COMPUTING IN COMMUTATIVE ALGEBRA

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1. STANDARD BASES AND SINGULAR

SINGULAR is available, free of charge, as a binary programme for most common hardware and software platforms. Release versions of SINGULAR can be downloaded through ftp from our FTP site

`ftp://www.mathematik.uni-kl.de/pub/Math/Singular/`,

or, using your favourite WWW browser, from

`http://www.singular.uni-kl.de/download.html`

The basis of SINGULAR is multivariate polynomial factorization and standard bases computations.

We explain first of all the notion of a Gröbner basis (with respect to any ordering) as the basis for computations in localizations of factorrings of polynomial rings. The presentation of a polynomial as a linear combination of monomials is unique only up to an order of the summands, due to the commutativity of the addition. We can make this order unique by choosing a total ordering on the set of monomials. For further applications it is necessary, however, that the ordering is compatible with the semigroup structure on Mon_n.

We give here only the important definitions, theorems and examples. Proofs can be found in [7]. The SINGULAR examples can be found on the CD in [7].

Definition 1.1. *A* monomial ordering *or* semigroup ordering *is a total (or linear) ordering $>$ on the set of monomials* $\mathrm{Mon}_n = \{x^\alpha \mid \alpha \in \mathbf{N}^n\}$ *in n variables satisfying*

$$x^\alpha > x^\beta :\implies: x^\gamma x^\alpha > x^\gamma x^\beta$$

for all $\alpha, \beta, \gamma \in \mathbf{N}^n$. We say also $>$ is a monomial ordering on $A[x_1, \dots, x_n]$, *A any ring, meaning that $>$ is a monomial ordering on* Mon_n.

Definition 1.2. *Let $>$ be a fixed monomial ordering. Write $f \in K[x]$, $f \neq 0$, in a unique way as a sum of non–zero terms*

$$f = a_\alpha x^\alpha + a_\beta x^\beta + \cdots + a_\gamma x^\gamma, \quad x^\alpha > x^\beta > \cdots > x^\gamma,$$

and $a_\alpha, a_\beta, \dots, a_\gamma \in K$. We define:

(1) $LM(f) :=$ `leadmonom(f)`$:= x^\alpha$, *the* leading monomial *of f,*
(2) $LE(f) :=$ `leadexp(f)`$:= \alpha$, *the* leading exponent *of f,*
(3) $LT(f) :=$ `lead(f)`$:= a_\alpha x^\alpha$, *the* leading term *or* head *of f,*
(4) $LC(f) :=$ `leadcoef(f)`$:= a_\alpha$, *the* leading coefficient *of f*

(5) *tail* $(f) := f -$ `lead(f)`$= a_\beta x^\beta + \cdots + a_\gamma x^\gamma$, *the* tail.
(6) *ecart*$(f) := \deg(f) - \deg(LM(f))$.

SINGULAR Example 1.

```
ring A = 0,(x,y,z),lp;
poly f = y4z3+2x2y2z2+3x5+4z4+5y2;
f;                           //display f in a lex-ordered way
//-> 3x5+2x2y2z2+y4z3+5y2+4z4
leadmonom(f);                //leading monomial
//-> x5
leadexp(f);                  //leading exponent
//-> 5,0,0
lead(f);                     //leading term
//-> 3x5
leadcoef(f);                 //leading coefficient
//-> 3
f - lead(f);                 //tail
//-> 2x2y2z2+y4z3+5y2+4z4
```

Definition 1.3. *Let* $>$ *be a monomial ordering on* $\{x^\alpha \mid \alpha \in \mathbf{N}^n\}$.

(1) $>$ *is called a* global ordering *if* $x^\alpha > 1$ *for all* $\alpha \neq (0,\ldots,0)$,
(2) $>$ *is called a* local ordering *if* $x^\alpha < 1$ *for all* $\alpha \neq (0,\ldots,0)$,
(3) $>$ *is called a* mixed ordering *if it is neither global nor local.*

Lemma 1.4. *Let* $>$ *be a monomial ordering, then the following conditions are equivalent:*

(1) $>$ *is a well–ordering.*
(2) $x_i > 1$ *for* $i = 1,\ldots,n$.
(3) $x^\alpha > 1$ *for all* $\alpha \neq (0,\ldots,0)$, *that is,* $>$ *is global.*

In the following examples we fix an enumeration $x_1,\ldots,x_n$ of the variables, any other enumeration leads to a different ordering.

%beginenumerate GLOBAL ORDERINGS

(i) *Lexicographical ordering* $>_{lp}$ (also denoted by lex):

$$x^\alpha >_{lp} x^\beta : :\Longleftrightarrow \exists\, 1 \le i \le n : \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i\,.$$

(ii) *Degree reverse lexicographical ordering* $>_{dp}$ (denoted by degrevlex):

$$\begin{aligned} x^\alpha >_{dp} x^\beta \quad :\Longleftrightarrow: \quad & \deg x^\alpha > \deg x^\beta \\ & \text{or} : \big(\deg x^\alpha = \deg x^\beta \text{ and } \exists\, 1 \le i \le n : \\ & \qquad \alpha_n = \beta_n, \ldots, \alpha_{i+1} = \beta_{i+1},\, \alpha_i < \beta_i\big)\,, \end{aligned}$$

where $\deg x^\alpha = \alpha_1 + \cdots + \alpha_n$.

