  $f, g \in R \setminus \{0\}$  with  $LM(f) = x^\alpha$  and  $LM(g) = x^\beta$ , respectively. Set

$$\gamma := \text{lcm}(\alpha, \beta) := (\max(\alpha_1, \beta_1), \dots, \max(\alpha_n, \beta_n))$$

and let  $\text{lcm}(x^\alpha, x^\beta) := x^\gamma$  be the least common multiple of  $x^\alpha$  and  $x^\beta$ . We define the  $s$ -polynomial (spoly, for short) of  $f$  and  $g$  to be

$$\text{spoly}(f, g) := x^{\gamma-\alpha} f - \frac{LC(f)}{LC(g)} \cdot x^{\gamma-\beta} g.$$

If  $LM(g)$  divides  $LM(f)$ , say  $LM(g) = x^\beta$ ,  $LM(f) = x^\alpha$ , then the  $s$ -polynomial is particularly simple,

$$\text{spoly}(f, g) = f - \frac{LC(f)}{LC(g)} \cdot x^{\alpha-\beta} g,$$

and  $LM(\text{spoly}(f, g)) < LM(f)$ .

**Definition 1.9.** Let  $\mathcal{G}$  denote the set of all finite lists  $G \subset R = K[x]_{>}$ .

$$NF : R \times \mathcal{G} \rightarrow R, (f, G) \mapsto NF(f \mid G),$$

is called a normal form on  $R$  if, for all  $G \in \mathcal{G}$ ,

$$(0) \quad NF(0 \mid G) = 0,$$

and, for all  $f \in R$  and  $G \in \mathcal{G}$ ,

$$(1) \quad NF(f \mid G) \neq 0 \implies LM(NF(f \mid G)) \notin L(G).$$

(2) If  $G = \{g_1, \dots, g_s\}$ , then  $f$  has a standard representation with respect to  $NF(- \mid G)$ , that is, there exists a unit  $u \in R^*$  such that

$$uf - NF(f \mid G) = \sum_{i=1}^s a_i g_i, \quad a_i \in R, \quad s \geq 0,$$

satisfying  $LM(\sum_{i=1}^s a_i g_i) \geq LM(a_i g_i)$  for all  $i$  such that  $a_i g_i \neq 0$ .

The existence of a normal form is given by the following algorithm:

**Algorithm 1.10.**  $NF(f \mid G)$

Let  $>$  be any monomial ordering.

Input:  $f \in K[x]$ ,  $G$  a finite list in  $K[x]$

Output:  $h \in K[x]$  a polynomial normal form of  $f$  with respect to  $G$ .

- $h := f$ ;
- $T := G$ ;
- *while*( $h \neq 0$  and  $T_h := \{g \in T \mid LM(g) \mid LM(h)\} \neq \emptyset$ )
  - choose*  $g \in T_h$  with *ecart*( $g$ ) minimal;
  - if* (*ecart*( $g$ ) > *ecart*( $h$ ))
    - $T := T \cup \{h\}$ ;
    - $h := \text{spoly}(h, g)$ ;
- *return*  $h$ ;

**Theorem 1.11.** *Let  $I \subset R$  be an ideal and  $G = \{g_1, \dots, g_s\} \subset I$ . Then the following are equivalent:*

- (1)  $G$  is a standard basis of  $I$ .
- (2)  $NF(f \mid G) = 0$  if and only if  $f \in I$ .

We will explain now how to use standard bases to solve problems in algebra.

### Ideal membership

*Problem:* Given  $f, f_1, \dots, f_k \in K[x]$ , and let  $I = \langle f_1, \dots, f_k \rangle_R$ . We wish to decide whether  $f \in I$ , or not.

*Solution:* We choose any monomial ordering  $>$  such that  $K[x]_> = R$  and compute a standard basis  $G = \{g_1, \dots, g_s\}$  of  $I$  with respect to  $>$ .  $f \in I$  if and only if  $NF(f \mid G) = 0$ .

