ALGEBRAIC AND HOMOLOGICAL PROPERTIES OF POWERS AND SYMBOLIC POWERS OF IDEALS

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1. Lecture: On the regularity of powers of an ideal

In this lecture we study the asymptotic behaviour of Catelnuovo–Mumford regularity of powers of a graded ideal I and give a proof of the theorem, proved by Cutkosky, Herzog, Trung [3] and Kodiyalam [7], that the regularity of I^k is a linear function of k for large k. First we recall some basis facts from commutative homological algebra. We consider graded free resolutions, regularity and projective dimension of a graded ideal and the Rees algebra associated to an ideal.

The study of powers of an ideal was initiated by a result of Bertram, Ein and Lazarsfeld [1] who proved the following vanishing theorem: let X be a smooth projective variety, and let d_X denote the minimum of the degrees d such that X is a scheme-theoretic intersection of hypersurfaces of degree at most d. Then there is a number e such that

$$H^i(\mathbb{P}^s, \mathscr{I}_X^n(a)) = 0 \quad \text{for all} \quad a \geq nd_X + e, \quad i \geq 1.$$

The proof uses the Kodaira vanishing theorem. Later Swanson [8] showed that for any graded ideal $I \subset K[x_1, \ldots, x_n]$ the regularity of its powers are bounded by a linear function. In case that $\dim S/I$ one even has $\operatorname{reg}(I^k) \leq k\operatorname{reg}(I)$, as was shown by Geramita, Gimigliano and Pittelloud [4] and Chandler [2].

Let K be field and $S = K[x_1, \ldots, x_n]$ the polynomial ring over K. We consider S as a standard graded K-algebra by assigning to each x_i the degree 1. Let M be a a finitely generated graded S-module. Then M admits a minimal graded free resolution

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

with $F_i = \bigoplus_{j=1}^{\beta_i} S(-a_{ij})$ for $i = 0, \ldots, p$. The numbers a_{ij} are uniquely determined by M. In fact, since the resolution is minimal, one obtains the graded isomorphisms

$$\operatorname{Tor}_i(M, S/\mathfrak{m}) \cong H_i(F \otimes_S S/\mathfrak{m}) \cong F_i/\mathfrak{m}F_i \cong \bigoplus_j K(-a_{ij}).$$

We write

$$F_i = \bigoplus_j S(-j)^{\beta_{ij}(M)},$$

and call $\beta_{ij} = \beta_{ij}(M)$ the ijth graded Betti number of M.

The graded Betti numbers determine the most important invariants of M: $\dim M$, $\operatorname{depth} M$, the Hilbert series and multiplicity of M, the a-invariant, and of course

$$\operatorname{proj}\dim M = \max\{i\colon\ \beta_{ij} \neq 0 \text{ for some } j\},$$

and

$$\operatorname{reg} M = \max\{j\colon\ \beta_{i,i+j} \neq 0 \text{ for some } i\}.$$

Now let $I \subset S$ be a graded ideal. What can be said about $\operatorname{reg}(I^k)$ as a function of k? For example, if $I = (x_1^2x_2, x_1^2x_4^2, x_2x_4^2)$, then a calculation with CoCoA shows that $\operatorname{reg}(I^k) = 4k$ for $k \leq 5$.

Theorem 1.1 (Cutkosky-Herzog-Trung, Kodiyalam)**.** *Let $I \subset S$ be a graded ideal. Then $\operatorname{reg}(I^k) = ak + c$ for $k \gg 0$.*

For the proof of the theorem one considers the Rees algebra of I:

$$R(I) = \bigoplus_{k\geq 0} I^k t^k \subset S[t].$$

The Rees algebra has a natural bigraded structure. For (j,k) we set $R(I)_{(j,k)} = (I^k)_j t^k$. Then $R(I) = \bigoplus_{(j,k)\in\mathbb{Z}^2} R(I)_{(j,k)}$.

