Continuous time models in Finance and Stochastic calculus

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Hanoi IMAMIS-CIMPA School
April 23th - May 4th 2007

April 19, 2007

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# Chapter 1

# Brownian motion and heat equation

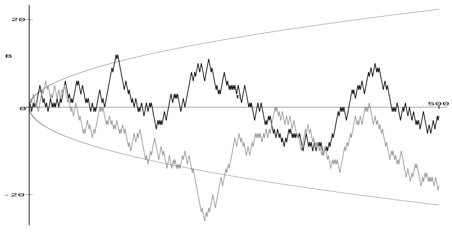
One of the first idea of continuous-time stochastic model go back to a botanist Robert Brown in 1827 when he discribed random movements of particles supended in a fluid. This explains the name *Brownian motion* given to the stochastic process that is the subject of this first lecture. In fact the first who have studied the Brownian Motion with a mathematical point of vue was probably Louis Bachelier in his famous thesis on the modelisation of the stock prices, defended in Paris in 1900. And many other well known scientists have contributed to the theory among them A. Einstein (1905), N. Wiener (1923), A. Kolmogorov (1933), P. Levy (1939) and K. Ito (1948). Brownan motions are also named *Wiener Processes*.

## 1.1 Brownian motion : definition and main properties

To model the dynamic of stock prices the idea is to consider the trajectory not as the trajectory of a deterministic dynamic but as one trajectories among a (possibly infinite) set of trajectories that could be realized, and to put on this set a probability measure. More precisely, a *stochastic process* is a map $\mathbb{T} \to L_0(\Omega, \mathcal{F}, \mathbb{P})$ from a time set $\mathbb{T}$ (for continuous time process $\mathbb{T} = \mathbb{R}^+$) to the space of bounded mesurable functions for a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}$. For a fixed $t \in \mathbb{T}$, $\omega \mapsto X_t(\omega)$ is a random variable on $\Omega$ and for a fixed $\omega$, $t \mapsto X_t(\omega)$ is a trajectory of the process.

**Definition:** A stochastic process is a (standard, one dimentional) Brownian motion $B_t : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ if

1. $B_0 = 0$
2. For all $0 \leq s < t$, $B_t - B_s$ has a gaussian distribution $\mathcal{N}(0, \sqrt{t-s})$
3. For all $0 < t_1 < t_2 < \ldots < t_n$, the random variables $\{B_{t_{i+1}} - B_{t_i}, i = 0, \ldots, n-1\}$ are independant
4. $B_t$ has continuous trajectories.



The main properties of the Brownian motion (BM) are :

1. Symetry property : if $(B_t)$ est un BM then $(-B_t)$ is also a BM.
2. Scale property : if $(B_t)$ is a BM than for all $c > 0$, the process $\frac{1}{c}(B_{c^2 t})$ is also a BM.
3. Translation property : if $(B_t)$ is a BM than for all $h$, $(B_{t+h}) - (B_h)$ is a BM.
4. Reversal property : if $(B_t)$ is a BM on $[0, T]$ than $(B_T) - (B_{T-t})$ is also a BM on $[0, T]$.
5. Behaviour at infinity : if $(B_t)$ is a BM than $(tB_{1/t})$ is a BM and a.s. : $\frac{B_t}{t} \to 0$.

Despite the fact that the trajectories of a Brownian motion are continuous by definition, they are a.s. *nowhere differentiable*. And we have also the two following caracteristics. First, if $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n = t$ is a subdivision $\sigma$ of the interval $[0, t]$, than the total variation, defined by

$$V_t(\omega) := \sup_\sigma \sum_i |B_{t_{i+1}}(\omega) - B_{t_i}(\omega)|$$

satisfy $V_t(\omega) = +\infty$ a.s. Second, if $\sigma_n$ is a sequence of partitions of $[0, t]$ such that $\lim_{n\to\infty} \|\sigma_n\| := \sup_i |t_{i+1} - t_i| = 0$, than

$$\lim_{n\to\infty} \sum_i |B_{t_{i+1}}(\omega) - B_{t_i}(\omega)|^2 = t.$$

## 1.2 Heat equation and Brownian motion

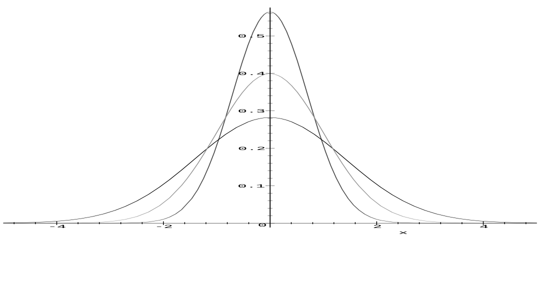
As for all $t > 0$ the probability distribution of $B_t$ is $\mathcal{N}(0, \sqrt{t})$, than it is easy to compute the expectation :

$$\mathbb{E}(B_t) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} y e^{-\frac{y^2}{2t}} dy = \int_{\mathbb{R}} y g(t, y) dy$$

where $g(t, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}$ is the gaussian pdf. But as the function $g(t, x)$ satisfies the Heat equation (easy to check by direct computations) :