### SINGULAR Example 2.

```
ring A = 0, (x,y), dp;
ideal I = x10+x9y2, y8-x2y7;
ideal J = std(I);
poly f = x2y7+y14;
reduce(f, J, 1);          //3rd parameter 1 avoids tail reduction
//-> -xy12+x2y7          //f is not in I
      f = xy13+y12;
reduce(f, J, 1);
//-> 0                    //f is in I
```

### Intersection with Subrings (Elimination of variables)

*Problem:* Given  $f_1, \dots, f_k \in K[x] = K[x_1, \dots, x_n]$ ,  $I = \langle f_1, \dots, f_k \rangle_{K[x]}$ , we should like to find generators of the ideal

$$I' = I \cap K[x_{s+1}, \dots, x_n], \quad s < n.$$

Elements of the ideal  $I'$  are said to be obtained from  $f_1, \dots, f_k$  by *eliminating*  $x_1, \dots, x_s$ . The following lemma is the basis for solving the elimination problem.

**Lemma 1.12.** Let  $>$  be an elimination ordering for  $x_1, \dots, x_s$  on the set of monomials  $\text{Mon}(x_1, \dots, x_n)$ , and let  $I \subset K[x_1, \dots, x_n]_{>}$  be an ideal. If  $S = \{g_1, \dots, g_k\}$  is a standard basis of  $I$ , then

$$S' := \{g \in S \mid LM(g) \in K[x_{s+1}, \dots, x_n]\}$$

is a standard basis of  $I' := I \cap K[x_{s+1}, \dots, x_n]_{>'}$ . In particular,  $S'$  generates the ideal  $I'$ .

**SINGULAR Example 3.**

```
ring A =0, (t,x,y,z), dp;
ideal I=t2+x2+y2+z2, t2+2x2-xy-z2, t+y3-z3;

eliminate(I,t);
//-> _[1]=x2-xy-y2-2z2      _[2]=y6-2y3z3+z6+2x2-xy-z2
```

Alternatively choose a product ordering:

```
ring A1=0, (t,x,y,z), (dp(1), dp(3));
ideal I=imap(A,I);
ideal J=std(I);
J;
//-> J[1]=x2-xy-y2-2z2      J[2]=y6-2y3z3+z6+2x2-xy-z2
//-> J[3]=t+y3-z3
```

**Radical Membership**

*Problem:* Let  $f_1, \dots, f_k \in K[x]_{>}$ ,  $>$  a monomial ordering on  $\text{Mon}(x_1, \dots, x_n)$  and  $I = \langle f_1, \dots, f_k \rangle_{K[x]_{>}}$ . Given some  $f \in K[x]_{>}$  we want to decide whether  $f \in \sqrt{I}$ . The following lemma, which is sometimes called *Rabinowich's trick*, is the basis for solving this problem.<sup>1</sup>

**Lemma 1.13.** Let  $A$  be a ring,  $I \subset A$  an ideal and  $f \in A$ . Then

$$f \in \sqrt{I} : \iff : 1 \in \tilde{I} := \langle I, 1 - tf \rangle_{A[t]}$$

where  $t$  is an additional new variable.

**SINGULAR Example 4.**

```
ring A =0, (x,y,z), dp;
ideal I=x5,xy3,y7,z3+xyz;
poly f =x+y+z;

ring B =0, (t,x,y,z), dp; //need t for radical test
ideal I=imap(A,I);
poly f =imap(A,f);
I=I,1-t*f;
std(I);
//-> _[1]=1                //f is in the radical
```

<sup>1</sup>We can even compute the full radical  $\sqrt{I}$ , but this is a much harder computation.

```
LIB"primdec.lib"; //just to see, we compute the radical
setring A;
radical(I);
//-> _[1]=z _[2]=y _[3]=x
```

### Intersection of Ideals

*Problem:* Given  $f_1, \dots, f_k, h_1, \dots, h_r \in K[x]$  and  $>$  a monomial ordering. Let  $I_1 = \langle f_1, \dots, f_k \rangle_{K[x]>}$  and  $I_2 = \langle h_1, \dots, h_r \rangle_{K[x]>}$ . We wish to find generators for  $I_1 \cap I_2$ .