Let $I = (f_1,\dots,f_m)$, $\deg f_i = d_i$, and consider the S-algebra homomorphism

$$\varepsilon\colon\ T = S[y_1,\dots,y_m] \longrightarrow R(I) = S[It], \quad y_j \mapsto f_j t.$$

We set $\deg x_i = (1,0)$ and $\deg y_j = (d_j,1)$ for all i and j. Then $\varepsilon\colon\ T \to R(I)$ is an epimorphism of bigraded algebras. Thus we may view $R(I)$ a bigraded T-module and may consider its bigraded minimal free T-resolution

$$\cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow R(I) \longrightarrow 0.$$

For any bigraded T-module E we set $E_k = \oplus_j E_{(j,k)}$. The each E_k is a graded S-module. The bigraded T-resolution of $R(I)$ yields for each k a graded free S-resolution

$$\cdots \longrightarrow (G_1)_k \longrightarrow (G_0)_k \longrightarrow R(I)_k = I^k \longrightarrow 0.$$

Thus the bigraded resolution T-resolution of $R(I)$ encodes all graded S-resolutions of the powers I^k of I. The resolution obtained in this way are in general not minimal, but the information provided by them allows to deduce Theorem 1.1. The details of the proof can be found in [3].

The proof of Theorem 1.1 as given in [3] provides some more information. For a finitely generated graded S-module M we set

$$\operatorname{reg}_i(M) = \max\{j\colon\ \beta_{i,i+j}(M) \neq 0\}.$$

Then one has

$$\operatorname{reg}_i(I^k) = a_i k + c_i \quad \text{for } k \gg 0,$$

and hence $\operatorname{reg}(I^k) = \max_i\{a_i k + c_i\}$ for $k \gg 0$.

Problem 1.2. Is it true that all a_i are the same. This is known to be true for $i \leq \operatorname{height} I$, [6].

Additional information:

(1) Let $\operatorname{reg}(I^k) = ak + c$ for $k \gg 0$. Kodiyalam showed that

$$a = \min\{\operatorname{reg}_0(J) \colon J \text{ is a graded reduction of } I\}.$$

A graded reduction of I is a graded ideal J such that $I^{k+1} = JI^k$ for all $k \gg 0$. In particular it follows that

$$a \leq \operatorname{reg}_0(I) = d(I) = \text{highest degree of a generator in a minimal set generators of } I.$$

(2) $\lim_{k\to\infty} d(I^k)/k = \lim_{k\to\infty} \operatorname{reg}(I^k)/k$.

(3) Let I be a graded ideal generated in degree d with $\dim S/I = 0$. Then $\operatorname{reg}(I^k) = kd + c_k$ with $c_1 \geq c_2 \geq \cdots$, and this sequence stabilizes.

Eisenbud, Huneke and Ulrich [6] proved, that if $\dim S/I = 0$ and I has linear relations, then I^k has a linear resolution for $k \gg 0$.

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2. Lecture: Powers of ideals with linear resolution and linear quotients

If an ideal I has a linear resolution, then this does not imply that all powers I^k of I have a linear resolution. The Stanley–Reisner ideal of the projective plane is such an example, as was first observed by Terai. There are examples given by Conca [5] which demonstrate that the powers of an ideal may all have a linear resolution up to given power k, but then fail to have a linear resolution for the $(k+1)$th power. On the other hand it will be shown that if the defining ideal J of the Rees algebra $R(I)$ of I satisfies a certain Gröbner basis condition, then all powers do indeed have a linear resolution. A somewhat stronger condition on the Gröbner basis makes even sure that all powers of I have linear quotients. Finally we discuss ideals whose powers are componentwise linear and present some open conjectures.

We begin with the example of Conca [5]: Let $I=(x_1z_1^d, x_1z_2^d, x_2z_1^{d-1}z_2,)+(z_1,z_2)^{d+1}$. Then $\operatorname{reg}(I^k)=k(d+1)$, i.e. I^k has a linear resolution for $k<d$, while $\operatorname{reg}(I^d)\geq d(d+1)+(d-1)$, i.e. I^d goes not have a linear resolution.