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

we have the following proposition (also easy to check by direct computations) :



**Proposition 1.1** *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function and let $u(t, x) := \mathbb{E} f(x + B_t)$ than, for all $t > 0$, $u(t, x)$ is $\mathcal{C}^\infty$ and satisfies the equation (1.1) with $u(0, x) = f(x)$.*

## 1.3 Geometric Brownian motion

The model introduced by Bachelier in 1900 for the stock prices is what we call now the *Brownian motion with drift*, just replacing the low of the increments $\mathcal{N}(0, \sqrt{t-s})$ by $\mathcal{N}(\mu, \sigma\sqrt{t-s})$. One of the drowback of this model is that the prices can become negative. The work of Bachelier was overlocked by scientists during more than sixty years and one has had to wait until the Nobel Price Samuelson to overcome this difficulty. Since Samuelson the usual model of stock prices is no longer a Brownian motion with drift but a *geometric Brownian motion*.

**Definition:** Let $(B_t)$ a Brownian motion and let $\mu$ and $\sigma$ two real numbers. The stochastic process

$$X_t := \mu t + \sigma B_t$$

is called a *Brownian motion with drift*. One has $\mathbb{E}(X_t) = \mu t$ and $Var(X_t) = \sigma^2 t$.

It is not difficult to generalize to $X_t$ the previous proposition : for $u(t,x) = \mathbb{E}f(t, x+X_t) = \mathbb{E}f(t, x+\mu t+\sigma B_t)$, the Heat equation (1.1) has to be replace by the equation

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial u}{\partial x} \tag{1.2}$$

with still the initiale condition $u(0,x) = f(x)$.

To go from the Brownian motion with drift to the geometric Brownian motion, the idea of Samuelson was to consider the dynamic of the returns (or relative increments $\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}$ ) instead of the dynamic of the prices themself. And more precisely he decided to measure the returns between two dates by the difference of the logarithm of the prices $\ln\frac{S_{t_{i+1}}}{S_{t_i}}$.

**Definition:** A process $(Y_t)$ is a *geometric Brownian motion* with parameters $\mu$ and $\sigma$ if the process $\ln Y_t$ is a Brownian motion with drift, namely (for *standard* geometric Brownian motion) if $Y_0 = 1$ and if

$$\ln Y_t = (\mu - \frac{\sigma^2}{2})t + \sigma B_t \text{ or } Y_t = \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma B_t\right).$$

It is easy to show that $\mathbb{E}(Y_t) = e^{\mu t}$ and $\mathbb{E}(Y_t^2) = e^{(2\mu+\sigma^2)t}$.

**Theorem 1.2** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continous bounded fonction and let $v(t,y) = \mathbb{E}f(yY_t)$, where $Y_t$ is a geometric Brownian motion with parameter $\mu$ et $\sigma$. Than $v(t,y)$ satisfy $v(0,y) = f(y)$ and the equation*

$$\frac{\partial v}{\partial t} = \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 v}{\partial y^2} + \mu y\frac{\partial v}{\partial y}. \tag{1.3}$$

| Stochastic process | PDE | PDE's solution |
|---|---|---|
| $B_t$ <br> Brownian motion | $\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \\ u(0,x) = f(x) \end{cases}$ <br> Heat equation | $u(t,x) = \mathbb{E}f(x+B_t)$ |
| $X_t = \mu t + \sigma B_t$ <br> Brownian with drift | $\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial u}{\partial x} \\ u(0,x) = f(x) \end{cases}$ | $u(t,x) = \mathbb{E}f(x+X_t)$ |
| $Y_t = \exp((\mu - \sigma^2/2)t + \sigma B_t)$ <br> Geometric Brownian motion | $\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\sigma^2 y^2\frac{\partial^2 v}{\partial y^2} + \mu y\frac{\partial v}{\partial y} \\ v(0,y) = f(y) \end{cases}$ | $v(t,y) = \mathbb{E}f(yY_t)$ |

# Chapter 2

# Black-Scholes model for stock prices

We will describe now the most important continuous time model in finance, the Black-Scholes model, and explain how one can price european Call and Put in such a model.