%le i

LOCAL ORDERINGS

(i) *Negative lexicographical ordering* $>_{ls}$:

$$x^\alpha >_{ls} x^\beta : \; :\Longleftrightarrow \; \exists\, 1 \le i \le n, \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i \,.$$

(ii) *Negative degree reverse lexicographical ordering*:

$$\begin{aligned} x^\alpha >_{ds} x^\beta \quad :\Longleftrightarrow: \quad & \deg x^\alpha < \deg x^\beta, \text{ where } \deg x^\alpha = \alpha_1 + \cdots + \alpha_n, \\ & \text{or : } \big(\deg x^\alpha = \deg x^\beta \text{ and } \exists\, 1 \le i \le n : \\ & \qquad \alpha_n = \beta_n, \ldots, \alpha_{i+1} = \beta_{i+1},\, \alpha_i < \beta_i\big). \end{aligned}$$

Let $>$ be a monomial ordering on the set of monomials $\mathrm{Mon}(x_1, \ldots, x_n) = \{x^\alpha \mid \alpha \in \mathbf{N}^n\}$, and $K[x] = K[x_1, \ldots, x_n]$ the polynomial ring in n variables over a field K. Then the leading monomial function LM has the following properties for polynomials $f, g \in K[x] \setminus \{0\}$:

(1) $\mathrm{LM}(gf) = \mathrm{LM}(g)\mathrm{LM}(f)$.
(2) $\mathrm{LM}(g+f) \le \max\{\mathrm{LM}(g), \mathrm{LM}(f)\}$ with equality if and only if the leading terms of f and g do not cancel.

In particular, it follows that

$$S_> := \{u \in K[x] \setminus \{0\} \mid \mathrm{LM}(u) = 1\}$$

is a multiplicatively closed set.

Definition 1.5. *For any monomial ordering $>$ on* $\mathrm{Mon}(x_1, \ldots, x_n)$*, we define*

$$K[x]_> := S_>^{-1} K[x] = \left\{ \frac{f}{u} \,\middle|\, f, u \in K[x], LM(u) = 1 \right\},$$

the localization of $K[x]$ with respect to $S_>$ and call $K[x]_>$ the ring associated to $K[x]$ and $>$.

Note that $S_> = K^*$ if and only if $>$ is global and $S_> = K[x] \setminus \langle x_1, \ldots, x_n \rangle$ if and only if $>$ is local.

Definition 1.6. *Let $>$ be any monomial ordering:*

(1) *For $f \in K[x]_>$ choose $u \in K[x]$ such that $LT(u) = 1$ and $uf \in K[x]$. We define*

$$\begin{aligned} LM(f) &:= LM(uf), \\ LC(f) &:= LC(uf), \\ LT(f) &:= LT(uf), \\ LE(f) &:= LE(uf), \end{aligned}$$

and $tail(f) = f - LT(f)$.

(2) *For any subset $G \subset K[x]_>$ define the ideal*

$$L_>(G) := L(G) := \langle LM(g) \mid g \in G \setminus \{0\} \rangle_{K[x]} \,.$$

$L(G) \subset K[x]$ is called the leading ideal *of G.*

Definition 1.7. *Let $I \subset R = K[x]_>$ be an ideal.*

(1) *A finite set $G \subset R$ is called a* standard basis *of I if*

$$G \subset I, \text{ and } L(I) = L(G)\,.$$

That is, G is a standard basis, if the leading monomials of the elements of G generate the leading ideal of I, or, in other words, if for any $f \in I \setminus \{0\}$ there exists a $g \in G$ satisfying $LM(g) \mid LM(f)$.

(2) *If $>$ is global, a standard basis is also called a* Gröbner basis.

(3) *If we just say that G is a standard basis, we mean that G is a standard basis of the ideal $\langle G \rangle_R$ generated by G.*

Standard bases can be characterized using the notion of the normal form. We need the following definitions:

Definition 1.8. *Let $f, g \in R \setminus \{0\}$ with $LM(f) = x^\alpha$ and $LM(g) = x^\beta$, respectively. Set*

$$\gamma := \operatorname{lcm}(\alpha, \beta) := \big(\max(\alpha_1, \beta_1), \ldots, \max(\alpha_n, \beta_n)\big)$$

and let $\operatorname{lcm}(x^\alpha, x^\beta) := x^\gamma$ *be the least common multiple of x^α and x^β. We define the s–*polynomial *(*spoly*, for short) of f and g to be*

$$spoly(f, g) := x^{\gamma-\alpha} f - \frac{LC(f)}{LC(g)} \cdot x^{\gamma-\beta} g\,.$$

If $LM(g)$ divides $LM(f)$, say $LM(g) = x^\beta$, $LM(f) = x^\alpha$, then the s–polynomial is particularly simple,

$$spoly(f, g) = f - \frac{LC(f)}{LC(g)} \cdot x^{\alpha-\beta} g\,,$$

and $LM\big(spoly(f, g)\big) < LM(f)$.

Definition 1.9. *Let $\mathcal{G}$ denote the set of all finite lists $G \subset R = K[x]_>$.*

$$NF : R \times \mathcal{G} \to R,\ (f, G) \mapsto NF(f \mid G)\,,$$

is called a normal form on R if, for all $G \in \mathcal{G}$,

(0) $NF(0 \mid G) = 0$,

and, for all $f \in R$ and $G \in \mathcal{G}$,

(1) $NF(f \mid G) \neq 0 \Longrightarrow LM\big(NF(f \mid G)\big) \notin L(G)$.

