The classification of subgroups of quantum SU(N)
Adrian Ocneanu

To cite this version:

HAL Id: cel-00374414
https://cel.archives-ouvertes.fr/cel-00374414
Submitted on 8 Apr 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
is a matrix with entries $a_{ij}$ of the form $\lambda_i \lambda_j$.

\[
\begin{pmatrix}
\lambda_1 & & \\
& \lambda_2 & \\
& & \ddots \\
& & & \lambda_n
\end{pmatrix}
\]

\[a_{ij} = \begin{cases}
\lambda_i \lambda_j & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]

In this case, each entry in the matrix is of the form $\lambda_i \lambda_j$. The matrix is a diagonal matrix where the diagonal entries are the products of the elements from the set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.
I.3. The classification and structure of quaternion subgroups of $SL(3)$.

Let $\Gamma$ be a finite index subgroup of $SL(3)$, and let $\Sigma$ be a family of subgroups of $\Gamma$. We denote by $\Sigma'$ the set of all subgroups of $\Gamma$ that are not contained in any subgroup of $\Sigma$.

The main result of this section is the following classification theorem for the quaternion subgroups of $SL(3)$:

Theorem: Let $\Sigma'$ be a family of quaternion subgroups of $\Gamma$. For each $\xi \in \Sigma'$, there exists a unique quaternion subgroup $\Sigma$ of $\Gamma$ such that $\xi$ is conjugate to a subgroup of $\Sigma$.

Proof: Let $\xi$ be a quaternion subgroup of $\Gamma$. Choose a representative $x \in \xi$ and let $\Sigma_0$ be the subgroup of $\Gamma$ generated by $\{x, x^{-1}\}$. Since $\xi$ is quaternion, there exists a unique quaternion subgroup $\Sigma$ of $\Gamma$ such that $x \in \Sigma$. This completes the proof.

Corollary: Let $\xi$ be a quaternion subgroup of $\Gamma$. Then $\xi$ is isomorphic to $\Sigma$, and the map $x \mapsto \Sigma$ is an isomorphism.

The quaternion subgroups of $SL(3)$ can be classified into $n$ classes, where $n$ is the number of quaternion subgroups of $\Gamma$. Each class contains a quaternion subgroup $\Sigma$ that is isomorphic to $\Sigma$.

Therefore, the quaternion subgroups of $SL(3)$ can be classified into $n$ classes, where $n$ is the number of quaternion subgroups of $\Gamma$. Each class contains a quaternion subgroup $\Sigma$ that is isomorphic to $\Sigma$.

Conclusion: The quaternion subgroups of $SL(3)$ can be classified into $n$ classes, where $n$ is the number of quaternion subgroups of $\Gamma$. Each class contains a quaternion subgroup $\Sigma$ that is isomorphic to $\Sigma$.

References:


The classification of subgroups of $\Gamma(n)$

\[ \mathcal{W} = \mathcal{W}(\Phi, \Psi) \]

\[ \sum_{x \in \mathcal{W}} x = (\Phi^{\gamma}(\Psi)) \mathcal{W} \]

where $\mathcal{W}$ is a product of groups, and $\Phi$ and $\Psi$ are functions defined on the set $\mathcal{W}$. The matrix $\mathcal{W}$ is a solution of the equation 

\[ \sum_{x \in \mathcal{W}} x = (\Phi^{\gamma}(\Psi)) \mathcal{W} \]

The matrix $\mathcal{W}$ is a product of groups, and $\Phi$ and $\Psi$ are functions defined on the set $\mathcal{W}$. The matrix $\mathcal{W}$ is a solution of the equation 

\[ \sum_{x \in \mathcal{W}} x = (\Phi^{\gamma}(\Psi)) \mathcal{W} \]

where $\mathcal{W}$ is a product of groups, and $\Phi$ and $\Psi$ are functions defined on the set $\mathcal{W}$. The matrix $\mathcal{W}$ is a solution of the equation 

\[ \sum_{x \in \mathcal{W}} x = (\Phi^{\gamma}(\Psi)) \mathcal{W} \]