## 2.1 The model

The Black-Scholes model for a stocks price is the continuous version of the CRR discret model. One begins with a constant interest rate, $r$, and an asset $(S_t)_{t\geq 0}$ that will be the underlying asset of an option $(C_t)_{t\geq 0}$. One assume that the dynamic of $S_t$ is $S_t = S_0 Y_t$ where $Y_t$ is a geometric Brownian motion with parameters $r$ and $\sigma$ ($\sigma$ is also chosen constant). Thus at time $t$ the stock price is

$$S_t = S_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma B_t\right) \tag{2.1}$$

The reason to take the interest rate $r$ as drift parameter of $\ln Y_t$ is because we want the present value of $S_t$, denoted $\tilde{S}_t = e^{-rt}S_t$, to be a martingale (we will explain this point later). Now if $C_T = \varphi(S_T)$ is the payoff of a european option (for example, $\varphi(S_T = (S_T - K)^+$ for a Call option) written on $S_t$, than the *fundamental formula of option pricing* is given by the following theorem :

**Theorem 2.1** *In a Black-Scholes model (2.1) for $S_t$ with constant interest rate $r$, the price $C_t$ at time $t$ of an european option $(T, \varphi(S_T))$ is given by*

$$C_t = e^{-r(T-t)}\mathbb{E}(\varphi(S_T)/\mathcal{F}_t). \tag{2.2}$$

For the moment, we will not prove this theorem. We will come back later on it to explain why the price is indeed, as in the discret case, equal to this expectation. Now we will only see how to compute this price in the cas $S_t$ follows such a model.

## 2.2 The PDE

To derive the Black-Scholes PDE, we will first transform the expectation of the theorem 2.1. Assume that at $t = 0$, $S_t = y$. The value of the option at time $t$ and knowing that $S_t = y$ is

$$C(t, y) = e^{-r(T-t)}\mathbb{E}\varphi(\frac{yS_{T-t}}{S_0}).$$

Indeed $S_T = S_0 exp(rT + \sigma B_T) = S_0 exp(rt + \sigma B_t) exp(r(T-t) + \sigma(B_T - B_t)) = y exp(r(T-t) + \sigma(B_T - B_t))$, and, as $B_T - B_t$ has the same low as $B_{T-t}$ according to the definition of Brownian motion, $S_T$ can be replace in the expectation by $\frac{yS_{T-t}}{S_0}$.

Now let $\tau = T - t$. Using the fact that $\frac{yS_\tau}{S_0}$ is a geometric Brownian motion and the theorem 1.2, $v(\tau, y) := e^{r\tau}C(\tau, y) = \mathbb{E}(\varphi(\frac{yS_\tau}{S_0}))$ satisfies the following PDE :

$$\frac{\partial v}{\partial \tau} = \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 v}{\partial y^2} + \mu y \frac{\partial v}{\partial y}$$

with $v(0, y) = \varphi(y)$. Thus $C(\tau, y) = e^{-r\tau}v(\tau, y)$ satisfies

$$\frac{\partial C}{\partial \tau} = -rC + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 C}{\partial y^2} + \mu y \frac{\partial C}{\partial y}$$

with $C(0, y) = \varphi(y)$. And finaly, replacing $\tau$ by $T - t$ (but without changing the notation for $C$), we obtain the famous Black-Scholes PDE :

$$\frac{\partial C}{\partial \tau} = -\frac{1}{2}\sigma^2 y^2 \frac{\partial^2 C}{\partial y^2} - \mu y \frac{\partial C}{\partial y} + rC$$

with $C(T, y) = \varphi(y)$.

| Stochastic process | PDE | PDE's solution |
|---|---|---|
| $\begin{cases} S_t = S_0 \exp((r - \sigma^2/2)t + \sigma B_t) \\ C_T = \varphi(S_T) \end{cases}$ Black-Scholes model | $\begin{cases} \frac{\partial C}{\partial t} = -\frac{1}{2}\sigma^2 y^2 \frac{\partial^2 C}{\partial y^2} - ry\frac{\partial C}{\partial y} + rC \\ C(T, y) = \varphi(y) \end{cases}$ Black-Scholes EDP | $\begin{cases} C(t, y) = e^{-r\tau}\mathbb{E}\varphi(y\frac{S_T}{S_0}) \\ \tau = T - t \end{cases}$ |

## 2.3 The Black-Scholes formula

The explicite computation of the expectation of the formula (2.1) or the computation of the explicite solution of this partial differential equation gives the famous *Black-Scholes formula :*

**Proposition 2.2** *In a Black-Scholes model, the price of a Call option at time $t$ if the value of the underlying asset is $x$, the exercice date $T$ and the exercice price $K$, is given by*

$$C(t, x) = x\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2) \tag{2.3}$$