(2) *If $G = \{g_1, \ldots, g_s\}$, then f has a* standard representation *with respect to $NF(- \mid G)$, that is, there exists a unit $u \in R^*$ such that*

$$uf - NF(f \mid G) = \sum_{i=1}^{s} a_i g_i,\ \ a_i \in R,\ \ s \geq 0\,,$$

satisfying $LM(\Sigma_{i=1}^{s} a_i g_i) \geq LM(a_i g_i)$ for all i such that $a_i g_i \neq 0$.

The existence of a normal form is given by the following algorithm:

Algorithm 1.10. NF$(f \mid G)$

Let $>$ be any monomial ordering.

Input: $f \in K[x]$, *G a finite list in $K[x]$*

Output: $h \in K[x]$ *a polynomial normal form of f with respect to G.*

- $h := f$;
- $T := G$;
- *while*$(h \neq 0$ *and* $T_h := \{g \in T \mid LM(g) \mid LM(h)\} \neq \emptyset)$
  choose $g \in T_h$ *with* $ecart(g)$ *minimal;*
  if $(ecart(g) > ecart(h))$
  $T := T \cup \{h\}$;
  $h := spoly(h, g)$;
- *return h;*

Theorem 1.11. *Let $I \subset R$ be an ideal and $G = \{g_1, \ldots, g_s\} \subset I$. Then the following are equivalent:*

(1) *G is a standard basis of I.*
(2) *$NF(f \mid G) = 0$ if and only if $f \in I$.*

We will explain now how to use standard bases to solve problems in algebra.

Ideal membership

Problem: Given $f, f_1, \ldots, f_k \in K[x]$, and let $I = \langle f_1, \ldots, f_k \rangle_R$. We wish to decide whether $f \in I$, or not.

Solution: We choose any monomial ordering $>$ such that $K[x]_> = R$ and compute a standard basis $G = \{g_1, \ldots, g_s\}$ of I with respect to $>$. $f \in I$ if and only if $\mathrm{NF}(f \mid G) = 0$.

SINGULAR Example 2.

```
ring A  = 0,(x,y),dp;
ideal I = x10+x9y2,y8-x2y7;
ideal J = std(I);
poly f  = x2y7+y14;
reduce(f,J,1);        //3rd parameter 1 avoids tail reduction
//-> -xy12+x2y7        //f is not in I
     f  = xy13+y12;
reduce(f,J,1);
//-> 0                //f is in I
```

Intersection with Subrings (Elimination of variables)

Problem: Given $f_1, \ldots, f_k \in K[x] = K[x_1, \ldots, x_n]$, $I = \langle f_1, \ldots, f_k \rangle_{K[x]}$, we should like to find generators of the ideal

$$I' = I \cap K[x_{s+1}, \ldots, x_n], \quad s < n\,.$$

Elements of the ideal I' are said to be obtained from $f_1, \ldots, f_k$ *by eliminating* $x_1, \ldots, x_s$. The following lemma is the basis for solving the elimination problem.

Lemma 1.12. *Let $>$ be an elimination ordering for $x_1,\dots,x_s$ on the set of monomials* $\mathrm{Mon}(x_1,\dots,x_n)$*, and let $I \subset K[x_1,\dots,x_n]_>$ be an ideal. If $S = \{g_1,\dots,g_k\}$ is a standard basis of I, then*

$$S' := \{g \in S \mid LM(g) \in K[x_{s+1},\dots,x_n]\}$$

is a standard basis of $I' := I \cap K[x_{s+1},\dots,x_n]_{>'}$. In particular, S' generates the ideal I'.

SINGULAR Example 3.

```
ring A =0,(t,x,y,z),dp;
ideal I=t2+x2+y2+z2,t2+2x2-xy-z2,t+y3-z3;

eliminate(I,t);
//-> _[1]=x2-xy-y2-2z2        _[2]=y6-2y3z3+z6+2x2-xy-z2
```

Alternatively choose a product ordering:

```
ring A1=0,(t,x,y,z),(dp(1),dp(3));
ideal I=imap(A,I);
ideal J=std(I);
J;
//-> J[1]=x2-xy-y2-2z2     J[2]=y6-2y3z3+z6+2x2-xy-z2
//-> J[3]=t+y3-z3
```

Radical Membership

Problem: Let $f_1,\dots,f_k \in K[x]_>$, $>$ a monomial ordering on $\mathrm{Mon}(x_1,\dots,x_n)$ and $I = \langle f_1,\dots,f_k\rangle_{K[x]_>}$. Given some $f \in K[x]_>$ we want to decide whether $f \in \sqrt{I}$. The following lemma, which is sometimes called *Rabinowich's trick*, is the basis for solving this problem. [1]

Lemma 1.13. *Let A be a ring, $I \subset A$ an ideal and $f \in A$. Then*

$$f \in \sqrt{I} :\iff: 1 \in \tilde{I} := \langle I, 1-tf\rangle_{A[t]}$$

where t is an additional new variable.