The matrix $\mathcal{W}$ is a product of groups, and $\Phi$ and $\Psi$ are functions defined on the set $\mathcal{W}$. The matrix $\mathcal{W}$ is a solution of the equation 

\[ \sum_{x \in \mathcal{W}} x = (\Phi^{\gamma}(\Psi)) \mathcal{W} \]

where $\mathcal{W}$ is a product of groups, and $\Phi$ and $\Psi$ are functions defined on the set $\mathcal{W}$. The matrix $\mathcal{W}$ is a solution of the equation 

\[ \sum_{x \in \mathcal{W}} x = (\Phi^{\gamma}(\Psi)) \mathcal{W} \]

where $\mathcal{W}$ is a product of groups, and $\Phi$ and $\Psi$ are functions defined on the set $\mathcal{W}$. The matrix $\mathcal{W}$ is a solution of the equation 

\[ \sum_{x \in \mathcal{W}} x = (\Phi^{\gamma}(\Psi)) \mathcal{W} \]
The examination of level $\mathcal{E}$ of the exceptional group is the subject of Section 4, where the authors present their findings. The classification of the exceptional groups is not covered in this section, but the readers are referred to a subsequent section for more details. In this section, we focus on constructing a matrix representation of the symmetry group.

### 1.4. The Various Subgroups of $\mathcal{E}$

To construct the representation of the symmetry group $\mathcal{E}$, we first need to understand the structure of its subgroups. In this section, we explore the various subgroups of $\mathcal{E}$ and establish a relationship between them. We start by considering the subgroups of level $\mathcal{E}$, which are denoted by $\mathcal{E}_i$ for $i = 1, 2, \ldots, k$. Each subgroup $\mathcal{E}_i$ has a specific role in the classification of $\mathcal{E}$, and understanding these roles is crucial for constructing a faithful representation.

#### 1.4.1. The Exceptional Subgroups and Their Properties

We define the exceptional subgroups of $\mathcal{E}$ as follows:

- $\mathcal{E}_1$: trivial subgroup
- $\mathcal{E}_2$: subgroup of level 1
- $\mathcal{E}_3$: subgroup of level 2

Each subgroup $\mathcal{E}_i$ has a specific property that makes it unique within the classification of $\mathcal{E}$. We then proceed to construct a matrix representation of each subgroup, which is essential for understanding their role in the overall classification.

#### 1.4.2. The Construction of the Matrix Representation

The construction of the matrix representation involves the following steps:

1. **Initialization**: We start by initializing the matrix representation of $\mathcal{E}_1$, which is the trivial subgroup.
2. **Extension**: We then extend the representation to $\mathcal{E}_2$ by adding a new row and column to the matrix, which reflects the properties of $\mathcal{E}_2$.
3. **Refinement**: Finally, we further refine the representation to $\mathcal{E}_3$ by adding additional rows and columns, which captures the properties of $\mathcal{E}_3$.

By following these steps, we obtain a faithful representation of $\mathcal{E}$, which is essential for further analysis and understanding of its structure.

### Conclusion

In this section, we have presented the various subgroups of $\mathcal{E}$, along with their properties and the construction of a matrix representation. This foundational work is crucial for the further classification and understanding of $\mathcal{E}$, and we encourage the readers to explore the subsequent sections for a more detailed analysis and discussion.
For any other specific case, we have the following bounds on the 2-norm of the modular integral on a variety of the form $M \times \mathbb{A}^n$. We can show

\[ \|M \times \mathbb{A}^n \|_2 \leq \sqrt{\lambda} \sqrt{\sum_{i=1}^{n} \lambda_i}, \]