*where* $d_1 = \frac{1}{\sigma\sqrt{T-t}}(\ln\frac{x}{K} + (r + \frac{\sigma^2}{2})(T - t))$, $d_2 = \frac{1}{\sigma\sqrt{T-t}}(\ln\frac{x}{K} + (r - \frac{\sigma^2}{2})(T - t))$ *and* $\mathcal{N}$ *is the gaussian distribution fonction* $\mathcal{N}(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{\xi^2}{2}}d\xi$. *Moreover the* delta *(derivative of the option price with respect to the stock price), also called* hedge ratio, *is given by*

$$\frac{\partial C}{\partial x} = \mathcal{N}(d_1).$$

**Remark:** To apply this formula to compute really an option price, one has to know several parameters. The parameters $T$ and $K$ are written in the contract of the option, the parameters $t$ and $x$ are known ($t$ is today and $x$ the time $t$ value of the underlying stock). It remains the parameter $r$ and $\sigma$. For $r$, one assume that it remain constant between $t$ and $T$ which is already a big simplification (it is not true at all..). But the major problem comes from $\sigma$, called *volatility*, that can not be observed (and varie a lot !). The way the practitioner used to choose the volatility when they want to price a new option is to deduce it from the market prices of the other traded options on the same underlying stock. Indeed from the formula (2.3) one can deduce that the option price is a non decreasing function of the volatility and thus at each market price corresponds a unique value of $\sigma$ (inverse image of it by the formula). This value of the parameter $\sigma$, called the *implied volatility*, corresponds to the anticipations of this value by the market. Unfortunately, it appears that the implied volatility is not constant (it changes with $T$ and $K$) as it was assumed in the model !

# Chapter 3

# Stochastic integral

To understand the results of the Black-Scholes theory and go further in the study of the continuous time models in finance we need more tools in stochastic calculus and first of all we need to define stochastic integrals.

## 3.1 Stochastic integral

In the discret models, one define the *Profit and Loss* of a strategy $\alpha_t$ on a stock $S_t$ as the quantity :

$$P\&L_t^S(\alpha) = \sum_{s\in]0..T]\delta t} \alpha_s \delta S_s \text{ , where } \delta S_s := S_s - S_{s-\delta t}.$$

The financial meaning is quite natural : if at time $s-\delta t$ you take a position of $\alpha_s$ stocks at the price $S_{s-\delta t}$, at time $s$ the price of the stock has changed of $\delta S_s = S_s - S_{s-\delta t}$, so you made a profit (or loss) of $\alpha_s \delta S_s$, and then you take a new position $\alpha_{s+\delta t}$ for the next time step with the information available, represented by $\mathcal{F}_s^S$. So $P\&L_t^S(\alpha)$ is just the total profit and loss up to time $t$ of the part of your portfolio invested in the stock $S_t$. Actually, $S_t - S_0$ is just $P\&L_t^S(1)$, the profit and loss of the *buy-and-hold strategy*.

If we try to extend this definition to the case of continuous time models (for example if $S_t$ is a Brownian motion), then it is not clear that it will be possible because this sum corresponds to the total variation of $\alpha$ along the trajectories of $S_t$ and we know that this total variation is a.s. infinite. Indeed we are in a hopeless situation if we try to give a meaning to this for each trajectory separatly. But, it was the discovery of Itô in 1048, if we relax our demand, then a Stochastic integral

$$P\&L_t^S(\alpha) = \int_0^t \alpha_s dS_s$$

should be defined trajectorywise.

To this aim, considere first the space of square integrable stochastic process :

**Definition:** Let $\mathbb{H}^2(\Omega\times\mathbb{R}^+)$ the set of all square integrable stochastic process $X_t$ (i.e. such that $\mathbb{E}\int X^2(t,.)dt < \infty$), adapted[1] to the filtration $(\mathcal{F}_t)$.

This set is an Hilbert space for the inner product $< X_t, Y_t >= \mathbb{E}\int X_t Y_t dt$. The most simple processes in this Hilbert space $\mathbb{H}^2(\Omega\times\mathbb{R}^+)$, called *elementary processes*, are the processes $\varphi(t,\omega) = \sum_0^{n-1} X_i(\omega)\mathbb{I}_{]t_i,t_{i+1}]}(t)$, where $t_0 < t_1 < \ldots < t_n$ is a discretization and $(X_i)_{i=0..n}$ a family of $\mathcal{F}_{t_i}^B$-mesurable and square integrable rv. Other exemples are the deterministic functions

[1] A stochastic process $X_t$ is said to be *adapted* to a filtration $(\mathcal{F}_t)$ if, for all $t\in\mathbb{R}^+$, $X_t$ is $\mathcal{F}_t-$mesurable. And a *filtration* is a increasing family $(\mathcal{F}_t, t\in\mathbb{R}^+)$ of right continuous, in the sens where $\mathcal{F}_t = \bigcap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$, subalgebra of $\Omega$ ; for any stochastic process $X_t$, one can define the associated *natural filtration* by $\mathcal{F}_t^X = \bigcap_{\varepsilon>0}\sigma(X_s, s\le t+\varepsilon)$, where $\sigma(X)$ is the subalgebra generated by the rv $X$ (in particular, $\mathcal{F}_t^B$ denotes the natural filtration of the Brownian motion $(B_t)$).

of $B_t$ such as $f(t, B_t)$. The sub set of all elementary processes is important because it is dense in $\mathbb{H}^2(\Omega \times \mathbb{R}^+)$ (i.e. all square integrable process is the limit of a sequence of elementary processes). The following theorem is also a definition of the stochastic integral.