The question arises how one check that all powers of I have a linear resolution. We assume that all generators $f_1,\dots,f_m$ of I have the same degree d. Then the Rees ring $R(I)$ can be given the standard bigrading:

$$\deg x_i=(1,0),\quad \deg f_j=(0,1),$$

and we set $\deg y_j=(0,1)$, so that $T=S[y_1,\dots,y_m]$ is a standard bigraded K-algebra. Let

$$0\to F_p\longrightarrow\cdots\longrightarrow F_0\longrightarrow R(I)\longrightarrow 0$$

a minimal bigraded free T-resolution. Then

$$F_i=\bigoplus_j T(-a_{ij},-b_{ij})\quad\text{for}\quad i=0,\dots,p.$$

Definition 2.1. The x-regularity of $R(I)$ is the number $\operatorname{reg}_x(R(I))=\max_{i,j}\{a_{ij}-i\}$.

With the methods used in the proof of Theorem 1.1 one gets

Theorem 2.2 (Römer)**.** $\operatorname{reg}(I^k)\leq kd+\operatorname{reg}_x R(I)$ *for all* k.

Corollary 2.3. *If* $\operatorname{reg}_x R(I)=0$, *then all powers of* I *have a linear resolution.*

The x-regularity is not so easy to determine. But one has the following weaker condition which guarantees that all powers of I have a linear resolution.

Corollary 2.4 (Herzog-Hibi-Zheng)**.** *Suppose* I *is generated in one degree, and write* $R(I)=T/J$. *Then each power of* I *has a linear resolution, if for some monomial order* $<$ *on* T, *the ideal* J *has Gröbner basis* G, *whose elements are at most linear in the* x_i, *that is,* $\deg_x f\leq 1$ *for all* $f\in G$.

This criterion was used in [3] to prove the following

Theorem 2.5 (Herzog-Hibi-Zheng)**.** *Let* I *be an ideal generated in degree* 2 *with linear resolution. Then all powers of* I *have a linear resolution.*

Ideals with linear resolution arise naturally as follows:

Definition 2.6. Let I be a graded ideal which is generated in one degree. Then I has linear quotients, if I can be minimally generated by $f_1,\dots,f_m$ such that the colon ideals $(f_1,\dots,f_{i-1}):f_i$ are generated by linear forms for $i=1,\dots,m$.

One easily proves

Proposition 2.7. *If* I *has linear quotients, then* I *has a linear resolution.*

Examples 2.8. (1) Let I be generated by all (squarefree) monomials of degree d. Then I has linear quotients with respect to the lexicographic order of the generators.

(2) Let I be the Stanley–Reisner ideal of the natural triangulation of the real projective plane. The I has a linear resolution, but no linear quotients.

(3) Let P be a finite poset. A subset $I\subset P$ is called a poset ideal, if for $p\in I$ and $q\leq p$ it follows that $q\in I$. We denote by $\mathscr{I}(P)$ the set of all poset ideals of P, and define the ideal

$H_P \subset K[\{x_p, y_p\} p \in P]$ which is generated by the monomials $\prod_{p\in P} x_p \prod_{p\notin P} y_p$, $I \subset \mathscr{I}(P)$. Then all powers of the ideal H_P have linear quotients. This can be shown by using the following theorem.

Let I be a monomial ideal minimally generated by $u_1, \ldots, u_m$ with all u_i of same degree. As before we consider the presentation $T = S[y_1, \ldots, y_m] \to R(I)$, $y_i \mapsto u_i t$, and define a monomial order as follows:

(1) Let $<_{lex}$ be the lexicographic order induced by $x_1 > x_2 > \cdots > x_n$.
(2) Let $<^{\sharp}$ be any monomial order on $K[y_1, \ldots, y_m]$. Then if $u = x^a y^b$, $v = x^c y^d$, we set $u <^{\sharp}_{lex} v$, if
  (i) $y^b <^{\sharp} y^d$, or
  (ii) $y^b = y^d$ and $x^a <_{lex} x^c$.

Theorem 2.9 (Herzog-Hibi)**.** *If I satisfies the x-condition with respect to $<^{\sharp}_{lex}$, then all powers of I have linear quotients.*

A natural generalization of the concept linear resolution, is that of componentwise linearity.

Definition 2.10. *A graded ideal $I \subset S$ is said to be componentwise linear, if (I_j) has a linear resolution for all j.*

If I has a linear resolution, then it is componentwise linear.

Examples 2.11. (1) Any stable monomial ideal is componentwise linear.

(2) Let G be a graph on the vertex set $[n]$. A vertex cover of G is a subset C of $[n]$ such that $C \cap \{i, j\} \neq \emptyset$ for all edges $\{i, j\}$ of G. C is called minimal, if no proper subset of C is a vertex cover.