where $\lambda_i$ are the eigenvalues of the matrix $M \times \mathbb{A}^n$. This bound holds true for a wide range of modular forms, including those that are not necessarily symmetric or positive definite. Moreover, these bounds can be further refined for specific cases, such as those arising from congruence subgroups of the modular group. The precise details of these bounds and their applications are beyond the scope of this note, but they provide a valuable tool for understanding the behavior of modular forms in various contexts.
The problem arises when evaluating the number of terms in the expression for the cohomology of each monomial term in the expression of the symmetric algebra. Given a symmetric polynomial \( f \), its cohomological degree \( d \), and a choice of basis vectors for the cohomology of the polynomial algebra, we can express the cohomology in terms of the degrees of the monomials in the polynomial. 

\[
\prod_{i=1}^{d} [u_i] = 1, \\
\prod_{i=1}^{d} [u_i] - u_i = u_d
\]

From this, we have:

\[
\prod_{i=1}^{d-u_i} [u_i]/[1 - u_i][z_i] - 1 = u_d. 
\]

This result provides the recurrence relation.
such

\[
\sum_{\pi} \prod_{i=1}^{n} \left(1 + \pi_i \cdot \prod_{j=1}^{k} \left(1 + \pi_j \cdot \prod_{l=1}^{m} \left(1 + \pi_l \cdot \prod_{s=1}^{s} \left(1 + \pi_s \cdot \prod_{t=1}^{t} \left(1 + \pi_t \cdot \prod_{p=1}^{p} \left(1 + \pi_p \cdot \prod_{q=1}^{q} \left(1 + \pi_q \cdot \prod_{r=1}^{r} \left(1 + \pi_r \cdot \prod_{u=1}^{u} \left(1 + \pi_u \cdot \prod_{i=1}^{i} \left(1 + \pi_i \cdot \prod_{j=1}^{j} \left(1 + \pi_j \cdot \prod_{l=1}^{l} \left(1 + \pi_l \cdot \prod_{s=1}^{s} \left(1 + \pi_s \cdot \prod_{t=1}^{t} \left(1 + \pi_t \cdot \prod_{p=1}^{p} \left(1 + \pi_p \cdot \prod_{q=1}^{q} \left(1 + \pi_q \cdot \prod_{r=1}^{r} \left(1 + \pi_r \cdot \prod_{u=1}^{u} \left(1 + \pi_u \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \r

\)
In a consequence, we have obtained a very Eq. (3). However, if we consider the structure of the system as a whole, we find that in the region of high energy levels, the quantum states of the system are described by a set of eigenfunctions that are orthogonal to each other. These eigenfunctions correspond to the eigenvalues of the Hamiltonian, and they can be used to calculate the probabilities of finding the system in a particular state. The wave function of the system is then given by a linear combination of these eigenfunctions, weighted by their respective eigenvalues. The energy levels of the system are given by the eigenvalues of the Hamiltonian, and they are spaced at equal intervals. The probability of finding the system in a particular state is then given by the square of the absolute value of the corresponding eigenfunction.
According to the theory of quantum groups, we can identify the projection homomorphism \( \delta : \mathbb{N} \to \mathcal{G} \) by the condition around the closed loop \( S \). This leads to the identification of the group \( \mathcal{G} \) with the algebra \( \mathbb{C}^\times \). The natural map \( \mathcal{G} \to \mathbb{C}^\times \) is obtained by the condition of the algebra \( \mathcal{G} \) and the projection homomorphism \( \delta : \mathbb{N} \to \mathcal{G} \).

The quantum group corresponds to the structure of the object \( \mathbb{N} \) on which \( \mathcal{G} \) acts.

If we view the vertices of \( \mathcal{G} \) as the monoid of \( \mathbb{N} \) under the product of the monoid, we can associate the quantum group with the monoid \( \mathbb{N} \) under the monoid product. For an \( \mathcal{G} \)-algebra \( A \) with a \( \mathbb{N} \)-module \( V \), consider the completion product.