**Theorem 3.1** *Let $B_t$ a Brownian motion and $(\mathcal{F}_t^B)$ its natural filtration. To any stochastic process $X_t \in \mathbb{H}^2(\Omega \times \mathbb{R}^+)$, is associated a square integrable rv $I(X_t)$, denoted $I(X_t) = \int X_t dB_t$, such that $\mathbb{E}(I(X_t)) = 0$ and $\mathbb{E}(I(X_t))^2 = \mathbb{E}(\int X_t^2 dt)$. The process $I(X_t)$ is call* a stochastique integral *or* Itô intégrale.

The idea of the proof is simple : the integral is first defined on the subset of elementary processes and, as the application $I$ is an isometry, it can be extended to all $\mathbb{H}^2(\Omega \times \mathbb{R}^+)$ by density.

This integrale can also be defined, as usual, as a function of it upper bound by $\int_0^t X_t dB_t := \int X_t \mathbb{I}_{]0,t]} dB_t$. The main properties are :

1. $\int_t^{t'} X_t dB_t =^{ps} \int_t^s X_t dB_t + \int_s^{t'} X_t dB_t$ (Chasles relation)
2. $\int_t^{t'} (\alpha X_t + Y_t) dB_t =^{ps} \alpha \int_t^{t'} X_t dB_t + \int_t^{t'} Y_t dB_t$ (linearity).
3. $\mathbb{E}\left(\int_t^{t'} X_t dB_t\right) = 0$ and Var $\left(\int_t^{t'} X_t dB_t\right) = \left(\int_t^{t'} X_t^2 dt\right)$.
4. $\int_t^{t'} X_t dB_t$ is a rv $\mathcal{F}_{t'}^B-$mesurable.

## 3.2 Martingales

We have already understood through the discret models approach that martingales play a crucial role in Finance. The intuitive idea goes back to Bachelier : he explained in 1900 that as the stock prices (or any prices) correspond to an agreement between two parties (market prices), and as their anticipations on the unknown future prices probably differ, the only fair price for both parties is equal to the expectation of all the future prices, given the information available today.

**Definition:** A integrable process $(M_t)$ (i.e. such that $\mathbb{E}(|M_t|) < +\infty$ for all $t$) adapted to the filtration $\mathcal{F}_t$ is

- a $(\mathcal{F}_t-)$ *martingale* if $\mathbb{E}(M_t/\mathcal{F}_s) = M_s$ for all $s \leq t$
- a $(\mathcal{F}_t-)$ *surmartingale* if $\mathbb{E}(M_t/\mathcal{F}_s) \leq M_s$ for all $s \leq t$
- a $(\mathcal{F}_t-)$ *sousmartingale* if $\mathbb{E}(M_t/\mathcal{F}_s) \geq M_s$ for all $s \leq t$

Notice that if $M_t$ is a martingale than for all $t$, $\mathbb{E}(M_t) = \mathbb{E}(M_0)$.

Easy computations with conditional expectation show that if $\mathcal{F}_t^B$is the natural filtration of the Brownian motion $B_t$ than the processes $B_t$, $B_t^2 - t$ and $exp(\sigma B_t - \frac{\sigma^2}{2} t)$ are $\mathcal{F}_t^B-$martingales. An other important example of $\mathcal{F}_t^B-$martingale is the stochastic integral $\int_0^t X_t dB_t$.

It is not difficult to prove that indeed a stochastic integrale is a martingale but what is true also, and more difficult to prove, is that any martingale is in fact a stochastic integral.

**Theorem 3.2** *Let $M_t$ be a square integrable martingale with respect to the natural filtration of a Brownian motion $\mathcal{F}_t^B$. There exists a square integrable adapted process $\alpha_t$ such that for all $t$ one has a.s. :*

$$M_t = M_0 + \int_0^t \alpha_s dB_s.$$

## 3.3 Itô processes

We have already meet three examples of Itô processes, the Bronian motion, the Brownian motion with drift and the geometric Brownian motion. They all belong to the following family :

**Definition:** Let $X_t$ be a stochastic process. If there exist two adapted processes $\mu_t$ and $\sigma_t$ such that $\mathbb{E}(\int_0^t |\mu_s| ds) < +\infty$ et $\mathbb{E}(\int_0^t \sigma_s^2 ds) < +\infty$ a.s. and if

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \varphi_s dB_s \tag{3.1}$$

than $X_t$ is an *Itô process*.

The first terme $\int_0^t \mu_s ds$ is the *finite variation part* and the second terme $\int_0^t \sigma_s dB_s$ is the *martingale part*. The decomposition in these two parts is unique and thus an Itô process is a martingale if and only if its finite variation part is zero.

The equation (3.1) can also be written simply $dX_t = \mu_t dt + \sigma_t dB_t$.