SINGULAR Example 4.

```
ring A =0,(x,y,z),dp;
ideal I=x5,xy3,y7,z3+xyz;
poly f =x+y+z;

ring B =0,(t,x,y,z),dp; //need t for radical test
ideal I=imap(A,I);
poly f =imap(A,f);
I=I,1-t*f;
std(I);
//-> _[1]=1              //f is in the radical
```

[1] We can even compute the full radical $\sqrt{I}$, but this is a much harder computation.

```
LIB"primdec.lib";   //just to see, we compute the radical
setring A;
radical(I);
//-> _[1]=z   _[2]=y     _[3]=x
```

Intersection of Ideals

Problem: Given $f_1,\dots,f_k,h_1,\dots,h_r \in K[x]$ and $>$ a monomial ordering. Let $I_1 = \langle f_1,\dots,f_k\rangle K[x]_>$ and $I_2 = \langle h_1,\dots,h_r\rangle K[x]_>$. We wish to find generators for $I_1 \cap I_2$.

Consider the ideal $J := \langle tf_1,\dots,tf_k,(1-t)h_1,\dots,(1-t)h_r\rangle (K[x]_>)[t]$.

Lemma 1.14. *With the above notations,* $I_1 \cap I_2 = J \cap K[x]_>$.

SINGULAR Example 5.

```
ring A=0,(x,y,z),dp;
ideal I1=x,y;
ideal I2=y2,z;
intersect(I1,I2);         //the built-in SINGULAR command
//-> _[1]=y2    _[2]=yz    _[3]=xz

ring B=0,(t,x,y,z),dp;   //the way described above
ideal I1=imap(A,I1);
ideal I2=imap(A,I2);
ideal J=t*I1+(1-t)*I2;
eliminate(J,t);
//-> _[1]=yz  _[2]=xz   _[3]=y2
```

Quotient of Ideals

Problem: Let I_1 and $I_2 \subset K[x]_>$. We want to compute

$$I_1 : I_2 = \{g \in K[x]_> \mid gI_2 \subset I_1\}\,.$$

Since, obviously, $I_1 : \langle h_1,\dots,h_r\rangle = \bigcap_{i=1}^r (I_1 : \langle h_i\rangle)$, we can compute $I_1 : \langle h_i\rangle$ for each i. The next lemma shows a way to compute $I_1 : \langle h_i\rangle$.

Lemma 1.15. *Let $I \subset K[x]_>$ be an ideal, and let $h \in K[x]_>$, $h \neq 0$. Moreover, let $I \cap \langle h\rangle = \langle g_1\cdot h,\dots,g_s\cdot h\rangle$. Then $I : \langle h\rangle = \langle g_1,\dots,g_s\rangle_{K[x]_>}$.*

SINGULAR Example 6.

```
ring A=0,(x,y,z),dp;
ideal I1=x,y;
ideal I2=y2,z;
quotient(I1,I2);         //the built-in SINGULAR command
//-> _[1]=y    _[2]=x
```

Kernel of a Ring Map

Let $\varphi : R_1 := (K[x]_{>_1})/I \to (K[y]_{>_2})/J =: R_2$ be a ring map defined by polynomials $\varphi(x_i) = f_i \in K[y] = K[y_1,\dots,y_m]$ for $i = 1,\dots,n$ (and assume that the monomial orderings satisfy $1 >_2 \mathrm{LM}(f_i)$ if $1 >_1 x_i$.

Define $J_0 := J \cap K[y]$, and $I_0 := I \cap K[x]$. Then φ is induced by

$$\tilde{\varphi} : K[x]/I_0 \to K[y]/J_0\,, \quad x_i \mapsto f_i\,,$$

and we have a commutative diagram

$$\begin{tikzcd} K[x]/I_0 \arrow[r, "\tilde{\varphi}"] \arrow[d, hook] & K[y]/J_0 \arrow[d, hook] \\ R_1 \arrow[r, "\varphi"] & R_2\,. \end{tikzcd}$$

Problem: Let I, J and φ be as above. Compute generators for $\mathrm{Ker}(\varphi)$.

Solution: Assume that $J_0 = \langle g_1,\dots,g_s\rangle_{K[y]}$ and $I_0 = \langle h_1,\dots,h_t\rangle_{K[x]}$.
Set $H := \langle h_1,\dots,h_t, g_1,\dots,g_s, x_1 - f_1,\dots,x_n - f_n\rangle \subset K[x,y]$, and compute $H' := H \cap K[x]$ by eliminating $y_1,\dots,y_m$ from H. Then H' generates $\mathrm{Ker}(\varphi)$ by the following lemma.

Lemma 1.16. *With the above notations,* $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\tilde{\varphi})R_1$ *and*

$$\mathrm{Ker}(\tilde{\varphi}) = \big(I_0 + \langle g_1,\dots,g_s, x_1 - f_1,\dots,x_n - f_n\rangle_{K[x,y]} \cap K[x]\big) \bmod I_0\,.$$

In particular, if $>_1$ *is global, then* $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\tilde{\varphi})$.

SINGULAR Example 7.

```
ring A=0,(x,y,z),dp;
ring B=0,(a,b),dp;
map phi=A,a2,ab,b2;
ideal zero;                //compute the preimage of 0
setring A;
preimage(B,phi,zero);      //the built-in SINGULAR command
//-> _[1]=y2-xz

ring C=0,(x,y,z,a,b), dp; //the method described above
ideal H=x-a2, y-ab, z-b2;
eliminate(H,ab);
//-> _[1]=y2-xz
```

2. Lecture: Polynomial solving and primary decomposition

Solvability of Polynomial Equations

Problem: Given $f_1, \dots, f_k \in K[x_1, \dots, x_n]$, we want to assure whether the system of polynomial equations

$$f_1(x) = \cdots = f_k(x) = 0$$

has a solution in $\overline{K}^n$, where $\overline{K}$ is the algebraic closure of K.