Consider the ideal  $J := \langle tf_1, \dots, tf_k, (1-t)h_1, \dots, (1-t)h_r \rangle_{(K[x]>)[t]}$ .

**Lemma 1.14.** *With the above notations,  $I_1 \cap I_2 = J \cap K[x]>$ .*

### SINGULAR Example 5.

```
ring A=0,(x,y,z),dp;
ideal I1=x,y;
ideal I2=y^2,z;
intersect(I1,I2); //the built-in SINGULAR command
//-> _[1]=y^2 _[2]=yz _[3]=xz
```

```
ring B=0,(t,x,y,z),dp; //the way described above
ideal I1=imap(A,I1);
ideal I2=imap(A,I2);
ideal J=t*I1+(1-t)*I2;
eliminate(J,t);
//-> _[1]=yz _[2]=xz _[3]=y^2
```

### Quotient of Ideals

*Problem:* Let  $I_1$  and  $I_2 \subset K[x]>$ . We want to compute

$$I_1 : I_2 = \{g \in K[x]> \mid gI_2 \subset I_1\}.$$

Since, obviously,  $I_1 : \langle h_1, \dots, h_r \rangle = \bigcap_{i=1}^r (I_1 : \langle h_i \rangle)$ , we can compute  $I_1 : \langle h_i \rangle$  for each  $i$ . The next lemma shows a way to compute  $I_1 : \langle h_i \rangle$ .

**Lemma 1.15.** *Let  $I \subset K[x]>$  be an ideal, and let  $h \in K[x]>$ ,  $h \neq 0$ . Moreover, let  $I \cap \langle h \rangle = \langle g_1 \cdot h, \dots, g_s \cdot h \rangle$ . Then  $I : \langle h \rangle = \langle g_1, \dots, g_s \rangle_{K[x]>}$ .*

### SINGULAR Example 6.

```
ring A=0,(x,y,z),dp;
ideal I1=x,y;
ideal I2=y^2,z;
quotient(I1,I2); //the built-in SINGULAR command
//-> _[1]=y _[2]=x
```



### Kernel of a Ring Map

Let  $\varphi : R_1 := (K[x]_{>_1})/I \rightarrow (K[y]_{>_2})/J =: R_2$  be a ring map defined by polynomials  $\varphi(x_i) = f_i \in K[y] = K[y_1, \dots, y_m]$  for  $i = 1, \dots, n$  (and assume that the monomial orderings satisfy  $1 >_2 \text{LM}(f_i)$  if  $1 >_1 x_i$ ).

Define  $J_0 := J \cap K[y]$ , and  $I_0 := I \cap K[x]$ . Then  $\varphi$  is induced by

$$\tilde{\varphi} : K[x]/I_0 \rightarrow K[y]/J_0, \quad x_i \mapsto f_i,$$

and we have a commutative diagram

$$\begin{array}{ccc} K[x]/I_0 & \xrightarrow{\tilde{\varphi}} & K[y]/J_0 \\ \downarrow & & \downarrow \\ R_1 & \xrightarrow{\varphi} & R_2. \end{array}$$

*Problem:* Let  $I, J$  and  $\varphi$  be as above. Compute generators for  $\text{Ker}(\varphi)$ .

*Solution:* Assume that  $J_0 = \langle g_1, \dots, g_s \rangle_{K[y]}$  and  $I_0 = \langle h_1, \dots, h_t \rangle_{K[x]}$ .

Set  $H := \langle h_1, \dots, h_t, g_1, \dots, g_s, x_1 - f_1, \dots, x_n - f_n \rangle \subset K[x, y]$ , and compute  $H' := H \cap K[x]$  by eliminating  $y_1, \dots, y_m$  from  $H$ . Then  $H'$  generates  $\text{Ker}(\varphi)$  by the following lemma.