# Chapter 4

# Itô formula

The Itô formula is the first and the main tool of the *stochastic calculus*. It allows to answer to the following question : if $X_t$ is an Itô process such that $dX_t = \mu_t dt + \sigma_t dB_t$ and $f$ a function, what can we say about the process $Y_t = f(X_t)$ ? Is it still an Itô process and how to compute it ?

The Itô formula corresponds for stochastic process to the *Taylor formula* for deterministic dynamics but in some sens one can say that it is easier because it has only three terms (and indeed no remainder).

## 4.1 The formula

**Theorem 4.1 (Itô Formula)** *Let $X_t$ be an Itô process defined by (3.1), i.e. such that $dX_t = \mu_t dt + \sigma_t dB_t$ and let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a function with first and second order continuous derivatives in $t$ and $x$. Than the process $Y_t := f(t, X_t)$ is still an Itô process and a.s. :*

$$Y_t = Y_0 + \int_0^t \frac{\partial f}{\partial t}(s, X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)dX_s + \int_0^t \frac{\varphi_t^2}{2}\frac{\partial^2 f}{\partial x^2}(s, X_s)ds.$$

Notice that in the formula, the stochastic integral $\int_0^t \frac{\partial f}{\partial x}(t, X_t)dX_t$ is to be understood as a simplified notation of $\int_0^t \frac{\partial f}{\partial x}(s, X_s)\alpha_s ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\varphi_s dB_s$. This is the way to extend the stochastic integral from the Brownian motion to any Itô process. In fact, stochastic integral can be generalized to a large classe of processes, the *semi-martingales* (sum of a (local) martingale and a process with (locally) bounded variation.

Notice that the Itô formula can be written simply

$$dY_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{\varphi_t^2}{2}\frac{\partial^2 f}{\partial x^2}dt.$$

For example for $X_t = B_t$ and $f(x) = x^2$,

$$d(B_t^2) = 0dt + 2B_t dB_t + \frac{1}{2}(2)dt$$

and than $d(B_t^2) = 2B_s dB_s + dt$. The "additional" term $dt$ is called the *Itô term*. In the stochastic calculus, one also have a *integration by part* formula :

**Proposition 4.2** *Let $X_t$ and $\overline{X}_t$ be two Itô processes such that $dX_t = \mu_t dt + \sigma_t dB_t$ and $d\overline{X}_t = \overline{\mu}_t dt + \overline{\sigma}_t dB_t$. Then*

$$X_t\overline{X}_t = X_0\overline{X}_0 + \int_0^t X_s d\overline{X}_s + \int_0^t \overline{X}_s dX_s + \int_0^t \sigma_s\overline{\sigma}_s ds.$$

Ther proof is straightforward. One apply the Itô formula to $(X_t + \overline{X}_t)^2$, $(X_t)^2$ and $(\overline{X}_t)^2$ and than one make the difference between the first square and the next two.

## 4.2 The Black-Scholes model

We have introduced the Black-Scholes model of stock prices $S_t = S_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t)$ saying that we have chosen $\mu = r$ because we wanted the discounted prices $\tilde{S}_t$ to be a martingale. Now we can explain this point. Indeed, if $Y_t$ is the geometric Brownian motion $Y_t = (\mu - \frac{\sigma^2}{2})t + \sigma B_t$ than its differential is simply $dY_t = (\mu - \frac{\sigma^2}{2})dt + \sigma dB_t$ and thus the Itô formula applied to $S_t = S_0 exp Y_t$ gives

$$S_t = S_0 + \int_0^t S_s dY_s + \frac{\sigma^2}{2}\int_0^t S_s ds.$$

Replacing $dY_t$ by its expression, one get

$$S_t = S_0 + \int_0^t \mu S_s dt + \sigma \int_0^t S_s dB_s$$

we obtain $dS_t = \mu S_t dt + \sigma S_t dB_t$. Notice that if $\mu \neq 0$, then $S_t$ is not a martingale. But its discounted value, $\tilde{S}_t = e^{-rt} S_t$ will be a martingale if and only if $\mu = r$. Indeed, using the *integration by part* rule (4.2), applied to the stochastic process $S_t$ and the deterministic function $e^{-rt}$ (in this case the Itô term is zero), one has

$$d\tilde{S}_t = d(e^{-rt}S_t) = -re^{-rt}S_t dt + e^{-rt}dS_t = (\mu - r)S_t dt + e^{-rt}dS_t.$$

Thus if one choose, as we did, $S_t = S_0 \exp((r - \frac{\sigma^2}{2})t + \sigma B_t)$ as a model for the stock prices, then the discounted stock prices $\tilde{S}_t$ is a martingale.

## 4.3 Stochastic differential equation

**Definition:** A *stochastic differential equation* (SDE) is an equation such that

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t \tag{4.1}$$

where the solutions $(X_t)$ are adapted to the natural filtration of $B_t$ stochastic processthat satisfy a.s. the equation for all $t$ in $[t_0, t^+[$. The initial condition $X_0$ could be a given number or, in general, a r.v..