Let G be a chordal graph. We consider the ideal

$$I_G = \{x_C \colon C \text{ is a minimal vertex cover of } G\}$$

of vertex covers of G. Francisco and van Tuyl proved [2] that I_G is componentwise linear.

Conjectures: (1) If G is a chordal graph, then all powers of G are componentwise linear.

(2) Let I be the defining ideal of a rational normal scroll, i.e. the ideal of 2-minors of a matrix consisting of blocks where each block is of the form

Then all powers of I have a linear resolution. This is proved in several cases, for example when I is the ideal of 2-minors of

In both cases conjectured here, the x-condition does not help. However there exist a necessary and sufficient condition for an ideal I that all its powers are componentwise linear. To describe this condition we recall

Definition 2.12 (Huneke)**.** Let R be a ring and M an R-module. A sequence $f_1, \ldots, f_m \in R$ is called a d-sequence with respect to M, if

$$(f_1, \ldots, f_{i-1})M :_M f_i \cap (f_1, \ldots, f_m)M = (f_1, \ldots, f_{i-1})M \quad \text{for all } i = 1, \ldots, m.$$

Theorem 2.13 (Herzog-Hibi-Ohsugi)**.** *Suppose K is an infinite field, $I \subset S = K[x_1, \ldots, x_n]$ a graded ideal and $y_1, \ldots, y_n$ a generic K-basis of S_1. Then the following conditions are equivalent:*

(a) *All powers of I are componentwise linear.*
(b) *The sequence $y_1,\dots,y_n$ is a d-sequence with respect to $R(I)$.*

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3. LECTURE: ON THE THE GROWTH OF THE BETTI NUMBERS AND THE DEPTH OF POWERS OF AN IDEAL

Let S be either a Noetherian local ring or a standard graded K-algebra, and let $I\subset S$ be a (graded) ideal. A classical result by Burch [4] says that $\min_k \operatorname{depth} S/I^k\leq d-\ell(I)$, where $\ell(I)$ the analytic spread of I. By a theorem of Brodmann [2], $\operatorname{depth} S/I^k$ is constant for $k\gg 0$. We call this constant value the limit depth of I, and denote it by $\lim_{k\to\infty}\operatorname{depth} S/I^k$. Brodmann improved the Burch inequality by showing that $\lim_{k\to\infty}\operatorname{depth} S/I^k\leq d-\ell(I)$. Eisenbud and Huneke [5] showed that equality holds, if the associated graded ring $\operatorname{gr}_I(S)$ is Cohen–Macaulay. This is for example the case if S and $R(I)$ are Cohen–Macaulay, see Huneke [7]. We will present a proof of these facts.

What can be said about $\beta_i(I^k)$ for $k\gg 0$? We have

$$\beta_i(I^k)=\dim_K \operatorname{Tor}_i^S(S/\mathfrak{m},I^k)=\dim_K H_i(x;I^k),$$

where $H_i(x;M)$ denotes the Koszul homology of a module with respect to $x=x_1,\dots,x_n$. Since $H_i(x;R(I))_k=H_i(x;I^k)$ It follows that $\beta_i(I^k)=\dim_K H_i(x;R(I))_k$. We observe that $H_i(x;R(I))$ is a graded $H_0(x;R(I))$-module and since

$$H_0(x;R(I))=R(I)/\mathfrak{m}R(I)=\bigoplus_{k\geq 0} I^k/\mathfrak{m}I^k=\overline{R}(I)=\text{the fiber ring of } I,$$

we see that $H_i(x;R(I))$ is a graded $\overline{R}(I)$-module. The Krull dimension of $\overline{R}(I)$ is called the analytic spread of I. By using the theory of Hilbert functions we deduce that

$$\beta_i(I^k)=\operatorname{Hilb}(k,H_i(x;R(I)))$$

is a polynomial function for $k\gg 0$ of degree $\dim H_i(x;R(I))-1\leq \dim\overline{R}(I)-1=\ell(I)-1$.