Let $I = \langle f_1, \dots, f_k \rangle_{K[x]}$, then the question is whether the algebraic set $V(I) \subset \overline{K}^n$ is empty or not.

Solution: By Hilbert's Nullstellensatz, $V(I) = \emptyset$ if and only if $1 \in I$. We compute a Gröbner basis G of I with respect to any global ordering on $\mathrm{Mon}(x_1, \dots, x_n)$ and normalize it (that is, divide every $g \in G$ by $\mathrm{LC}(g)$). Since $1 \in I$ if and only if $1 \in L(I)$, we have $V(I) = \emptyset$ if and only if 1 is an element of a normalized Gröbner basis of I. Of course, we can avoid normalizing, which is expensive in rings with parameters. Since $1 \in I$ if and only if G contains a non–zero constant polynomial, we have only to look for an element of degree 0 in G.

SINGULAR Example 8.

```
ring  A=0,(x,y,z),lp;
ideal I=x2+y+z-1,
        x+y2+z-1,
        x+y+z2-1;
ideal J=groebner(I);    //the lexicographical Groebner basis
J;
//-> J[1]=z6-4z4+4z3-z2       J[2]=2yz2+z4-z2
//-> J[3]=y2-y-z2+z           J[4]=x+y+z2-1
```

We use the multivariate solver based on triangular sets.

```
LIB"solve.lib";
list s1=solve(I,6);
//-> // name of new current ring: AC
s1;
//-> [1]:          [2]:      [3]:             [4]:      [5]:
//->    [1]:          [1]:      [1]:             [1]:      [1]:
//->        0.414214     0         -2.414214       1         0
//->    [2]:          [2]:      [2]:             [2]:      [2]:
//->        0.414214     0         -2.414214       0         1
//->    [3]:          [3]:      [3]:             [3]:      [3]:
//->        0.414214     1         -2.414214       0         0
```

If we want to compute the zeros with multiplicities then we use 1 as a third parameter for the command:

```
setring A;
list s2=solve(I,6,1);
s2;
//-> [1]:                      [2]:
//->     [1]:                      [1]:
//->         [1]:                      [1]:
//->             [1]:                      [1]:
//->                 -2.414214                 0
//->             [2]:                      [2]:
//->                 -2.414214                 1
//->             [3]:                      [3]:
//->                 -2.414214                 0
//->         [2]:                      [2]:
//->             [1]:                      [1]:
//->                 0.414214                  1
//->             [2]:                      [2]:
//->                 0.414214                  0
//->             [3]:                      [3]:
//->                 0.414214                  0
//->     [2]:                          [3]:
//->         1                             [1]:
//->                                           0
//->                                       [2]:
//->                                           0
//->                                       [3]:
//->                                           1
//->                                   [2]:
//->                                       2
```

The output has to be interpreted as follows: there are two zeros of multiplicity 1 and three zeros ($(0,1,0)$, $(1,0,0)$, $(0,0,1)$) of multiplicity 2.

Definition 2.1.

(1) *A maximal ideal $M \subset K[x_1,\dots,x_n]$ is called in* general position *with respect to the lexicographical ordering with $x_1 > \cdots > x_n$, if there exist $g_1,\dots,g_n \in K[x_n]$ with $M = \langle x_1 + g_1(x_n),\dots,x_{n-1}+g_{n-1}(x_n),\ g_n(x_n)\rangle$.*

(2) *A zero–dimensional ideal $I \subset K[x_1,\dots,x_n]$ is called in* general position *with respect to the lexicographical ordering with $x_1 > \cdots > x_n$, if all associated primes $P_1,\dots,P_k$ are in general position and if $P_i \cap K[x_n] \neq P_j \cap K[x_n]$ for $i \neq j$.*

Proposition 2.2. *Let K be a field of characteristic 0, and let $I \subset K[x]$, $x = (x_1,\dots,x_n)$, be a zero–dimensional ideal. Then there exists a non–empty, Zariski open subset $U \subset K^{n-1}$*

such that for all $\underline{a} = (a_1, \dots, a_{n-1}) \in U$, the coordinate change $\varphi_{\underline{a}} : K[x] \to K[x]$ defined by $\varphi_{\underline{a}}(x_i) = x_i$ if $i < n$, and

$$\varphi_{\underline{a}}(x_n) = x_n + \sum_{i=1}^{n-1} a_i x_i$$

has the property that $\varphi_{\underline{a}}(I)$ is in general position with respect to the lexicographical ordering defined by $x_1 > \cdots > x_n$.

Proposition 2.3. *Let $I \subset K[x_1, \dots, x_n]$ be a zero–dimensional ideal. Let $\langle g \rangle = I \cap K[x_n]$, $g = g_1^{\nu_1} \dots g_s^{\nu_s}$, g_i monic and prime and $g_i \neq g_j$ for $i \neq j$. Then*

(1) $I = \bigcap_{i=1}^{s} \langle I, g_i^{\nu_i} \rangle$.