**Examples:**

- The solution of $dX_t = X_t(\mu dt + \sigma dB_t)$ such that $X_0 = 1$ is the standard geometric Brownian motion $X_t = \exp(\mu t + \sigma B_t)$.
- More generally, the solutions of the SDE $dX_t = X_t(\mu(t)dt + \sigma(t)dB_t)$ are $X_t = X_0 \exp(\int_0^t (\mu(s) - \frac{1}{2}\sigma^2(s))ds + \int_0^t \sigma(s)dB_s)$.
- The solutions of the SDE $dX_t = -aX_t dt + \sigma dB_t$, called Langevin equation, are the processes $X_t = X_0 e^{-at} + \sigma \int_0^t e^{-a(t-s)} dB_s$, called Ornstein-Uhlenbeck processes.
-

As for the ordinary differential equations, we also have for SDE an existence and unicity theorem :

**Theorem 4.3** *Consider a stochastic differential equation (4.1) and assume that the two functions $\mu$ et $\sigma$ are continuous and that it exists a constant $C$ such that*

1. $|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq C|x - y|$
2. $|\mu(t,x)| + |\sigma(t,x)| \leq C(1 + |x|)$

*Than for all $T > 0$, the equation (4.1) has a unique solution $\mathcal{F}_t^B$−adapted in* $[0, T]$ *(*unique *means that if $X_t$ and $\overline{X}_t$ are two solutions, than $\forall t \in [0, T]$, $X_t =^{ps} \overline{X}_t$).*

# Chapter 5

# Arbitrage pricing

In this lecture, we will come back to the *fondamental option pricing formula* (2.1) for a Black-Scholes model. There is various way to derive this formula, we propose here two of them, the first use the Itô formula to prove that the price must be a solution of the Black-Scholes PDE. The second shows the existence of an hedging portfolio having this value and claims that, because of no arbitrage asumption, this value must be the good one.

## 5.1 The Black-Scholes market

Let $S_t = S_0 \exp((r - \frac{\sigma^2}{2})t + \sigma B_t)$ the Black-Scholes model of a stock prices and $C_t$ the price of an option $(T, \varphi(T))$ on $S_t$. In our market, we need a second asset, denoted $S_t^0$ with a deterministic dynamics given by $S_t^0 = S_0^0 e^{rt}$, with usually $S_0^0 = 1$.

Let $(\Omega, P, \mathcal{F}_t^B)$ be the filtered probability space where the Brownian motion $B_t$ is defined. We already notice that the discounted value of the stock price, $\tilde{S}_t$, is a martingale with respect to this filtration $\mathcal{F}_t^B$.

**Definition:** A european option $(T, \varphi(S_T))$ on the stock $(S_t)$, is simply a $\mathcal{F}_T^B$-mesurable and positive r.v.. For example, $\varphi(S_T) = (S_T - K)^+$ for a *european Call* and $\varphi(S_T) = (K - S_T)^+$ for a *european Put*.

Recall that in a Black-Scholes model, the following fondamental pricing formula holds :

**Theorem 5.1 (Fondamental pricing formula)** *In this Black-Scholes market, the value $C_t$ at time $t \in [0, T]$ of a european option $(T, \varphi(S_T))$ on the stock $(S_t)$ is given by*

$$C_t = \mathbb{E}(e^{-r(T-t)}\varphi(S_T)/\mathcal{F}_t).$$

**Definition:** In our model a *hedging strategy* is a couple $(\alpha_t^0, \alpha_t)$ of two $\mathcal{F}_t^B$-adapted r.v. corresponding to the investment into the two non risky and risky assets $S_t^0$ and $S_t$. With such a strategy, one build a portfolio $\Pi$ with the time $t$ value given by $\Pi_t = h_t^0 S_t^0 + h_t S_t$. We only consider strategies having positive and square integrable componantes.

**Definition:** A strategy $(\alpha_t^0, \alpha_t)$ corresponding to the portefolio $\Pi$ is called *self-financing* if for all $t \in [0, T]$, $d\Pi_t = \alpha_t^0 dS_t^0 + \alpha_t dS_t$. If a self-financing strategy $(\alpha_t^0, \alpha_t)$ satisfies $\Pi_T = \varphi(S_T)$, one says that it *duplicate* the option $(T, \varphi(S_T))$. A portfolio that is self-financing and that duplicate an option is called an *hedging* portfolio for this option.

It is easy to show that :

**Proposition 5.2** *A strategy $(\alpha_t^0, \alpha_t)$ of a portfolio $\Pi_t$ is self-financing if and only if $\tilde{\Pi}_t = \tilde{\Pi}_0 + \int_0^t \alpha_s d\tilde{S}_s$ (or using the simplified notation $d\tilde{\Pi}_t = \alpha_t d\tilde{S}_t$).*

## 5.2 Delta hedging

Assume that the option value $C_t$ can be written as a function $C_t = C(t, S_t)$ of $t$ and $S$ and that this function has continuous second derivatives.