**Lemma 1.16.** *With the above notations,  $\text{Ker}(\varphi) = \text{Ker}(\tilde{\varphi})R_1$  and*

$$\text{Ker}(\tilde{\varphi}) = (I_0 + \langle g_1, \dots, g_s, x_1 - f_1, \dots, x_n - f_n \rangle_{K[x,y]} \cap K[x]) \text{ mod } I_0.$$

*In particular, if  $>_1$  is global, then  $\text{Ker}(\varphi) = \text{Ker}(\tilde{\varphi})$ .*

### SINGULAR Example 7.

```
ring A=0, (x,y,z), dp;
ring B=0, (a,b), dp;
map phi=A, a2, ab, b2;
ideal zero; //compute the preimage of 0
setring A;
preimage(B, phi, zero); //the built-in SINGULAR command
//-> _[1]=y2-xz

ring C=0, (x,y,z,a,b), dp; //the method described above
ideal H=x-a2, y-ab, z-b2;
eliminate(H, ab);
//-> _[1]=y2-xz
```

## 2. LECTURE: POLYNOMIAL SOLVING AND PRIMARY DECOMPOSITION

### Solvability of Polynomial Equations

*Problem:* Given  $f_1, \dots, f_k \in K[x_1, \dots, x_n]$ , we want to assure whether the system of polynomial equations

$$f_1(x) = \dots = f_k(x) = 0$$

has a solution in  $\bar{K}^n$ , where  $\bar{K}$  is the algebraic closure of  $K$ .

Let  $I = \langle f_1, \dots, f_k \rangle_{K[x]}$ , then the question is whether the algebraic set  $V(I) \subset \bar{K}^n$  is empty or not.

*Solution:* By Hilbert's Nullstellensatz,  $V(I) = \emptyset$  if and only if  $1 \in I$ . We compute a Gröbner basis  $G$  of  $I$  with respect to any global ordering on  $\text{Mon}(x_1, \dots, x_n)$  and normalize it (that is, divide every  $g \in G$  by  $\text{LC}(g)$ ). Since  $1 \in I$  if and only if  $1 \in L(I)$ , we have  $V(I) = \emptyset$  if and only if 1 is an element of a normalized Gröbner basis of  $I$ . Of course, we can avoid normalizing, which is expensive in rings with parameters. Since  $1 \in I$  if and only if  $G$  contains a non-zero constant polynomial, we have only to look for an element of degree 0 in  $G$ .

### SINGULAR Example 8.

```
ring A=0,(x,y,z),lp;
ideal I=x2+y+z-1,
      x+y2+z-1,
      x+y+z2-1;
ideal J=groebner(I); //the lexicographical Groebner basis
J;
//-> J[1]=z6-4z4+4z3-z2      J[2]=2yz2+z4-z2
//-> J[3]=y2-y-z2+z         J[4]=x+y+z2-1
```

We use the multivariate solver based on triangular sets.

```
LIB"solve.lib";
list s1=solve(I,6);
//-> // name of new current ring: AC
s1;
//-> [1]:          [2]:          [3]:          [4]:          [5]:
//->   [1]:          [1]:          [1]:          [1]:          [1]:
//->     0.414214      0          -2.414214      1          0
//->   [2]:          [2]:          [2]:          [2]:          [2]:
//->     0.414214      0          -2.414214      0          1
//->   [3]:          [3]:          [3]:          [3]:          [3]:
//->     0.414214      1          -2.414214      0          0
```

If we want to compute the zeros with multiplicities then we use 1 as a third parameter for the command:

```

setring A;
list s2=solve(I,6,1);
s2;
//-> [1]: [2]:
//-> [1]: [1]:
//-> [1]: [1]:
//-> -2.414214 0
//-> [2]: [2]:
//-> -2.414214 1
//-> [3]: [3]:
//-> -2.414214 0
//-> [2]: [2]:
//-> [1]: [1]:
//-> 0.414214 1
//-> [2]: [2]:
//-> 0.414214 0
//-> [3]: [3]:
//-> 0.414214 0
//-> [2]: [3]:
//-> 1 [1]:
//-> 0
//-> [2]:
//-> 0
//-> [3]:
//-> 1
//-> [2]:
//-> 2

```

The output has to be interpreted as follows: there are two zeros of multiplicity 1 and three zeros  $((0, 1, 0), (1, 0, 0), (0, 0, 1))$  of multiplicity 2.