If I is in general position with respect to the lexicographical ordering with $x_1 > \cdots > x_n$, then

(2) $\langle I, g_i^{\nu_i} \rangle$ *is a primary ideal for all i.*

SINGULAR Example 9 (zero–dim primary decomposition)**.**
We give an example for a zero-dimensional primary decomposition.

```
option(redSB);
ring R=0,(x,y),lp;
ideal I=(y2-1)^2,x2-(y+1)^3;
```

The ideal I is not in general position with respect to `lp`, since the minimal associated prime $\langle x^2 - 8, y - 1 \rangle$ is not.

```
map phi=R,x,x+y;     //we choose a generic coordinate change
map psi=R,x,-x+y;    //and the inverse map
I=std(phi(I));
I;
//-> I[1]=y7-y6-19y5-13y4+99y3+221y2+175y+49
//-> I[2]=112xy+112x-27y6+64y5+431y4-264y3-2277y2-2520y-847
//-> I[3]=56x2+65y6-159y5-1014y4+662y3+5505y2+6153y+2100
factorize(I[1]);
//-> [1]:
//->    _[1]=1
//->    _[2]=y2-2y-7
//->    _[3]=y+1
//-> [2]:
//->    1,2,3

ideal Q1=std(I,(y2-2y-7)^2); //the candidates for the
                             //primary ideals
ideal Q2=std(I,(y+1)^3);     //in general position
Q1; Q2;

//-> Q1[1]=y4-4y3-10y2+28y+49   Q2[1]=y3+3y2+3y+1
```

```
//-> Q1[2]=56x+y3-9y2+63y-7      Q2[2]=2xy+2x+y2+2y+1
                                 Q2[3]=x2

factorize(Q1[1]);   //primary and general position test
                    //for Q1
//-> [1]:
//->    _[1]=1
//->    _[2]=y2-2y-7
//-> [2]:
//->    1,2

factorize(Q2[1]);   //primary and general position test
                    //for Q2
//-> [1]:
//->    _[1]=1
//->    _[2]=y+1
//-> [2]:
//->    1,3
```

Both ideals are primary and in general position.

```
Q1=std(psi(Q1));      //the inverse coordinate change
Q2=std(psi(Q2));      //the result
Q1; Q2;

//-> Q1[1]=y2-2y+1     Q2[1]=y2+2y+1
//-> Q1[2]=x2-12y+4    Q2[2]=x2
```

We obtain that I is the intersection of the primary ideals Q_1 and Q_2 with associated prime ideals $\langle y-1, x^2-8\rangle$ and $\langle y+1, x\rangle$.

The following proposition reduces the higher dimensional case to the zero-dimensional case:

Proposition 2.4. *Let $I \subset K[x]$ be an ideal and $u \subset x = \{x_1, \dots, x_n\}$ be a maximal independent set of variables[2] with respect to I.*

(1) *$IK(u)[x \smallsetminus u] \subset K(u)[x \smallsetminus u]$ is a zero–dimensional ideal.*
(2) *Let $S = \{g_1, \dots, g_s\} \subset I \subset K[x]$ be a Gröbner basis of $IK(u)[x \smallsetminus u]$, and let $h := \operatorname{lcm}\big(LC(g_1), \dots, LC(g_s)\big) \in K[u]$, then*

$$IK(u)[x \smallsetminus u] \cap K[x] = I : \langle h^\infty \rangle,$$

and this ideal is equidimensional of dimension $\dim(I)$.

[2] It is maximal such that $I \cap K[u] = \langle 0 \rangle$.

(3) *Let* $IK(u)[x \smallsetminus u] = Q_1 \cap \cdots \cap Q_s$ *be an irredundant primary decomposition, then also* $IK(u)[x \smallsetminus u] \cap K[x] = (Q_1 \cap K[x]) \cap \cdots \cap (Q_s \cap K[x])$ *is an irredundant primary decomposition.*

Finally we explain how to compute the radical.

Proposition 2.5. *Let* $I \subset K[x_1,\dots,x_n]$ *be a zero–dimensional ideal and* $I \cap K[x_i] = \langle f_i \rangle$ *for* $i = 1,\dots,n$. *Moreover, let* g_i *be the squarefree part of* f_i, *then* $\sqrt{I} = I + \langle g_1,\dots,g_n \rangle$.

The higher dimensional case can be reduced similarly to the primary decomposition to the zero-dimensional case.

3. Lecture: Invariants

The computation of the Hilbert function will be discussed and explained. Let K be a field.

Definition 3.1. *Let* $A = \bigoplus_{\nu \geq 0} A_\nu$ *be a Noetherian graded K–algebra, and let* $M = \bigoplus_{\nu \in \mathbb{Z}} M_\nu$ *be a finitely generated graded A–module. The* Hilbert function $H_M : \mathbb{Z} \to \mathbb{Z}$ *of M is defined by*

$$H_M(n) := \dim_K(M_n),$$

and the Hilbert–Poincaré series HP_M *of M is defined by*

$$HP_M(t) := \sum_{\nu \in \mathbb{Z}} H_M(\nu) \cdot t^\nu \in \mathbb{Z}[[t]][t^{-1}].$$

Theorem 3.2. *Let* $A = \bigoplus_{\nu \geq 0} A_\nu$ *be a graded K–algebra, and assume that A is generated, as K–algebra, by* $x_1,\dots,x_r \in A_1$. *Then, for any finitely generated (positively) graded A–module* $M = \bigoplus_{\nu \geq 0} M_\nu$,

$$HP_M(t) = \frac{Q(t)}{(1-t)^r} \text{ for some } Q(t) \in \mathbb{Z}[t].$$

Note that SINGULAR has a command which computes the numerator $Q(t)$ for the Hilbert–Poincaré series:

SINGULAR Example 10.

```
ring A=0,(t,x,y,z),dp;
ideal I=x5y2,x3,y3,xy4,xy7;
intvec v = hilb(std(I),1);
v;
//-> 1,0,0,-2,0,0,1,0
```

We obtain $Q(t) = t^6 - 2t^3 + 1$.