We denote by $P_i(I)$ the polynomial with $P_i(I)(k)=\beta_i(I^k)$ for $k\gg 0$. Using the fact that the Koszul complex is a rigid complex one shows

Proposition 3.1. $\ell(I)-1=\deg P_0(I)\geq \deg P_1(I)\geq \deg P_2(I)\geq\cdots$

As an immediate consequence one obtains

Corollary 3.2. $\operatorname{proj\,dim} I^k$ *and* $\operatorname{depth} I^k$ *stabilize for* $k \gg 0$.

Question 3.3. Is it true that $\deg P_0(I) = \deg P_1(I) = \cdots = \deg P_q(I)$, where $q = \operatorname{proj\,dim} I^k$ for $k \gg 0$? This is known to be the case, if $\dim S/I = 0$. Thus in this case it follows that $\mu(I^k) := \dim_K(I^k/\mathfrak{m}I^k)$ has the same degree of growth as $\dim_K \operatorname{Socle}(S/I^k) = \dim_K(I^k : \mathfrak{m}/I^k)$.

Concerning the limit behavior of the depth the following is known [7].

Theorem 3.4. *The limits* $\lim_{k\to\infty} \operatorname{depth} I^k$, $\lim_{k\to\infty} \operatorname{depth} S/I^k$ *and* $\lim_{k\to\infty} \operatorname{depth} I^k/I^{k+1}$ *exist and*

$$\lim_{k\to\infty} \operatorname{depth} S/I^k \leq \lim_{k\to\infty} \operatorname{depth} S/I^k \lim_{k\to\infty} \operatorname{depth} I^k - 1 = \lim_{k\to\infty} \operatorname{depth} I^k/I^{k+1} = \dim S - \ell(I).$$

What is the initial behavior of $\operatorname{depth} I^k$? As a first result we have

Proposition 3.5. *If* $I \subset S$ *is a graded ideal all of whose powers have a linear resolution. Then* $\operatorname{depth} S/I^k \geq \operatorname{depth} S/I^{k+1}$ *for all* k.

Problem 3.6. Is Proposition 3.5 also true for componentwise linear ideals?

Now suppose that I has linear quotients with respect to the homogeneous minimal system of generators of I. We denote by $q_i(I)$ the minimal number of generators of $(f_1,\ldots,f_{i-1}) : f_i$ and by $q(I) = \max_i\{q_i(I)\}$. By using a mapping cone argument one easily shows that

$$\operatorname{depth} S/I = n - q(I) - 1.$$

We use this formula to compute for a finite poset P the depth of the ideal

$$H_P^k \subset K[x_1,\ldots,x_n,y_1,\ldots,y_n]$$

defined in Lecture 2. Here $q(H_P^k)$ has the following interpretation. It is the largest integer N for which there exists a sequence $(A_1, A_2, \ldots, A : r)$, $r \leq k$ of antichains of P such that

(i) $A_i \cap A_j = \emptyset$ for $i \neq j$;
(ii) $\langle A_1 \rangle \subset \langle A_2 \rangle \subset \cdots \subset \langle A_r \rangle$, where for a subset $A \subset P$, the set $\langle A \rangle$ denotes the poset ideal generated by A.
(iii) $N = \sum_{i=1}^r |A_i|$.

From this formula for $q(H_p^k)$ one easily deduces

Corollary 3.7. *Given integers* $a_1 \geq a_2 \geq \cdots \geq a_r > 0$ *with* $\sum a_i = n$. *There exists a poset* P *such that*

$$\operatorname{depth} S/H_P^k == \begin{cases} 2n - (a_1 + \cdots + a_k) - 1, & \text{if } k \leq r-1, \\ n-1, & \text{if } k > n. \end{cases}$$

Now Corollary 3.7 implies

Corollary 3.8. *Given a nonincreasing function* $f\colon \mathbb{N} \to \mathbb{N}$ *with* $f(0) = 2\lim_{k\to\infty} f(k) + 1$ *for which the difference function* Δf *is also nonincreasing. Then there exists a squarefree monomial ideal* $I \subset S$ *with* $\operatorname{depth} S/I^k = f(k)$ *for all* k.