**Definition 2.1.**

- (1) A maximal ideal  $M \subset K[x_1, \dots, x_n]$  is called in general position with respect to the lexicographical ordering with  $x_1 > \dots > x_n$ , if there exist  $g_1, \dots, g_n \in K[x_n]$  with  $M = \langle x_1 + g_1(x_n), \dots, x_{n-1} + g_{n-1}(x_n), g_n(x_n) \rangle$ .
- (2) A zero-dimensional ideal  $I \subset K[x_1, \dots, x_n]$  is called in general position with respect to the lexicographical ordering with  $x_1 > \dots > x_n$ , if all associated primes  $P_1, \dots, P_k$  are in general position and if  $P_i \cap K[x_n] \neq P_j \cap K[x_n]$  for  $i \neq j$ .

**Proposition 2.2.** Let  $K$  be a field of characteristic 0, and let  $I \subset K[x]$ ,  $x = (x_1, \dots, x_n)$ , be a zero-dimensional ideal. Then there exists a non-empty, Zariski open subset  $U \subset K^{n-1}$

such that for all  $\underline{a} = (a_1, \dots, a_{n-1}) \in U$ , the coordinate change  $\varphi_{\underline{a}} : K[x] \rightarrow K[x]$  defined by  $\varphi_{\underline{a}}(x_i) = x_i$  if  $i < n$ , and

$$\varphi_{\underline{a}}(x_n) = x_n + \sum_{i=1}^{n-1} a_i x_i$$

has the property that  $\varphi_{\underline{a}}(I)$  is in general position with respect to the lexicographical ordering defined by  $x_1 > \dots > x_n$ .

**Proposition 2.3.** Let  $I \subset K[x_1, \dots, x_n]$  be a zero-dimensional ideal. Let  $\langle g \rangle = I \cap K[x_n]$ ,  $g = g_1^{v_1} \dots g_s^{v_s}$ ,  $g_i$  monic and prime and  $g_i \neq g_j$  for  $i \neq j$ . Then

$$(1) I = \bigcap_{i=1}^s \langle I, g_i^{v_i} \rangle.$$

If  $I$  is in general position with respect to the lexicographical ordering with  $x_1 > \dots > x_n$ , then

$$(2) \langle I, g_i^{v_i} \rangle \text{ is a primary ideal for all } i.$$

**SINGULAR Example 9** (zero-dim primary decomposition).

We give an example for a zero-dimensional primary decomposition.

```
option(redSB);
ring R=0, (x,y), lp;
ideal I=(y^2-1)^2, x^2-(y+1)^3;
```

The ideal  $I$  is not in general position with respect to  $lp$ , since the minimal associated prime  $\langle x^2 - 8, y - 1 \rangle$  is not.

```
map phi=R,x,x+y; //we choose a generic coordinate change
map psi=R,x,-x+y; //and the inverse map
I=std(phi(I));
I;
//-> I[1]=y^7-y^6-19y^5-13y^4+99y^3+221y^2+175y+49
//-> I[2]=112xy+112x-27y^6+64y^5+431y^4-264y^3-2277y^2-2520y-847
//-> I[3]=56x^2+65y^6-159y^5-1014y^4+662y^3+5505y^2+6153y+2100
factorize(I[1]);
//-> [1]:
//-> _[1]=1
//-> _[2]=y^2-2y-7
//-> _[3]=y+1
//-> [2]:
//-> 1,2,3

ideal Q1=std(I,(y^2-2y-7)^2); //the candidates for the
//primary ideals
ideal Q2=std(I,(y+1)^3); //in general position
Q1; Q2;

//-> Q1[1]=y^4-4y^3-10y^2+28y+49 Q2[1]=y^3+3y^2+3y+1
```

```

//-> Q1[2]=56x+y3-9y2+63y-7      Q2[2]=2xy+2x+y2+2y+1
                                   Q2[3]=x2

factorize(Q1[1]); //primary and general position test
                //for Q1

//-> [1]:
//->   _[1]=1
//->   _[2]=y2-2y-7
//-> [2]:
//->   1,2

factorize(Q2[1]); //primary and general position test
                //for Q2

//-> [1]:
//->   _[1]=1
//->   _[2]=y+1
//-> [2]:
//->   1,3