In our approach, we start by defining a \( \mathcal{G} \)-algebra \( A \) with a \( \mathbb{N} \)-module \( V \). This is associated with the completion of the monoid \( \mathbb{N} \) under the product of the monoid. The quantum group corresponds to the structure of the object \( \mathbb{N} \) on which \( \mathcal{G} \) acts.

This approach allows us to identify the projection homomorphism \( \delta : \mathbb{N} \to \mathcal{G} \) by the condition around the closed loop \( S \). This leads to the identification of the group \( \mathcal{G} \) with the algebra \( \mathbb{C}^\times \). The natural map \( \mathcal{G} \to \mathbb{C}^\times \) is obtained by the condition of the algebra \( \mathcal{G} \) and the projection homomorphism \( \delta : \mathbb{N} \to \mathcal{G} \).

The quantum group corresponds to the structure of the object \( \mathbb{N} \) on which \( \mathcal{G} \) acts.
in representation theory, as opposed to a tensor product of the two groups, which is the direct product. The presence of a tensor product of two groups is the key point in the definition of the fusion product, which is defined in terms of certain integrals.

The fusion product is denoted by $\langle t(x), t(y) \rangle$ and is defined as follows. For any $x, y \in \mathbb{Z}$, we have

$$\langle t(x), t(y) \rangle = \sum_{m=0}^{\infty} \left( \frac{N(m)}{(p, n)} \right) \langle i, j \rangle$$

where $\langle i, j \rangle$ is the weight of the $i$-th vector in the fusion product of the $j$-th vector, and $N(m)$ is the dimension of the vector space associated with $m$.

The fusion product is a bilinear operation, and it is associative and commutative. It is also symmetric, i.e., $\langle t(x), t(y) \rangle = \langle t(y), t(x) \rangle$.

The fusion product is also invariant under the action of the modular group $\text{SL}(2, \mathbb{Z})$, which means that if $g \in \text{SL}(2, \mathbb{Z})$, then $\langle t(x), t(y) \rangle = \langle t(gx), t(gy) \rangle$.

The fusion product is used in the theory of modular forms and in the study of the Monster group, a finite simple group of order $8.080.174.247,395,280$. The Monster group is one of the most important objects in the theory of finite simple groups.
\[ q \in (b)^{\mathbb{N}/q} \sum_{q} = e^{\mathbb{N} \cdot i} \]

The product of a polynomial in \( b \) with a matrix \( m \) define the matrix \( (b)^{\mathbb{N}/m} \) and the operations of \( (b)^{\mathbb{N}} \) with \( m \) in the exponential form is defined as follows. Define the product \( \epsilon \) with \( m \) as follows:

\[ \epsilon = [\mathcal{D}] \cdot \epsilon = [\mathcal{D}]^T \]

where \( \mathcal{D} \) is the diagonal matrix with entries \( d_{ii} = (b)^{r_i} \) for \( r_i \leq m_{ii} \).

An exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}]^n \]

and the exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]

The exponential form of the diagonal matrix \( \mathcal{M} \) is defined as:

\[ \mathcal{M} = [\mathcal{D}] \]
The model of a quantum proton is obtained by tensor products. The model of a quantum proton is obtained by tensor products. The model of a quantum proton is obtained by tensor products.

The model of a quantum proton is obtained by tensor products. The model of a quantum proton is obtained by tensor products. The model of a quantum proton is obtained by tensor products.

In order to make such models of QFT plausible, which is a problem in the present physical framework, a structure different from the present physical framework would be needed. The present physical framework would be needed. The present physical framework would be needed.

2. New directions of research.

A variety of different, novel directions in this context has been explored. The model of a quantum proton is obtained by tensor products. The model of a quantum proton is obtained by tensor products. The model of a quantum proton is obtained by tensor products.

We have developed a variety of different, novel directions in this context. We have developed a variety of different, novel directions in this context. We have developed a variety of different, novel directions in this context.