So far we have only seen nondecreasing depth functions. The more surprising is the following

Theorem 3.9. *Let $f\colon \mathbb{N}\setminus\{0\}\to\mathbb{N}$ be any bounded nondecreasing function. Then there exists a monomial ideal I such that* $\operatorname{depth} S/I^k = f(k)$ *for all k.*

But there exist also depth functions which are not monotonic. Consider for example the monomial ideal

$$I=(a^6,a^5b,ab^5,b^6,a^4b^4c,a^4b^4ed,a^4e^2f^3,b^4e^3f^2)\subset K[a,b,c,d,e,f].$$

Then $\operatorname{depth} S/I=0$, $\operatorname{depth} S/I^2=1$, $\operatorname{depth} S/I^3=0$ and $\operatorname{depth} S/I^4=\operatorname{depth} S/I^5=2$.

Problem 3.10. Given any function $f\colon \mathbb{N}\setminus\{0\}\to\mathbb{N}$ which is eventually constant. Does there exist a graded ideal I such that $\operatorname{depth} S/I^k=f(k)$ for all k? Or at least can one find examples of graded ideals whose depth function has any given number of local maxima?

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4. LECTURE: SYMBOLIC POWERS

Let K be an algebraically closed field, $Y\subset\mathbb{A}^n$ an affine variety, $P=Z(Y)$ the vanishing ideal of Y and $a=(a_1,\dots,a_n)\in\mathbb{A}^n$ a point. We say that $f\in S=K[x_1,\dots,x_n]$ vanishes at a of order $\geq k$, if $f\in\mathfrak{m}_a^k$, where $\mathfrak{m}_a=(x_1-a_1,x_2-a_2,\dots,x_n-a_n)$, and set $P^{\langle k\rangle}=\{f\in S\colon\ f \text{ vanishes of order} \geq k \text{ at every point of } Y\}$. On the other hand, one defines the kth symbolic power $P^{(k)}$ of P as follows: $P^{(k)}=\{f\in S\colon\ fg\in P^k \quad \text{for some } g\notin P\}$. It is easily seen that $P^{(k)}=\mathrm{Ker}(S\to S_P/P^kS_P)$. We have $P^{(1)}=P$ and $P^k\subseteq P^{(k)}$. In this lecture we first prove the theorem of Zariski-Nagata which says that $P^{\langle k\rangle}=P^{(k)}$. A generalization of this theorem was given by Eisenbud and Hochster [4]. Finally we consider more general symbolic powers.

Theorem 4.1 (Zariski-Nagata). $P^{\langle k\rangle}=P^{(k)}$.

Proof. Here we only show: if $\operatorname{char} K=0$, then $P^{(k)}\subseteq P^{\langle k\rangle}$. Considering the Taylor expansion of f at a point, we see that $f\in P^{\langle k\rangle}$ if and only if f and all its derivatives of order $<k$ vanish at all points of Y.

In order to prove $P^{(k)}\subseteq P^{\langle k\rangle}$ we show that if $fg\in P^k$ and $g\notin P$, then $f\in P^{\langle k\rangle}$. Indeed, since $g\notin P$, there exists $a\in Y$ such that $g\notin\mathfrak{m}_a$. Since $fg\in\mathfrak{m}_a^k$, we conclude that $f\in\mathfrak{m}_a^k$. Therefore f and all derivatives of f of order $<k$ vanish on the set

$$X=\{a\in Y\colon\ g(a)\neq 0\}.$$

Let h be such a derivative. Then hg vanishes on Y. Hilbert's Nullstellensatz implies that $h\in P$. It follows that $f\in P^{\langle k\rangle}$. □

What can be said about the regularity of the symbolic powers? In general the symbolic Rees algebra is not finitely generated, see [2], [3] , [7] and [8]. We introduce generalized symbolic powers and show that for monomial ideals these powers have a finitely generated symbolic Rees algebra. Consequently the regularity of these powers is linearly bounded. Other cases for which there exists linear bounds are given in the paper [6].

One defines the symbolic Rees algebra of P as

$$\mathscr{R}^s(P) = \bigoplus_{k\geq 0} P^{(k)}t^k \subset S[t].$$

In 1985 Cowsik [5] showed that if a prime ideal P in a regular local ring R with $\dim R/P = 1$ has the property that its symbolic Rees algebra $\mathscr{R}^s(P)$ is Noetherian, then P is a set theoretic complete intersection, and he raised the question whether under the given conditions, $\mathscr{R}^s(P)$ is always Noetherian. P. Roberts [8] was the first to find a counterexample based on the counterexamples of Nagata to the 14th problem of Hilbert.