```

Both ideals are primary and in general position.

```

Q1=std(psi(Q1)); //the inverse coordinate change
Q2=std(psi(Q2)); //the result
Q1; Q2;

//-> Q1[1]=y2-2y+1      Q2[1]=y2+2y+1
//-> Q1[2]=x2-12y+4    Q2[2]=x2

```

We obtain that  $I$  is the intersection of the primary ideals  $Q_1$  and  $Q_2$  with associated prime ideals  $\langle y-1, x^2-8 \rangle$  and  $\langle y+1, x \rangle$ .

The following proposition reduces the higher dimensional case to the zero-dimensional case:

**Proposition 2.4.** *Let  $I \subset K[x]$  be an ideal and  $u \subset x = \{x_1, \dots, x_n\}$  be a maximal independent set of variables<sup>2</sup> with respect to  $I$ .*

- (1)  $IK(u)[x \setminus u] \subset K(u)[x \setminus u]$  is a zero-dimensional ideal.
- (2) Let  $S = \{g_1, \dots, g_s\} \subset I \subset K[x]$  be a Gröbner basis of  $IK(u)[x \setminus u]$ , and let  $h := \text{lcm}(LC(g_1), \dots, LC(g_s)) \in K[u]$ , then

$$IK(u)[x \setminus u] \cap K[x] = I : \langle h^\infty \rangle,$$

and this ideal is equidimensional of dimension  $\dim(I)$ .

---

<sup>2</sup>It is maximal such that  $I \cap K[u] = \langle 0 \rangle$ .

- (3) Let  $IK(u)[x \setminus u] = Q_1 \cap \cdots \cap Q_s$  be an irredundant primary decomposition, then also  $IK(u)[x \setminus u] \cap K[x] = (Q_1 \cap K[x]) \cap \cdots \cap (Q_s \cap K[x])$  is an irredundant primary decomposition.

Finally we explain how to compute the radical.

**Proposition 2.5.** Let  $I \subset K[x_1, \dots, x_n]$  be a zero-dimensional ideal and  $I \cap K[x_i] = \langle f_i \rangle$  for  $i = 1, \dots, n$ . Moreover, let  $g_i$  be the squarefree part of  $f_i$ , then  $\sqrt{I} = I + \langle g_1, \dots, g_n \rangle$ .

The higher dimensional case can be reduced similarly to the primary decomposition to the zero-dimensional case.

### 3. LECTURE: INVARIANTS

The computation of the Hilbert function will be discussed and explained. Let  $K$  be a field.

**Definition 3.1.** Let  $A = \bigoplus_{v \geq 0} A_v$  be a Noetherian graded  $K$ -algebra, and let  $M = \bigoplus_{v \in \mathbb{Z}} M_v$  be a finitely generated graded  $A$ -module. The Hilbert function  $H_M : \mathbb{Z} \rightarrow \mathbb{Z}$  of  $M$  is defined by

$$H_M(n) := \dim_K(M_n),$$

and the Hilbert–Poincaré series  $HP_M$  of  $M$  is defined by

$$HP_M(t) := \sum_{v \in \mathbb{Z}} H_M(v) \cdot t^v \in \mathbb{Z}[[t]][t^{-1}].$$

**Theorem 3.2.** Let  $A = \bigoplus_{v \geq 0} A_v$  be a graded  $K$ -algebra, and assume that  $A$  is generated, as  $K$ -algebra, by  $x_1, \dots, x_r \in A_1$ . Then, for any finitely generated (positively) graded  $A$ -module  $M = \bigoplus_{v \geq 0} M_v$ ,

$$HP_M(t) = \frac{Q(t)}{(1-t)^r} \text{ for some } Q(t) \in \mathbb{Z}[t].$$

Note that SINGULAR has a command which computes the numerator  $Q(t)$  for the Hilbert–Poincaré series:

#### SINGULAR Example 10.

```
ring A=0, (t,x,y,z), dp;
ideal I=x5y2,x3,y3,xy4,xy7;
intvec v = hilb(std(I),1);
v;
//-> 1,0,0,-2,0,0,1,0
```

We obtain  $Q(t) = t^6 - 2t^3 + 1$ .