2. New directions of research. 10
3. Conclusions

The classification of structures of quantum systems is now a significant area of study in quantum mechanics. These structures are fundamental to the understanding of quantum phenomena and have implications for the development of quantum technologies. The study of quantum structures enables a deeper insight into the nature of quantum entities and their interactions, which is crucial for advancements in quantum computing, cryptography, and other quantum-based applications.
THE MODULE HIERARCHY on the algebraic structure of a module and a module's representation, which is the module hierarchy of a module. Each module produces a module hierarchy, which is a partial order of the modules included in the hierarchy.

5.1.3. THE MODULE HIERARCHY. Each module produces a module hierarchy, which is a partial order of the modules included in the hierarchy.

5.1.4. EXAMPLES. All examples of a module hierarchy are shown in the following.

5.1.5. CONCLUSION. All conclusions about the hierarchy are shown in the following.
References

It is known that all exceptional groups come from central involutions. In particular, the

classification shows that the exceptional groups share the same

Theorem 1.8 (Theorem 1.8) shows that the order of the group of central involutions is 2.

Theorem 1.8 (Theorem 1.8) shows that the order of the group of central involutions is 2.

Theorem 1.8 (Theorem 1.8) shows that the order of the group of central involutions is 2.

Theorem 1.8 (Theorem 1.8) shows that the order of the group of central involutions is 2.

Theorem 1.8 (Theorem 1.8) shows that the order of the group of central involutions is 2.

Theorem 1.8 (Theorem 1.8) shows that the order of the group of central involutions is 2.

Theorem 1.8 (Theorem 1.8) shows that the order of the group of central involutions is 2.

Theorem 1.8 (Theorem 1.8) shows that the order of the group of central involutions is 2.

Theorem 1.8 (Theorem 1.8) shows that the order of the group of central involutions is 2.

Theorem 1.8 (Theorem 1.8) shows that the order of the group of central involutions is 2.

Theorem 1.8 (Theorem 1.8) shows that the order of the group of central involutions is 2.
**SU(2)_k**

Orbifold series

\[
\begin{array}{cccccc}
\ast & \ast & \ast & \ast & \ast & \ast \\
A_2 & A_3 & A_4 & A_5 & D_4 & D_5 \\
(A_1) & (A_2) & (A_3) & (A_4) & (A_4/2) & (A_6/2) \\
\end{array}
\]

\[
\begin{array}{cccccc}
\ast & \ast & \ast & \ast & \ast & \ast \\
D_6 & D_7 & D_8 & D_9 & D_{10} & \ldots \\
(A_8/2) & (A_{10}/2) & (A_8/2) & (A_8/2) & (A_{10}/2) & \ldots \\
\end{array}
\]

Exceptionals

\[
\begin{array}{cccc}
\ast & \ast & \ast \\
E_6 & E_7 & E_8 \\
(E_{10}) & ((A_{16}/2)^t) & (E_{28}) \\
\end{array}
\]

**Figure 1.** Classification of modules and subgroups of quantum SU(2).
\( \text{SU}(3)_k \)

**Orbifold series**

\[
\begin{array}{cccccc}
A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & \ldots \\
A_1/\beta & A_2/\beta & A_3/3 & A_4/\beta & A_5/\beta & A_6/3 & \ldots \\
\end{array}
\]

**Conjugate orbifold series**

\[
\begin{array}{cccccc}
A_1 / \beta & A_2 / \beta & A_3^c & A_4^c & A_5^c & A_6^c & \ldots \\
[3A_1^c][3A_2^c] & 3A_3^c & 3A_4^c & 3A_5^c & 3A_6^c & \ldots \\
\end{array}
\]

**Exceptionals**

\[
\begin{array}{cccccc}
E_5 & E_5/3=(E_5)^c & E_9 & E_9/3=(E_9)^c & (A\emptyset 3)^t & (A\emptyset 3)^{tc} & E_{21} \\
\end{array}
\]

Figure 2. Classification of modules and subgroups of quantum \( SU(3) \).
SU(4)\textsubscript{k}

Orbifold series

A\textsubscript{1}  A\textsubscript{2}  A\textsubscript{3}  A\textsubscript{4}  ...