The latter output has to be interpreted as follows: if $\mathtt{v} = (v_0,\dots,v_d,0)$ then $Q(t) = \sum_{i=0}^{d} v_i t^i$.

Theorem 3.3. *Let $>$ be any monomial ordering on $K[x] := K[x_1,\dots,x_r]$, and let $I \subset K[x]$ be a homogeneous ideal. Then*

$$HP_{K[x]/I}(t) = HP_{K[x]/L(I)}(t)\,,$$

where $L(I)$ is the leading ideal of I with respect to $>$.

Examples how to compute the Hilbert polynomial, the Hilbert–Samuel function, the degree respectively and the multiplicity and the dimension of an ideal can be found in [7]. As above all computations are reduced to compute the corresponding invariants for the leading ideal.

4. Lecture: Homological Algebra

Here we will show different approaches how to test Cohen–Macaulayness using SINGULAR. More details about the underlying theory can be found in [7].

SINGULAR Example 11 (first test for Cohen–Macaulayness)**.**
Let $(A,\mathfrak{m})$ be a local ring, $\mathfrak{m} = \langle x_1,\dots,x_n\rangle$. Let M be an A–module given by a presentation $A^\ell \to A^s \to M \to 0$. To check whether M is Cohen–Macaulay we use that the equality

$$\begin{aligned}\dim\big(A/\operatorname{Ann}(M)\big) = \dim(M) &= \operatorname{depth}(M)\\ &= n - \sup\{i \mid H_i(x_1,\dots,x_n,M) \neq 0\}\,.\end{aligned}$$

is necessary and sufficient for M to be Cohen–Macaulay. The following procedure computes $\operatorname{depth}(\mathfrak{m},M)$*, where $\mathfrak{m} = \langle x_1,\dots,x_n\rangle \subset A = K[x_1,\dots,x_n]_>$ and M is a finitely generated A–module with $\mathfrak{m}M \neq M$.*

The following procedures use the procedures Koszul Homology from `homolog.lib` and Ann from `primdec.lib` to compute the Koszul Homology $H_i(x_1,\dots,x_n,M)$ and the annihilator $\operatorname{Ann}(M)$. They have to be loaded first.

```
LIB "homolog.lib";
proc depth(module M)
{
   ideal m=maxideal(1);
   int n=size(m);
   int i;
   while(i<n)
   {
     i++;
     if(size(KoszulHomology(m,M,i))==0){return(n-i+1);}
   }
   return(0);
}
```

Now the test for Cohen–Macaulayness is easy.

```
LIB "primdec.lib";
proc CohenMacaulayTest(module M)
{
  return(depth(M)==dim(std(Ann(M))));
}
```
The procedure returns 1 if M is Cohen–Macaulay and 0 if not.

As an application, we check that a complete intersection is Cohen–Macaulay and that $K[x,y,z]_{\langle x,y,z\rangle}/\langle xz,yz,z^2\rangle$ is not Cohen–Macaulay.

```
ring R=0,(x,y,z),ds;
ideal I=xz,yz,z2;
module M=I*freemodule(1);
CohenMacaulayTest(M);
//-> 0

I=x2+y2,z7;
M=I*freemodule(1);
CohenMacaulayTest(M);
//-> 1
```

SINGULAR Example 12 (second test for Cohen–Macaulayness)**.**

Let $A = K[x_1,\dots,x_n]_{\langle x_1,\dots,x_n\rangle}/I$. Using Noether normalization, we may assume that $A \supset K[x_{s+1},\dots,x_n]_{\langle x_{s+1},\dots,x_n\rangle} =: B$ is finite. We choose a monomial basis $m_1,\dots,m_r \in K[x_1,\dots,x_s]$ of $A\big|_{x_{s+1}=\dots=x_n=0}$.

Then $m_1,\dots,m_r$ is a minimal system of generators of A as B–module. A is Cohen–Macaulay if and only if A is a free B–module, that is, there are no B–relations between $m_1,\dots,m_r$, in other words, $syz_A(m_1,\dots,m_r) \cap B^r = \langle 0\rangle$. This test can be implemented in SINGULAR *as follows:*

```
proc isCohenMacaulay(ideal I)
{
   def A    = basering;
   list L   = noetherNormal(I);
   map phi  = A,L[1];
   I        = phi(I);
   int s    = nvars(basering)-size(L[2]);
   execute("ring B=("+charstr(A)+"),x(1..s),ds;");
   ideal m  = maxideal(1);
   map psi  = A,m;
   ideal J  = std(psi(I));
   ideal K  = kbase(J);
   setring A;
   execute("
     ring C=("+charstr(A)+"),("+varstr(A)+"),(dp(s),ds);");
```

```
   ideal I  = imap(A,I);
   qring D  = std(I);
   ideal K  = fetch(B,K);
   module N = std(syz(K));
   intvec v = leadexp(N[size(N)]);
   int i=1;
   while((i<s)&&(v[i]==0)){i++;}
   setring A;
   if(!v[i]){return(0);}
   return(1);
}
```