The latter output has to be interpreted as follows: if  $v = (v_0, \dots, v_d, 0)$  then  $Q(t) = \sum_{i=0}^d v_i t^i$ .

**Theorem 3.3.** *Let  $>$  be any monomial ordering on  $K[x] := K[x_1, \dots, x_r]$ , and let  $I \subset K[x]$  be a homogeneous ideal. Then*

$$HP_{K[x]/I}(t) = HP_{K[x]/L(I)}(t),$$

where  $L(I)$  is the leading ideal of  $I$  with respect to  $>$ .

Examples how to compute the Hilbert polynomial, the Hilbert–Samuel function, the degree respectively and the multiplicity and the dimension of an ideal can be found in [7]. As above all computations are reduced to compute the corresponding invariants for the leading ideal.

#### 4. LECTURE: HOMOLOGICAL ALGEBRA

Here we will show different approaches how to test Cohen–Macaulayness using SINGULAR. More details about the underlying theory can be found in [7].

**SINGULAR Example 11** (first test for Cohen–Macaulayness).

*Let  $(A, \mathfrak{m})$  be a local ring,  $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ . Let  $M$  be an  $A$ –module given by a presentation  $A^\ell \rightarrow A^s \rightarrow M \rightarrow 0$ . To check whether  $M$  is Cohen–Macaulay we use that the equality*

$$\begin{aligned} \dim(A/\text{Ann}(M)) &= \dim(M) = \text{depth}(M) \\ &= n - \sup\{i \mid H_i(x_1, \dots, x_n, M) \neq 0\}. \end{aligned}$$

*is necessary and sufficient for  $M$  to be Cohen–Macaulay. The following procedure computes  $\text{depth}(\mathfrak{m}, M)$ , where  $\mathfrak{m} = \langle x_1, \dots, x_n \rangle \subset A = K[x_1, \dots, x_n]_{>}$  and  $M$  is a finitely generated  $A$ –module with  $\mathfrak{m}M \neq M$ .*

The following procedures use the procedures Koszul Homology from `homolog.lib` and `Ann` from `primdec.lib` to compute the Koszul Homology  $H_i(x_1, \dots, x_n, M)$  and the annihilator  $\text{Ann}(M)$ . They have to be loaded first.

```
LIB "homolog.lib";
proc depth(module M)
{
  ideal m=maxideal(1);
  int n=size(m);
  int i;
  while(i<n)
  {
    i++;
    if(size(KoszulHomology(m,M,i))==0){return(n-i+1);}
  }
  return(0);
}
```

Now the test for Cohen–Macaulayness is easy.

```
LIB "primdec.lib";
proc CohenMacaulayTest(module M)
{
  return(depth(M)==dim(std(Ann(M))));
}
```

The procedure returns 1 if  $M$  is Cohen–Macaulay and 0 if not.

As an application, we check that a complete intersection is Cohen–Macaulay and that  $K[x, y, z]_{\langle x, y, z \rangle} / \langle xz, yz, z^2 \rangle$  is not Cohen–Macaulay.

```
ring R=0, (x, y, z), ds;
ideal I=xz, yz, z2;
module M=I*freemodule(1);
CohenMacaulayTest(M);
//-> 0
```

```
I=x2+y2, z7;
M=I*freemodule(1);
CohenMacaulayTest(M);
//-> 1
```

**SINGULAR Example 12** (second test for Cohen–Macaulayness).

Let  $A = K[x_1, \dots, x_n]_{\langle x_1, \dots, x_n \rangle} / I$ . Using Noether normalization, we may assume that  $A \supset K[x_{s+1}, \dots, x_n]_{\langle x_{s+1}, \dots, x_n \rangle} =: B$  is finite. We choose a monomial basis  $m_1, \dots, m_r \in K[x_1, \dots, x_s]$  of  $A|_{x_{s+1}=\dots=x_n=0}$ .