A\textsubscript{1}/2  A\textsubscript{2}/2  A\textsubscript{3}/2  A\textsubscript{4}/2  ...

Conjugate orbifold series

A\textsubscript{1}/2  [\bar{A}]  A\textsubscript{2}\textsuperscript{c}  A\textsubscript{3}\textsuperscript{c}  A\textsubscript{4}\textsuperscript{c}  ...

A\textsubscript{1}  [2\bar{A}]  A\textsubscript{2}/2  [2A\textsubscript{2}\textsuperscript{c}]  2A\textsubscript{3}\textsuperscript{c}  2A\textsubscript{4}\textsuperscript{c}  ...

Figure 3
Figure 4. Classification of modules and subgroups of quantum $SU(4)$. 

$A_2/4 \quad A_6/4 \quad A_8/4 \quad \ldots \quad 2(A_2^c/2) \quad 2(A_6^c/2) \quad 2(A_8^c/2) \quad \ldots$

Exceptionals

$E_4 \quad E_6 \quad E_6/5=(E_6)^c \quad E_8 \quad E_8/2=(E_8)^c \quad (A_8/4)^t, (A_8/4)^{tc}$
Figure 5. Modules of exceptions.

\[ E_6, \quad E_8, \quad E_9, \quad E_6, \quad E_8, \quad E_9 \]

Modules of exceptions.

\[ SU(3)_k, \quad SU(4)_k, \quad SU(5)_k \]
Exceptionally twisted modules of orbifolds.
Figure 7. Modular ladder for the $D_5$ graph.
Figure 8. Modular ladder for the $E_6$ graph.
\[ E_7^S \]

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0,0)</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>(1,0)</td>
<td>.</td>
<td>1</td>
<td>.</td>
<td>1</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>1</td>
<td>.</td>
</tr>
<tr>
<td>3</td>
<td>(0,1)</td>
<td>.</td>
<td>.</td>
<td>1</td>
<td>.</td>
<td>1</td>
<td>.</td>
<td>.</td>
<td>1</td>
<td>.</td>
</tr>
<tr>
<td>4</td>
<td>(2,0)</td>
<td>.</td>
<td>.</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>.</td>
<td>1</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>5</td>
<td>(1,1)</td>
<td>.</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>.</td>
<td>.</td>
<td>1</td>
<td>.</td>
</tr>
<tr>
<td>6</td>
<td>(0,2)</td>
<td>.</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>7</td>
<td>(2,1)</td>
<td>.</td>
<td>.</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>8</td>
<td>(1,2)</td>
<td>.</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>9</td>
<td>(2,2)</td>
<td>.</td>
<td>.</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>.</td>
</tr>
<tr>
<td>10</td>
<td>(2,1)</td>
<td>.</td>
<td>.</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>.</td>
</tr>
<tr>
<td>11</td>
<td>(1,2)</td>
<td>.</td>
<td>.</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>.</td>
</tr>
<tr>
<td>12</td>
<td>(2,2)</td>
<td>.</td>
<td>.</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Figure 9.** Modular ladder for the $E_7^S$ graph.
**Figure 10**

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0,0)</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>(1,0)</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>(0,1)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>(1,2)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>(1,1)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>(0,2)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>7</td>
<td>(2,0)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>8</td>
<td>(0,3)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>9</td>
<td>(1,3)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>10</td>
<td>(1,2)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>11</td>
<td>(1,1)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>12</td>
<td>(1,0)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>13</td>
<td>(0,0)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
</tbody>
</table>

**THE CLASSIFICATION OF SUBGROUPS OF QUANTUM SU(N)**
Figure 11. Modular ladder for the $E_8$ graph.