As the above procedure uses `noetherNormal` from `algebra.lib`, we first have to load this library.

```
LIB"algebra.lib";
ring r=0,(x,y,z),ds;
ideal I=xz,yz;
isCohenMacaulay(I);
//-> 0

I=x2-y3;
isCohenMacaulay(I);
//-> 1
```

SINGULAR Example 13 (3rd test for Cohen–Macaulayness)**.**
We use the Auslander–Buchsbaum formula to compute the depth of M and then check if $\mathrm{depth}(M) = \dim(M) = \dim\big(A/\operatorname{Ann}(M)\big)$.

We assume that $A = K[x_1,\ldots,x_n]_{\langle x_1,\ldots,x_n\rangle}/I$ *and compute a minimal free resolution. Then* $\mathrm{depth}(A) = n - \mathrm{pd}_{K[x_1,\ldots,x_n]_{\langle x_1,\ldots,x_n\rangle}}(A)$*. If M is a finitely generated A–module of finite projective dimension, then we compute a minimal free resolution of M and obtain* $\mathrm{depth}(M) = \mathrm{depth}(A) - \mathrm{pd}_A(M)$.

```
proc projdim(module M)
{
   list l=mres(M,0);            //compute the resolution
   int i;
   while(i<size(l))
   {
     i++;
     if(size(l[i])==0){return(i-1);}
   }
}
```

Now it is easy to give another test for Cohen–Macaulayness.

```
proc isCohenMacaulay1(ideal I)
```

```
{
  int de=nvars(basering)-projdim(I*freemodule(1));
  int di=dim(std(I));
  return(de==di);
}

ring R=0,(x,y,z),ds;
ideal I=xz,yz;
isCohenMacaulay1(I);
//-> 0

I=x2-y3;
isCohenMacaulay1(I);
//-> 1

I=xz,yz,xy;
isCohenMacaulay1(I);
//-> 1
kill R;
```

The following procedure checks whether the depth of M is equal to d. It uses the procedure `Ann` from `primdec.lib`.

```
proc CohenMacaulayTest1(module M, int d)
{
  return((d-projdim(M))==dim(std(Ann(M))));
}

LIB"primdec.lib";
ring R=0,(x,y,z),ds;
ideal I=xz,yz;
module M=I*freemodule(1);
CohenMacaulayTest1(M,3);
//-> 0

I=x2+y2,z7;
M=I*freemodule(1);
CohenMacaulayTest1(M,3);
//-> 1
```

References

[1] Cox, D.; Little, J.; O'Shea, D.: Ideals, Varieties and Algorithms. Springer (1992).

[2] Decker, W.; Lossen, Chr.: Computing in Algebraic Geometry; A quick start using SINGULAR. Springer, (2006).

[3] Decker, W.; Greuel, G.-M.; Pfister, P.: Primary Decomposition: Algorithms and Comparisons. In: Algorithmic Algebra and Number Theory, Springer, 187–220 (1998).

[4] Decker, W.; Greuel, G.-M.; de Jong, T.; Pfister, G.: The Normalization: a new Algorithm, Implementation and Comparisons. In: Proceedings EUROCONFERENCE Computational Methods for Representations of Groups and Algebras (1.4. – 5.4.1997), Birkhäuser, 177–185 (1999).
[5] Dickenstein, A.; Emiris, I.Z.: Solving Polynomial Equations; Foundations, Algorithms, and Applications. Algorithms and Computations in Mathematics, Vol. 41, Springer, (2005).
[6] Eisenbud, D.; Grayson, D.; Stillman, M., Sturmfels, B.: Computations in Algebraic Geometry with Macaulay2. Springer, (2001).
[7] Greuel, G.-M.; Pfister G.: A Singular Introduction to Commutative Algebra. Springer 2008.
[8] Greuel, G.-M.; Pfister, G.: SINGULAR and Applications, Jahresbericht der DMV 108 (4), 167-196, (2006).
[9] Kreuzer, M.; Robbiano, L.: Computational Commutative Algebra 1. Springer (2000).
[10] Vasconcelos, W.V.: Computational Methods in Commutative Algebra and Algebraic Geometry. Springer (1998).

Computer Algebra Systems

[11] ASIR (Noro, M.; Shimoyama, T.; Takeshima, T.): `http://www.asir.org/`.
[12] CoCoA (Robbiano, L.): A System for Computation in Algebraic Geometry and Commutative Algebra. Available from `cocoa.dima.unige.it/cocoa`
[13] Macaulay 2 (Grayson, D.; Stillman, M.): A Computer Software System Designed to Support Research in Commutative Algebra and Algebraic Geometry. Available from `http://math.uiuc.edu/Macaulay2`.
[14] SINGULAR (Greuel, G.-M.; Pfister, G.; Schönemann, H.): A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern, free software under the GNU General Public Licence (1990-2007).
`http://www.singular.uni-kl.de`.