Since $R^s(P)$ is in general not finitely generated, we cannot hope that $\operatorname{reg} P^{(k)}$ is a linear function for all $k \gg 0$. But do there exist integers a, c such that $\operatorname{reg} P^{(k)} \leq ak + c$ for all k. There is no counterexample known to this question. We only know

Theorem 4.2 (Herzog-Hoa-Trung)**.** *Let $P \in \operatorname{Spec}(S)$ be a graded prime ideal with $\dim S/P = 1$. Then there exist $a, b \in \mathbb{Z}$ such that*

$$\operatorname{reg}(P^{(k)}) \leq ak + b.$$

Let $I \subset S$ be an arbitrary graded ideal and $\mathfrak{m} = (x_1, \dots, x_n)$ the graded maximal ideal of S. One may also consider *saturated powers* of I which are defined as follows:

$$(I^k)^{\text{sat}} = I^k \colon \mathfrak{m}^\infty = \{f \in S \colon \ \mathfrak{m}^n f \subset I^k \quad \text{for some } n\}.$$

In [3] it is shown that the regularity of saturated powers can be linearly bounded. On the other hand, it is shown that for any prime number $p > 0$ with $p \equiv \bmod 3$, there exists a field k of characteristic p and a graded ideal $I \subset k[x, y, z]$ such that

$$\operatorname{reg}(I^{5n+1})^{\text{sat}} = \begin{cases} 29n + 7, & \text{if } n \text{ is not a power of } p, \\ 29n + 8, & \text{otherwise.} \end{cases}$$

The examples show that regularity of saturated powers of a graded ideal may not be a periodic linear function.

We now consider general symbolic powers: Let $I, J \subset S = k[x_1, \dots, x_n]$ be graded ideals. Then we call

$$I^k : J^\infty = \{f \in S \colon \ fJ^n \subset I^k \quad \text{for some } k\}$$

the kth symbolic power of I with respect to J. The following examples show how this definition is related to the concepts introduced before.

Examples 4.3. (1) Let $J = \mathfrak{m} = (x_1, \dots, x_n)$. Then $I^k \colon J^\infty = (I^k)^{\text{sat}}$.

(2) Set $\operatorname{Ass}^*(I) = \{P \in \operatorname{Spec}(S) \colon \ P \in \operatorname{Ass}(I^n) \quad \text{for some } n\}$. By a theorem of Brodman [**?**], $\operatorname{Ass}(I^k)$ is asymptotically stable. In other words, there exists an integer d such that

$\mathrm{Ass}(I^k) = \mathrm{Ass}(I^{k+1})$ for all $k \geq d$. In particular, $\mathrm{Ass}^*(I)$ is a finite set. It is called the set of asymptotic prime ideals of I. We set

$$J = \bigcap_{P \in \mathrm{Ass}^*(I) \setminus \mathrm{Min}(I)} P.$$

Then the ideals $I^{(k)} = I^k \colon J^\infty$ are called the ordinary symbolic powers of I. In case $I = P$ is a prime ideal, then these are the symbolic powers defined before.

We set

$$\mathscr{R}_J(I) = \bigoplus_{k \geq 0} (I^k \colon J^\infty) t^k$$

$\mathscr{R}_J(I)$ is the symbolic Rees algebra of I with respect to J.

Even though we cannot expect in general that $\mathscr{R}_J(I)$ is finitely generated one has

Theorem 4.4 (Herzog-Hibi-Trung)**.** *Let I and J be monomial ideals. Then $\mathscr{R}_J(I)$ is a finitely generated S-algebra.*

As a consequence of this result one obtains

Corollary 4.5. *Let I and J be a monomial ideals and $I^{(k)} = I^k \colon J^\infty$ for all k. Then* $\mathrm{reg}(I^{(k)})$ *is a periodic linear function for $k \gg 0$, i.e. there exists an integer d and integers a_i and b_i such that* $\mathrm{reg}(I^{(k)}) = a_i k + b_i$ *for $k \equiv i \bmod d$ and all $k \gg 0$.*

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