Then  $m_1, \dots, m_r$  is a minimal system of generators of  $A$  as  $B$ -module.  $A$  is Cohen–Macaulay if and only if  $A$  is a free  $B$ -module, that is, there are no  $B$ -relations between  $m_1, \dots, m_r$ , in other words,  $\text{syz}_A(m_1, \dots, m_r) \cap B^r = \langle 0 \rangle$ . This test can be implemented in SINGULAR as follows:

```
proc isCohenMacaulay(ideal I)
{
  def A = basering;
  list L = noetherNormal(I);
  map phi = A, L[1];
  I = phi(I);
  int s = nvars(basering)-size(L[2]);
  execute("ring B="+charstr(A)+" ,x(1..s), ds;");
  ideal m = maxideal(1);
  map psi = A, m;
  ideal J = std(psi(I));
  ideal K = kbase(J);
  setring A;
  execute("
    ring C="+charstr(A)+" ,("+varstr(A)+") , (dp(s), ds);");
```



```

ideal I = imap(A,I);
qring D = std(I);
ideal K = fetch(B,K);
module N = std(syz(K));
intvec v = leadexp(N[size(N)]);
int i=1;
while((i<s)&&(v[i]==0)){i++;}
setring A;
if(!v[i]){return(0);}
return(1);
}

```

As the above procedure uses `noetherNormal` from `algebra.lib`, we first have to load this library.

```

LIB"algebra.lib";
ring r=0,(x,y,z),ds;
ideal I=xz,yz;
isCohenMacaulay(I);
//-> 0

```

```

I=x2-y3;
isCohenMacaulay(I);
//-> 1

```

**SINGULAR Example 13** (3rd test for Cohen–Macaulayness).

We use the Auslander–Buchsbaum formula to compute the depth of  $M$  and then check if  $\text{depth}(M) = \dim(M) = \dim(A/\text{Ann}(M))$ .

We assume that  $A = K[x_1, \dots, x_n]_{\langle x_1, \dots, x_n \rangle} / I$  and compute a minimal free resolution. Then  $\text{depth}(A) = n - \text{pd}_{K[x_1, \dots, x_n]_{\langle x_1, \dots, x_n \rangle}}(A)$ . If  $M$  is a finitely generated  $A$ -module of finite projective dimension, then we compute a minimal free resolution of  $M$  and obtain  $\text{depth}(M) = \text{depth}(A) - \text{pd}_A(M)$ .

```

proc projdim(module M)
{
  list l=mres(M,0);          //compute the resolution
  int i;
  while(i<size(l))
  {
    i++;
    if(size(l[i])==0){return(i-1);}
  }
}

```

Now it is easy to give another test for Cohen–Macaulayness.

```

proc isCohenMacaulay1(ideal I)

```

```

{
  int de=nvars(basering)-projdim(I*freemodule(1));
  int di=dim(std(I));
  return(de==di);
}

```

```

ring R=0,(x,y,z),ds;
ideal I=xz,yz;
isCohenMacaulay1(I);
//-> 0

```

```

I=x2-y3;
isCohenMacaulay1(I);
//-> 1

```

```

I=xz,yz,xy;
isCohenMacaulay1(I);
//-> 1
kill R;

```

The following procedure checks whether the depth of  $M$  is equal to  $d$ . It uses the procedure `Ann` from `primdec.lib`.

```

proc CohenMacaulayTest1(module M, int d)
{
  return((d-projdim(M))==dim(std(Ann(M))));
}

```

```

LIB"primdec.lib";
ring R=0,(x,y,z),ds;
ideal I=xz,yz;
module M=I*freemodule(1);
CohenMacaulayTest1(M,3);
//-> 0

```

```

I=x2+y2,z7;
M=I*freemodule(1);
CohenMacaulayTest1(M,3);
//-> 1

```

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