**Theorem 5.3** *In a Black-Scholes model, the price of an european option $C(t,S)$ is equal to the value of an hedging portfolio $\Pi_t$ with a componante $\Delta_t$ in the underlying stock and that satisfies*

$$\left\{ \begin{array}{rcl} \frac{\partial C}{\partial t}(t,S) + \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2}(t,S) + rS\frac{\partial C}{\partial S}(t,S) - rC(t,S) & = & 0 \\ C(T,S) & = & \varphi(S) \end{array} \right. \tag{5.1}$$

*and the quantity $\Delta(t,S)$ in the underlying stock is $C(t,S) = \frac{\partial C}{\partial S}(t,S)$.*

Consider a portfolio $\Pi_t$ of one long option $C_t$ position and one short position in some quantity $\Delta_t$ of the underlying stock $S_t$ :

$$\Pi_t = C(t,S_t) - \Delta_t S_t$$

If the quantity $\Delta_t$ can be chosen in such a way that the corresponding strategy is self-financing and completely hedge the risk of the portfolio (such a dynamic hedging is called a *delta hedging*), then the resulting portfolio will satisfy

$$d\Pi_t = r\Pi_t dt.$$

Now, using the Itô formula applying to the function $C$ and the previous proposition 5.2, one has :

$$d\Pi_t = dC(t,S_t) - \Delta_t dS_t = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS_t + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 C}{\partial S^2}dt - \Delta_t dS_t$$

Thus, to have no stochastic term (in $dS$) and only deterministic term (in dt)), we find that

$$\left\{ \begin{array}{rcl} \frac{\partial C}{\partial S} & = & \Delta_t \\ d\Pi_t & = & \left(\frac{\partial C}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2}\right)dt. \end{array} \right. \tag{5.2}$$

But as $d\Pi_t = r\Pi_t dt = r(C(t,S_t) - \Delta_t S_t)dt = rC(t,S_t)dt - r\frac{\partial C}{\partial S}S_t$, the Black-Scholes PDE follows.

The existence of a (even explicite !) solution of this equation shows the quantity $\Delta(t,S_t)$ can be computed in such a way that the portfolio $\Pi_t$ builded with exactly $\Delta(t,S_t)$ underlying stocks (and rebalancing continuously) will perfectly hedge the option $(T,\varphi(S_T))$. Now the price of such a portfolio must be equal to the option price because it will take the same final value, and thus must take the same time $t$ value for all $t < T$ (no arbitrage).

## 5.3 Existence of an hedging portfolio

There is a completely different way to derive the fundamental pricing formula that we will consider now.

Let us first assume that an self-financing strategy that duplicate the option exists. Using the proposition 5.2, we know that the correspondfing portfolio $\tilde{\Pi}_t$ satisfies $\tilde{\Pi}_t = \tilde{\Pi}_0 + \int_0^t \alpha_s d\tilde{S}_s$. As $dS_t = rS_t dt + \sigma S_t dB_t$, we have $d\tilde{S}_t = \sigma\tilde{S}_t dB_t$. Thus $\tilde{\Pi}_t = \tilde{\Pi}_0 + \int_0^t \alpha_s\sigma\tilde{S}_s dB_s$. Now, using the fact that as a stochastic integrale the portfolio $\tilde{\Pi}_t$ is a martingale, it follows that $\tilde{\Pi}_t = \mathbb{E}(\tilde{\Pi}_T/\mathcal{F}_t)$ and thus $\Pi_t = \mathbb{E}(\varphi(S_T)/\mathcal{F}_t)$.

As we know now that if a self-financing strategy that duplicate the option exists, the price of the corresponding portfolio has to be equal to the formula, it remains to prove the existence of such a portfolio, namely the existence of such a strategy. From the final value of the portfolio $\Pi_T = \varphi(S_T)$, one define $\tilde{\Pi}_t = \mathbb{E}(\tilde{\Pi}_T/\mathcal{F}_t)$ as a $\mathcal{F}_t^B-$martingale closed by it final values. Now using the martingale representation theorem 3.2, we know that there existe a (square integrable) stochastic process $\beta_t$ such that

$$\tilde{\Pi}_t = \tilde{\Pi}_0 + \int_0^t \beta_s dB_s.$$

If $\alpha_t := \frac{\beta_t}{\sigma \tilde{S}_t}$ and if $\alpha_t^0 := \tilde{\Pi}_t - \alpha_t \tilde{S}_t$ then $(\alpha_t^0, \alpha_t)$ is indeed a hedging strategy for the option we were looking for.

# Chapter 6

# Arbitrage probabilities

## 6.1 Uncomplete markets

## 6.2 Arbitrage-free markets

## 6.3 The two fundamental theorems

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